2.10 PUPILS GENERALISING

We attach three research papers on learners’ algebraic thinking. The first is a general paper on learners’ misconceptions, seen from a constructivist learning theory. The other two papers are about children’s generalisation thinking strategies.

PROBLEM 110: CHILDREN’S GENERALISING

... the research brings Good News and Bad News. The Good News is that, basically, students are acting like creative young scientists, interpreting their lessons through their own generalizations. The Bad News is that their methods of generalizing are often faulty.  

Steve Maurer, 1987

Discuss, using examples from your own class.

PROBLEM 111:
The following is asked to the learners:

Mapoela builds the following pattern of triangles. How many matches does she need to build 100 such triangles?

Many students solve the problem in the following way:

For four triangles we need 9 matches.

So for 100 triangles we need \((100 \div 4) \times 9 = 225\) matches.

Identify the mistake of the learners and explain the origin of the mistake.

Explain and describe a possible intervention to correct such a mistake.

PROBLEM 112: HOMEWORK

In your class children had to factorise \(x^2 - ax - bx + ab\) for homework.

You give the answer as \((x - a)(x - b)\).

Looking through their work, you notice that Nicola’s answer is \((x - b)(x - a)\) and that she has marked it wrong.

Also, Vusi’s answer is \((a - x)(b - x)\) and he has marked it wrong.

How do you react?

PROBLEM 113: FRACTIONS

Learners were asked to find a fraction between \(\frac{3}{4}\) and \(\frac{5}{6}\).

John answers: \(\frac{4}{5}\), adding: “because 4 is between 3 and 5, and 5 is between 4 and 6.”

How do you respond?
HANDLING PUPILS’ MISCONCEPTIONS

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This paper will briefly delineate a theory for learning mathematics as a basis to reflect on (some particular) misconceptions of pupils in mathematics. Such a theory should enable us

- to predict what errors pupils will typically make
- to explain how and why children make (these) errors
- to help pupils to resolve such misconceptions.

1. THE ROLE OF THEORY

Teachers are often wary of theory - they want something practical. Yet, as Dewey has said, “in the end, there is nothing as practical as a good theory.” How come? Theory is like a lens through which one views the facts; it influences what one sees and what one does not see. “Facts” can only be interpreted in terms of some theory. Without an appropriate theory, one cannot even state what the “facts” are. Let me illustrate with a story, taken from Davis (1984).

It is said that in Italy in the 1640’s, the water table had receded so far that a very deep well had to be sunk in order to reach water. This was done. Then pumps were fitted to the pipes, and ... disaster! ... no water poured out of the spigot. It was clear that something was wrong, but what? Their understanding of the situation, i.e. their theory of pumping water, was that it was the pumps that pulled (sucked) the water to the surface. So the fault had to be with the pumps. New pumps were installed ... better pumps were designed, built and installed ... still no water. Then still better pumps, then even better ones. But the result was always the same: no water emerged from the spigot. They were baffled.

Finally, in 1643, Evangelista Torricelli, who invented the barometer, presented an alternative explanation (theory): It was not, he said, the pumps that pulled the water up. The pumps merely evacuated air from the pipes, creating unequal pressures at the two ends of the column of water, after which it was the atmospheric pressure that pushed the water up the pipes. This explained the difficulty: the air pressure is about 1 kg/cm², which is enough to support a column of water 10 m high. It follows that if the water in a well is more than 10 m deep, it cannot be pumped to the surface using atmospheric pumps. Building better and better atmospheric pumps would not resolve the issue – and that probably led to the invention of hydraulic pumps, which could do the job.

Let us again consider the role of theory. First, one cannot even discuss the matter without using some theory to explain the situation. Second, the objective fact that no water came out of the pumps, like the fact that a car refuses to start, does not lead anywhere. Unless you can say why there is no water, or why the car will not start, you are unable to do anything to change the situation. And in order to say why, you must interpret the “facts” in terms of an appropriate theory. Third, notice how the two different theories differed in their interpretation of the “facts” and suggested – prescribed! – different remedies to resolve the issue: one remedy was doomed, while the other offered some hope.

Now, has this story anything to say about the subject under discussion, i.e. pupils’ misconceptions in mathematics? The fact is that our pupils often make mistakes in mathematics – don’t we know it! But unless we can say why they make these mistakes, we are unable to do something about it. And in order to say why, we must interpret these mistakes in terms of a theory – a learning theory. As teachers, all our interventions in the classroom are guided by some theory – be it conscious or subconscious – of how children learn mathematics. Different teachers hold different learning theories, and address pupils’ mistakes in different ways. Could it be that all our frustrated efforts at eliminating errors are due to embracing an inappropriate learning theory – that we are trying to build better and better atmospheric pumps?

An escapist route, which is nevertheless a theory, is to view many pupils as rather dim, that they are not capable of understanding, and should rather not take mathematics. In general, it is not very useful to think of children’s errors in terms of low intelligence, low mathematical aptitude, perceptual difficulties or learning disabilities. Of course these factors play a role, but if we are really concerned with helping individual children, such abstract
ideas won’t help – it is only when we work at the level of specific detail and get to know the specific roots of mistakes, that we are able to help.

The type of theory we adopt will also determine the importance of misconceptions for learning and teaching. Why should we care about pupils’ misconceptions? What is the role of pupils’ misconceptions in their learning? How will knowing what a pupil has got wrong help us to teach better?

2. LEARNING THEORY

I shall briefly outline two opposing learning theories, which will, by necessity, be both simple (presenting the ideas in oversimplified form) and simplistic (presenting the ideas in its most radical form), but which will illustrate different approaches to handling pupils’ misconceptions.

2.1 Behaviourism

The behaviourist or connectionist theory of learning relates to an empiricist philosophy of science, that all knowledge originates in experience. The traditional empiricist motto is “There is nothing in the mind that was not first in the senses.” Hence a person can obtain direct and absolute knowledge of any reality, because, through the senses, the image of that reality corresponds exactly with the reality (a replica or photo-copy).

Behaviourism therefore assumes that pupils learn what they are taught, or at least some subset of what they are taught, because it is assumed that knowledge can be transferred intact from one person to another. The pupil is viewed as a passive recipient of knowledge, an “empty vessel” to be filled, a blank sheet (tabula rasa) on which the teacher can write. Behaviourists, therefore, believe that knowledge is taken directly from experience, and that a pupil’s current knowledge is unnecessary to learning.

This theory sees learning as conditioning, whereby specific responses are linked with specific stimuli. According to Thorndike’s (1922) law of exercise, the more times a stimulus-induced response is elicited, the longer the learning (response) will be retained. The law of effect states that appropriate stimulus-response bonds are strengthened by success and reward (positive reinforcement) and inappropriate S-R bonds are weakened by failure (negative reinforcement). Consequently the organisation of learning must proceed from the simple to the complex, short sequences of small items of knowledge and exercise of these in turn through drill and practice. One learns by stockpiling, by accumulation of ideas (Bouvier, 1987).

From a behaviourist perspective, errors and misconceptions are not important, because it does not consider pupils’ current concepts as relevant to learning. Errors and misconceptions are seen rather like a faulty byte in a computer’s memory – if we don’t like what is there, it can simply be erased or written over, by telling the pupil the correct view of the matter (Strike, 1983). This perspective is succinctly put by Gagne (1983: 15):

> The effects of incorrect rules of computation, as exhibited in faulty performance, can most readily be overcome by deliberate teaching of correct rules ... This means that teachers would best ignore the incorrect performances and set about as directly as possible teaching the rules for correct ones.

2.2 Constructivism

A constructivist perspective on learning (e.g. Piaget, 1970; Skemp, 1979) assumes that concepts are not taken directly from experience, but that a person’s ability to learn from and what he learns from an experience depends on the quality of the ideas that he is able to bring to that experience. This is again the same idea as our introduction about the role of theory: observation is driven by theory, so the quality of the observation is determined by the quality of the pre-existing theory. Knowledge does not simply arise from experience. Rather, it arises from the interaction between experience and our current knowledge structures.

The student is therefore not seen as passively receiving knowledge from the environment; it is not possible that knowledge can be transferred ready-made and intact from one person to another. Therefore, although instruction clearly affects what children learn, it does not determine it, because the child is an active participant in the construction of his own knowledge. This construction activity involves the interaction of a child’s existing ideas and new ideas, i.e. new ideas are interpreted and understood in the light of that child’s own current knowledge, built up out of his previous experience. Children do not only interpret knowledge, but they organise and structure this knowledge into large units of interrelated concepts. We shall call such a unit of interrelated ideas in the child’s mind a schema. Such schemas are valuable intellectual tools, stored in memory, and which can be retrieved and utilised. Learning then basically involves the interaction between a child’s schemas and new ideas. This interaction involves two interrelated processes:
Assimilation: If some new, but recognisably familiar, idea is encountered, this new idea can be incorporated directly into an existing schema that is very much like the new idea, i.e. the idea is interpreted or re-cognised in terms of an existing (concept in a) schema. In this process the new idea contributes to our schemas by expanding existing concepts, and by forming new distinctions through differentiation.

Accommodation: Sometimes a new idea may be quite different from existing schemas; we may have a schema which is relevant, but not adequate to assimilate the new idea. Then it is necessary to re-construct and re-organise our schema. Such re-construction leaves previous knowledge intact, as part or subset or special case of the new modified schema (i.e. previous knowledge is never erased).

Thus to understand an idea means to incorporate it into an appropriate existing schema. However, sometimes some new idea may be so different from any available schema, that it is impossible to link it to any existing schema, i.e. assimilation or accommodation is impossible. In such a case the learner creates a new “box” and tries to memorise the idea. This is rote learning: because it is not linked to any previous knowledge it is not understood; it is isolated knowledge, therefore it is difficult to remember. Such rote learning is the cause of many mistakes in mathematics as pupils try to recall partially remembered and distorted rules.

To the constructivist learning is not, as for the behaviourist, a matter of adding, of stockpiling new concepts to existing ones. Rather, learning leads to changes in our schemas.

It is clear that the character of a pupil’s existing schemas will determine what he learns from experience or information and how it is understood. At the heart of a constructivist approach to teaching is an awareness of the interaction between a child’s current schemas and learning experiences, to look at learning from the perspective of the child, for the teacher to put himself in the child’s shoes, by considering the mental processes by which new knowledge is acquired. Because knowledge cannot be transferred ready-made, to support the child to construct his own knowledge, discussion, communication, reflection and negotiation are essential components of a constructivist approach to teaching.

From a constructivist perspective misconceptions are crucially important to learning and teaching, because misconceptions form part of a pupil’s conceptual structure that will interact with new concepts, and influence new learning, mostly in a negative way, because misconceptions generate errors.

I distinguish between slips, errors and misconceptions. Slips are wrong answers due to processing; they are not systematic, but are sporadically carelessly made by both experts and novices; they are easily detected and are spontaneously corrected. I shall not consider slips in the rest of the paper. Errors are wrong answers due to planning; they are systematic in that they are applied regularly in the same circumstances. Errors are the symptoms of the underlying conceptual structures that are the cause of errors. It is these underlying beliefs and principles in the cognitive structure that are the cause of systematic conceptual errors that I shall call misconceptions.

If we want to account for pupils’ misconceptions, we must look at pupils’ current schemas and how they interact with each other, with instruction and with experience.

In order to reflect on some typical misconceptions of children, it will be useful to look a little closer at cognitive functioning. We would think of something like the following over-simplified process in cognitive functioning when a pupil tries to solve a problem (Davis, 1983):

1. Some item(s) of information in the problem is (are) selected to act as a cue to trigger the retrieval of a seemingly appropriate schema in the cognitive structure (memory).
2. Specific information from the problem (“values”) are fed into appropriate “variables” in the retrieved schema. (If no values can be supplied, the schema will fill in values itself, from typical values in past experience. We call this a default evaluation.)
3. Some evaluative judgement of the suitability (the “goodness of fit”) of steps 1 and 2 are made (and cycling back where necessary).
4. If the judgement is that steps 1 and 2 have been successful, the result (i.e. the combination of cue information from the problem and the content of the schema) is used to continue.

The process can be illustrated using the solution of the following equation:

\[ x^2 - 5x + 2 = 0 \]

Step 1 consists of some visual cues in the equation, e.g. the exponent and the number of terms, that cause us to say (in effect): “Ah! It’s a quadratic equation!”, with the result that we retrieve from memory the “quadratic equation schema”, which has many items of information, but which also (hopefully!) contains the quadratic formula:
If \( ax^2 + bx + c = 0 \)
then \( x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \)

Step 2 involves looking at our specific problem \( x^2 - 5x + 2 = 0 \) and taking from it certain specific information to enter into the “variables” of our memorised formula in the schema. We see that “1” should be used as a replacement for \( a \), “-5” for \( b \) and “2” for \( c \).

Step 3 involves whatever checks we carry out in order to convince ourselves that this is all correct, after which use of the quadratic formula (step 4) easily produces the answer.

A pupil can fail to solve the problem for many reasons, e.g. in step 1:

- he may not posses the schema that is needed
- he may posses an appropriate schema, but the retrieval mechanism cannot locate it
- the retrieved schema is flawed or incomplete
- an inappropriate schema is retrieved.

It is important to realise that once step 4 has been reached, the solution of the problem is wholly determined by the combined information of the used cues and the content and structure of the retrieved schema, e.g. if in our example above the quadratic formula in the schema (memory) was flawed, our solution would be flawed. We say the solution is mediated by the schema.

Let us now look at some specific misconceptions by analysing in what ways current schemas mediate new learning leading to misconceptions. One should acknowledge, of course, that errors are also a function of other variables in the education process, including the teacher, the curriculum, social factors, affective factors, emotional factors, motivation, attitudes, and possible interactions among these variables. For this paper, however, I shall concentrate only on cognitive variables.

3. SOME MISCONCEPTIONS AND THEIR GENESIS

3.1 Patchwork

What order of difficulty do we expect in the following three additions for young pupils learning column addition?

(A) 523   (B) 593   (C) 586   (D) 586
+ 25         + 25         + 25         +325

Traditional analysis would suggest that (A) should be the easiest, since (B) involves an extra “carry”, (C) two “carries” and (D) involves an extra addition. Surprisingly, (A) is the most difficult for many children! Why should this be, and how do we account for the following frequent responses to (A)?

(E) 523   (F) 523   (G) 523
+ 25         + 25         + 25
748         948         48

Maybe we may diagnose that such pupils don’t understand place-value, or don’t understand “carrying”, or that they don’t know their number combinations, and we may remediate the problem by teaching these “missing concepts”, or by teaching and reteaching the correct procedure directly.

Yet, clinical research (e.g. Davis, 1984) suggests that the “misconception” is elsewhere, and that these errors are quite plausible, as seen from the perspective of the child, whose response is mediated by his existing schemas based on previous learning. To solve (A), the addition cue triggers the child’s addition schema, which may include an item to add column by column, but may also include that addition is a binary operation – to add one needs two numbers. But in (A) there is an isolated digit, only one number! When the pupil is blocked in his progress, he does some patchwork: he distorts the column-by-column rule in order to satisfy the need for two numbers (E and F), or ignores the left-column (G) so as not to violate his notion of addition as a binary operation. This analysis also explains why many pupils are more successful in (B) than in (A). From this analysis it can also be expected that the same phenomenon will show itself in subtraction, and, indeed, it does (e.g. 276 – 14 = 162).

It is clear that successful remediation will build on the pupil’s correct knowledge by introducing 0 as a placeholder in the isolated digit column, so that the child’s addition schema is extended to reconcile the conflicting column-by-column and two-numbers rules. Direct teaching of the correct procedure, on the other
hand, in no way eliminates the underlying cause of the erroneous behaviour and therefore does not change the schema. Although direct instruction may produce a change in performance, this change is often not permanent – before long the original schema reasserts itself, and the child’s behaviour reverts back to what it was before instruction. (See also 3.6)

Pupils often degenerate into distorting rules to allow a schema to overcome some obstacle. Here is a high school example:

If \( e + f = 8 \)

Then \( e + f + g = ? \)

If a pupil’s arithmetic addition schema is retrieved, it will require that numbers be added. Blocked in its progress because no values can be given for the letters, the schema will make a default evaluation and somehow manage to produce replacement numbers. In our research (Olivier, 1984), 58% of std 6 pupils supplied a numerical answer to the question. The most common responses were 12 (from \( 4 + 4 + 4 \)), 15 (from \( 3 + 5 + 7 \), introducing a relationship between alphabetic order and number order) and 15 (from \( 8 + g \), and \( g \) is the 7th letter of the alphabet!). This example again illustrates to what extent pupils’ attack on a “new” problem is influenced by their attempts to relate it to previously learned ideas, or, put differently, to what extent previously learned ideas actually guide or direct their response to a “new” problem.

3.2 Ordering decimals

The following question is from a recent std 5 mathematics competition:

Which number is the largest?

(A) 0,62  (B) 0,234  (C) 0,4 (D) 0,31  (E) 0,532

Response:  0,62 (38%)  0,532 (29%)  0,4 (25%)

Would we expect this result, and can we explain the errors and the sources of the errors?

Studies in Israel, the United States and France (Resnick et al, 1989; Nesher, 1987) have obtained similar results, and have shown that these errors are not slips, but that they are systematic errors based on children’s pre-existing valid knowledge. What pre-existing knowledge is at stake and how does it interfere with pupils’ attempts to order decimals?

First, pupils choosing 0,532 are using their valid knowledge of ordering whole numbers, e.g. “0,532 is bigger than 0,62 because 532 is bigger than 62, so the longest number is the biggest.”

Second, pupils choosing 0,4 are using their valid knowledge of ordering common fractions, e.g. “0,4 is bigger than 0,62 because tenths are bigger than hundredths, so the shortest number is the biggest.”

A child newly learning about decimals must build a schema of decimal numbers and relate it to previously acquired whole numbers, common fractions and measurement. This prior knowledge can support (there are many common features), but can also interfere (there are crucial differences) with the child’s construction of a correct and adequate schema for decimals. From results such as those illustrated above, it is only too clear that this prior knowledge is interfering with many children’s decimal concepts. This can be attributed to an overgeneralization of the properties of whole numbers and common fractions to decimals, but mainly it means that pupils have not accommodated their previous schemas to include decimals; children view the new system of decimals as “identical” to the previous systems, ignoring the differences between the systems, i.e. they distort the concept of decimals so that it can conveniently be assimilated into existing schemas. Their decimal schemas are therefore inadequate and defective.

I argue, however, that although children may have a defective decimal schema, they manage to cope quite well in the school situation. It is exactly this success that works against any attempt at accommodation, because pupils realise full well that they can handle decimals with their previous conceptions, so why make the effort to change? No accommodation is necessary. I am, in effect, saying that our limited mathematical expectations of pupils are partly to blame for the omnipresence of these misconceptions. First, our teaching does not emphasise conceptual understanding of decimals, but rather emphasises algorithmic expertise. We are satisfied if pupils can add, subtract, multiply and divide decimals. The point is that these procedures are mainly taught by rules that reduce the operations to operations on whole numbers (e.g. to multiply two decimals, ignore the comma and multiply the
whole numbers ...). No conceptual understanding of decimals are required for such rules; no wonder pupils think they can manage with their whole number ideas. Second, pupils having the two misconceptions mentioned, will have a 100% success rate when initially comparing decimals of equal length, thus reinforcing their misconception, and neither they nor their teacher will realise that they are obtaining correct results with a defective strategy. Furthermore, if exercises are not intentionally designed to diagnose and discriminate such misconceptions (e.g. 0.4 vs. 0.32 will not discriminate the fraction rule and 0.4 vs. 0.62 will not discriminate the whole number rule), pupils may have a high success rate, believing their mistakes were mere slips. This identifies an important perspective on pupils’ misconceptions, as explicated by Léonard and Sackur-Grisvard (1987):

Erroneous conceptions are so stable because they are not always incorrect. A conception that fails all the time cannot persist. It is because there is a local consistency and a local efficiency in a limited area, that those incorrect conceptions have stability. (p. 44)

For what problems are those conceptions mathematically correct? For what problems are they erroneous? It is only when we know the mathematical limits of the student’s misconception, that we will be able to know when their conceptions will fail, to prevent them, and eventually to teach them to students. (p. 45)

I mention two further interesting snippets from Resnick’s research. Apparently the whole-number misconception declines with age, while the fraction misconception is more persistent and increases with age. Different curriculum sequences produce different misconceptions, as illustrated by the finding that French children by and large avoid the fraction misconception and outperformed children in Israel and the United States; in France decimals are taught before common fractions. This, of course, confirms that children’s misconceptions derive from attempts to integrate new knowledge with already established knowledge.

3.3 Generalising over numbers

It is a well-known fact that pupils who have learned to solve quadratic equations by factoring, e.g.

\[ x^2 - 5x + 6 = 0 \]

\[ \Rightarrow (x - 3)(x - 2) = 0 \]

so, either \( x - 3 = 0 \) or \( x - 2 = 0 \), tend to make the following error:

\[ x^2 - 10x + 21 = 12 \]

\[ \Rightarrow (x - 7)(x - 3) = 12 \]

so, either \( x - 7 = 12 \) or \( x - 3 = 12 \).

This error is very difficult to eradicate – or is, at least, very difficult to eradicate permanently. Even with able pupils, receiving excellent instruction emphasising the special role of zero in the zero product principle, this error will continue to crop up in pupils’ work. Despite careful explanations of why it is an error and despite short-term elimination of the error, it keeps coming back. How do we explain it?

Matz (1980) presents a theory that explains the persistence of this error. There are two levels of procedures guiding cognitive functioning: **surface level procedures**, which are the ordinary rules of arithmetic and algebra, and **deep level procedures**, which create, modify, control and in general guide the surface level procedures. One such deep level guiding principle is the generalisation over numbers, which, in effect says that “the specific numbers don’t matter – you could use other numbers.” This is a very important and in most cases a very necessary observation, which comes naturally to children, e.g. when learning to add, say 52 + 43 by column addition, a child will never master arithmetic if he believed the method works only for 52 + 43. He must believe that the method also works for 34 + 23 and 46 + 21 or any other sum than 52 + 43, also for combinations he has never seen before. Thus, in order to learn arithmetic a pupil must have such a deep level procedure generalising over numbers.

This works very well; as a matter of fact too well: pupils have the natural tendency to overgeneralize over numbers. Because pupils are so accustomed to generalise over numbers, one can predict that errors will be made for any type of problem whose specific numerical values are critical. Overgeneralization of number and number properties may be the single most important underlying cause of pupil’s misconceptions.

This is exactly what happens in the case of the quadratic equation. In \((x - 3)(x - 2) = 0\), the numbers 3 and 2 are not critical to the method, but the 0 is! Pupils should therefore generalise:
\[(x - a)(x - b) = 0\]
\[\Rightarrow x - a = 0 \text{ or } x - b = 0 \quad \text{(1)}\]

Pupils who fail to realise the critical nature of the 0, treat it just as they do the other numbers and overgeneralize:
\[(x - a)(x - b) = c\]
\[\Rightarrow x - a = c \text{ or } x - b = c \quad \text{(2)}\]

Equation (2) would be a correct generalisation of equation (1) if generalising were appropriate in this case. Unfortunately it is not. It is probably the first important rule pupils have met where some specific number should not be generalised.

We all know this. What is interesting, is our awareness of the guiding deep level procedure of generalising over number as the cause of the error; the surface level procedures are operating correctly. This explains why the error is so obstinate and resistant to change, despite our best efforts, and despite pupils’ best intentions: it is not just a matter of learning; it cannot simply be erased from memory, because it is continually being re-created by a sensible deep level guiding principle. What is lacking is a critic – a danger signal that in this particular case the application of the deep level procedure is wrong, which probably only comes with experience of making such mistakes.

This example shows again the sensibility of pupils’ errors and how pupils’ misconceptions are not random, but originate in a consistent conceptual framework based on earlier acquired knowledge.

### 3.4 Generalising over operations

If you teach mathematics in primary school and probably even if you teach in high school, you will recognise the following as a frequent and persistent error (Van Lehn, 1982):

\[
\begin{align*}
263 & \quad 546 \\
-128 & \quad -375 \\
145 & \quad 231
\end{align*}
\]

To remediate the error one may try direct teaching of the correct algorithm and address issues such as place-value, “borrowing” and number combinations. Yet we all know that this error is extremely obstinate and resistant to change – it will recur again and again.

Are our diagnoses correct? If we want to account for pupils’ systematic errors from a constructivist perspective, analysing the procedures and their prerequisites is not sufficient. We must, especially, know how this new knowledge is embedded in a larger meaning system that the child already holds and from which he derives his guiding principles; we must analyse what knowledge in previous learning may be influencing a new idea.

Available research (e.g. Davis, 1984) suggests that one guiding principle is children’s erroneous conception that subtraction is commutative, i.e. the order does not matter, so 6 – 4 and 4 – 6 are the same, or rather they have the same answer.

Why would pupils think that subtraction is commutative? Again, it is an outcome of their experience influenced by correct previous learning. In the system of whole numbers in primary school children work only with 6 – 4; 4 – 6 only arises when we introduce negative numbers in std 6, so the need to discriminate between the two forms never really arises. We also know that children – and humans in general – generally do not discriminate any finer than is needed in a given situation, i.e. discriminations are not made where they are not presently needed (Davis, 1984).

The commutativity of subtraction is further reinforced by word problems containing phrases such as “the difference between Bill’s age and Mary’s age is 2 years”, without specifying who is older, so presumably 6 – 4 and 4 – 6 both produce 2 as result (actually, this is an early conception of absolute value!). Also, have pupils not often heard to always “take the smaller from the larger”, which is exactly what they are doing in the beginning examples? The point is that although children know 6 – 4 and 4 – 6 have different meanings, they may reason that the method to get the answer of 4 – 6 is to calculate 6 – 4, which is the only physical meaning they have available; so 6 – 4 and 4 – 6 are, by definition, equal in value. Even high school teachers will frequently find that pupils write \((30° + 40°) – 180° = 110°\) for an angle in a triangle, but calculate \(70° – 180° = 110°\).

But the main contributory influence for seeing subtraction as commutative is probably that pupils have extensive experience of the commutativity of addition and multiplication when learning their tables, and, in lieu of any contradictory evidence, they have no reason to expect that subtraction will behave otherwise. They are
overgeneralizing over operations. We can predict that the same misconception will show itself in division, and, indeed, it often does (although other misconceptions are induced by the intuitive meanings of the operations—see next paragraph). It is possible that the early introduction of calculators in the primary school may alleviate this particular misconception.

One of the largest and most frequently occurring class of errors in the high school, which Matz (1980) calls linear extrapolation errors, is illustrated by the following examples:

\[ \sqrt{a + b} = \sqrt{a} + \sqrt{b} \]
\[ (a + b)^2 = a^2 + b^2 \]
\[ a(bc) = (ab)(ac) \]
\[ \log(a + b) = \log a + \log b \]
\[ \sin(a + b) = \sin a + \sin b \]

I shall not delve into the problem here, except to say that these errors are probably grounded in an overgeneralization of the “distributive property”, which children encounter often in arithmetic and in introductory algebra, and where it is natural to work with each part independently, e.g.

\[ a(b + c) = ab + bc \]
\[ a(b - c) = ab - ac \]
\[ \frac{b+c}{a} = \frac{b}{a} + \frac{c}{a} \]
\[ (ab)^n = a^n b^n \]
\[ A \cup (B \cap C) = (A \cup B) \cap (A \cup C) \]

Putting it differently, these errors are an overgeneralization of the property \( f(a + b) = f(a) + f(b) \), which applies only when \( f \) is a linear function, to the form \( f(a * b) = f(a) * f(b) \), where \( f \) is any function and \( * \) any operation. This super-formula now acts as another deep level procedure, saying “work the parts separately”, so that the indicated errors are continually being re-created, which explains its obstinate recurrence.

As mentioned before, the error will probably only be resolved if the pupil develops (from experience of errors!) a critic that will recognize the deep level construction as an error. In this regard it is important to be aware of the conditions under which the deep level procedure may be applied.

### 3.5 Meanings

The following two problems differ markedly in difficulty to pupils (Bell et al, 1981; 1984). Why should this be? Can we predict and explain the difficulty?

(A) 1 liter of petrol costs R1.12. How much will it cost to fill a tank holding 3 litres?

(B) 1 liter of petrol costs R1.12. How much will it cost to fill a small tank which holds 0.53 litres?

Would we be surprised at a success rate of 27% for 13-year-olds for (B)? Ah, we would say, it is because pupils find decimals difficult! This explains nothing. What is it about decimals that pupils find difficult? Note that pupils were not required to perform the actual calculation, but only to indicate what operation was needed to solve the problem. So the difficulty does not lie in the calculation, but in the choice of operation. In Bell’s study, 63% of pupils erroneously chose division in (B). Can we explain that?

The mediating or driving misconception causing the error is that “multiplication makes bigger and division makes smaller”. So in (B) pupils reasoned that 0.53 \( \ell \) is less than 1 \( \ell \), so it should cost less than R1.12. So to make it smaller they are driven by their misconception to choose division as operation.

What are the origins, the roots of these misconceptions? Well, as you have come to expect at this stage – in experience of successful previous learning. When working with whole numbers, multiplication always makes bigger (except for 0 and 1, which may be discarded as special cases). So it is, again, a case of an overgeneralization from whole numbers (where it is true) to decimal numbers (and probably fractions and integers, where it is not generally true).

The question is: why do pupils not make the necessary accommodation after working with fractions and decimals for some time; why do they not notice that it is not valid for decimals and fractions? The answer probably lies in our emphasis on the procedures for the operations, where any checking is done, not by estimation, but by
repeating the same process. We therefore never focus our attention on the relative sizes of the factors and product in the multiplication of decimals. The misconception has no apparent detrimental effect on calculation, so we may not even notice it or be overly concerned about it. So the misconception is alive and well and influencing children’s problem solving!

But the roots of the misconception lies deeper. Consider the following two problems used by Fischbein et al (1985):

(A) From 1 quintal of wheat you get 0.75 quintal of flour. How much flour do you get from 15 quintals of wheat?
(B) 1 kg of a detergent is used in making 15 kg of soap. How much soap can be made from 0.75 kg of detergent?

The numbers in both problems are the same, yet (A) yielded 79% success and (B) only 27% (with 45% choosing division). How can that be explained? The difference can be ascribed to pupils’ implicit intuitive meaning of multiplication, namely repeated addition. In the repeated addition model of multiplication, multiplication necessarily makes bigger:

\[ 3 \times 5 = 5 + 5 + 5 \]

and multiplication is not commutative (or rather, the forms have different meanings):

\[ 5 \times 3 = 3 + 3 + 3 + 3 + 3 \]

In a repeated multiplication model 3 \times 0.47 has a meaning, but 0.47 \times 3 cannot be interpreted as repeated addition. Now notice that in (A) the model is 15 \times 0.75, which can be understood as repeated addition and therefore cannot be related to pupils’ intuitive meaning of multiplication. It is clear that pupils’ choice of operation is mediated by their original implicit model of multiplication.

It is again clear that senior pupils have not progressed, have not accommodated their understanding of the meaning of multiplication beyond their first ideas. To be able to cope with area of a rectangle, and problems relating to speed, price, etc. the meaning of multiplication must be extended to include other models of multiplication, e.g. the idea of rate. Do we try to teach it at all?

Here is an example where children’s interpretation of the meaning of symbols lead them astray:

In a certain college there are six times as many students as there are professors. Use S for the number of students and P for the numbers of professors to write an equation for the situation.

In our research (Olivier, 1984), 58% of std 8 pupils erroneously responded with \( P = 6S \), which means they are interpreting S as an abbreviation for “students” and P for “professors”. In essence, they are using letter symbols as labels, or as abbreviations for units as in 6 gram = 6 g, which is probably children’s first encounter with letter symbols. Pupils often carry this prior meaning into algebra, with disastrous results, as the example shows. This misconception of the meaning of letter symbols in algebra is reinforced when we treat \( 2x + 3x \) as mere abstract “letters” and \( a + b \) as apples and bananas. Pupils need to construct meaning for letters as numerical variables in order to cope with algebra.

3.6 Interference

Davis (1984) offers an alternative explanation for the students-professors error. He suggests that pupils may indeed have a numerical-variable schema available. But a schema such as the letter-as-label, which is acquired early and developed well may prove to be extremely persistent, so much that it may sometimes continue to be retrieved inappropriately long after one has become fully cognisant of the conditions under which it is or is not used. Put differently: a new appropriate schema may be available, but the old schema continues to exist. The source of such misconceptions lies in retrieving the wrong schema and not recognising the retrieval error. As for remediating the misconception, Davis advocates making sure that pupils are aware of both schemas and of the likelihood of incorrect choice.

This issue of the retrieval of an inappropriate schema is further illustrated by the following well-known teacher-pupil dialogue:

Teacher: What is four times four?
Pupil: Eight
Teacher: *How much is four plus four?*
Pupil: *Oh! It should be sixteen!*

How is this sequence to be explained? The addition schema is constructed first and is well developed. Thus, when a question is asked about multiplication, which is a more recent (and maybe less secure) piece of learning, the pupil replaces it with a question dealing with earlier (and presumably more securely learned) material. This replacement is also common in other cases, and we notice that it is nearly always the case that the replacement is with an earlier idea:

\[ 4 \times 4 \text{ becomes } 4 + 4 \]
\[ 2^3 \text{ becomes } 2 \times 3 \]
\[ 6 \div \frac{1}{2} \text{ becomes } 6 \times \frac{1}{2} \text{ or } 6 - 2 \]
\[ \frac{3x}{x} \text{ becomes } 2x \]

We notice also that the visual cues for the pairs of questions are very similar in nature. Maybe pupils are not discriminating the visual cues?

However, it is not always the case that previously learned skills interfere with new skills, but often also the other way around, e.g. \( x + x = 2x \) until pupils learn multiplication, then \( x + x \) suddenly becomes \( x^2 \).

Byers and Erlwanger (1985) suggest that this confusion should be sought in memory transformations and subjective organisation during retention. They suggest that many errors are due to attempts by students to simplify mathematical material. The student tries to introduce his own unity, coherence and consistency into material he has learned at different times, and to do so on the basis of hypotheses which appear to him to be both simple and sensible. Because in the event old and new concepts, strategies and algorithms tend to be confused and substituted for each other, the resulting errors are usually ascribed to “interference”.

Jerome Bruner has also noticed this confusion:

> ... when children give wrong numbers it is not so often that they are wrong, as that they are answering a different question. The teacher's job is to find out what question they are in fact answering.

Teachers must help pupils to differentiate between such cases and stress the conditions under which each is applicable.

4. DISCUSSION

(1) The most essential message of this paper is that we should have sympathy – more: empathy, with children for their errors and misconceptions in mathematics. If we understand the general principles of cognitive functioning from a constructivist perspective, we will realise that, for the most part, children do not make mistakes because they are stupid – their mistakes are rational and meaningful efforts to cope with mathematics. These mistakes are derivations from what they have been taught. Of course, these derivations are objectively illogical and wrong, but, psychologically, from the child’s perspective, they make a lot of sense (Ginsburg, 1977).

(2) We would probably all agree that mathematics is a cumulative subject, and that any new learning depends on previous learning. We would also agree that

- correct new learning depends on previous correct learning,

and also that

- incorrect new learning is often the result of previous incorrect learning.

What I have tried to show, is that

- incorrect new learning is mostly the result of previous correct learning.

Every misconception we have discussed had a legitimate origin in previous correct learning – each misconception was correct for some earlier task, as performed in some earlier domain of the curriculum.

(3) The source of misconceptions is mostly an overgeneralization of previous knowledge (that was correct in an earlier domain), to an extended domain (where it is not valid).
A schema acquired early and developed well is highly resistant to change.

Children do not easily accommodate new ideas when necessary, i.e. change their present schemas, but rather assimilate new ideas into existing schemas, which means that the new idea must to a certain extent be distorted to be “like” a previous idea.

Traditionally, the university blames the high school for poor teaching, the high school blames senior primary, who blames junior primary, who blames the home ... In our examples of misconceptions we have seen that pupils’ early learning is correct, but that it is exactly this correct learning that is eventually the source of later misconceptions. Where does the problem really lie? Either earlier learning must be changed so that pupils’ ideas will not later have to be changed (i.e. teach the “correct” notion from the start), or we must later make a special effort to prevent or remediate children’s misconceptions. Neither is easy. For early learning, take “multiplication makes bigger” as an example, which we have shown, originates from the early teaching of multiplication as repeated addition. This leaves us with a fundamental didactical dilemma. If we continue to introduce multiplication via repeated addition, we create a strong, and resistant, but incomplete meaning of multiplication that will come to conflict with later meanings of multiplication. On the other hand, repeated addition is probably the best introductory meaning available for multiplication, so we have little choice but to continue in this way. The notion of decimals before fractions is an interesting possibility. But it is not possible to teach decimals before whole numbers, or algebra before arithmetic! There is little to change, we must accept the possibility that early learning may, through overgeneralization, lead to misconceptions.

Can we prevent or remediate misconceptions later? Yes and no. Yes, later teaching is presently not adequately aware of the major cognitive leaps pupils must make in e.g. the transition from whole numbers to decimals and fractions, and from arithmetic to algebra. Our later teaching emphasises computational and manipulative dexterity at symbolic level, rather than conceptual understanding. This dexterity does not require pupils to accommodate their existing schemas – many pupils’ misconceptions are masked by adequate performance in mathematics. So, if later teaching really addressed the issues, we could prevent or remediate the mentioned misconceptions by helping pupils to integrate the new and the previous knowledge. No, because misconceptions may develop naturally as a product of typical human mental processing. Research shows that the initial intuitive ideas become so deeply rooted in the child’s mind that they continue to exert an unconscious control over mental behaviour even after the child has acquired formal mathematical notions of the idea that are solid and correct (e.g. Fischbein et al, 1985).

From a constructivist perspective the teacher cannot transmit knowledge ready-made and intact to the pupil. Errors and misconceptions are seen as the natural result of children’s efforts to construct their own knowledge, and these misconceptions are intelligent constructions based on correct or incomplete (but not wrong) previous knowledge. Misconceptions, therefore, cannot be avoided. Such errors and misconceptions should not be treated as terrible things to be uprooted, since this may confuse the child and shake his confidence in his previous knowledge. Instead, making errors is best regarded as part of the process of learning. We should create a classroom atmosphere that is tolerant of errors and misconceptions and exploit them as opportunities to enhance learning. In this regard direct teaching (“telling”) of missing concepts will not do. Rather teachers should help pupils to connect new knowledge to previous learning. Swan (1983), Nesher (1987) and Olivier (1988) describe a teaching approach that is designed to expose children’s misconceptions and provide a feedback mechanism that leads to cognitive conflict. Discussion, communication, reflection, and negotiation of meaning are essential features of a successful approach to resolve pupils’ misconceptions.

REFERENCES


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MOMENTS OF CONFLICT AND MOMENTS OF CONVICTION IN GENERALISING
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This paper reports on part of a study of students’ ability to handle algebraic generalisation problems. In this paper we focus and elaborate on moments when students grapple with deciding about the validity of their generalisations. We interviewed ten students near the end of grade 7. During the interviews we tried to create cognitive conflict by challenging the students’ justification for the methods they used and then documented their attempts to resolve such conflicts. We found that most students’ justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation.

INTRODUCTION
Number patterns, the relationship between variables, and generalisation are emphasised as important components of algebra curricula reform in many countries and also in South Africa. Much research has been done on children’s generalisation processes documenting children’s strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g. Garcia-Cruz and Martinon, 1997; Taplin, 1995; Orton and Orton, 1994; MacGregor and Stacey, 1993). Our own ongoing research confirms many of these findings.

However, little research has been done in analysing children’s thinking in the processes of generalising. For example, do students view their efforts at generalising as hypotheses? Do they realise the necessity to validate their methods and answers? How do they become convinced of the validity of generalisations? Garcia-Cruz and Martinon (1997) for example, report that most children they interviewed checked their rules. This was done either by counting or drawing or extending the numerical sequence. It is not clear from their report, however, whether their students spontaneously checked their answers because they felt the need for validation, or how they became convinced of the validity of their strategies and answers.

In this paper we focus and elaborate on such moments where students grapple with deciding about the validity of their generalisations. During interviews with children, we tried in several ways to create cognitive conflict by challenging their justification for the methods they used and then documented how they tried to resolve such conflicts.

RESEARCH CONTEXT
The research reported in this paper is part of an ongoing research project aimed at informing curriculum development. The project enlisted eight schools in the suburbs of Cape Town as project schools. Seven of the eight schools are in traditional black townships. All the children interviewed in this research came from one of these seven schools. As a baseline study for the project’s diagnostic purposes, we have been
collecting data on children's performance in mathematics using various tools. One of these tools is a written baseline test.

RESEARCH METHODOLOGY
As a first stage we wanted to gather data on the most mathematically competent students. The students were chosen on the basis of performance in the baseline test and the teacher's evaluation. We interviewed ten students near the end of grade 7. Each student was interviewed three times in 45-minute sessions by two of the researchers, twice individually and once in pairs. A fourth session took place in which the students were given two generalisation problems to do individually. All interviews were videotaped. In addition to the video protocols, written transcripts of the subjects’ verbal responses as well as their paper-and-pencil activities were used in the analysis.

THE PROBLEMS
We presented the students with a series of eight generalisation problems in which we varied the representation of the problems. Some problems were formulated in terms of numbers only (in the form of a table of values), some were formulated in terms of pictures only (in the form of a drawing of the situation) and some problems were formulated in terms of both pictures and numbers.

The questions were in each case basically the same, namely given the values of \(f(1), f(2), f(3),\) and \(f(4)\), we asked students to find the values of \(f(5), f(20)\) and \(f(100)\) and to explain and justify their answers and strategies. Six of the functions were linear functions of the form \(f(n) = an + b\), and two functions were simple quadratic functions of the form \(f(n) = n^2\). Here are two examples, “cans” and “matches”:

\((B1)\): Cans are packed to form pyramids.

The table shows how many cans are needed for different pyramids.

Complete the table.

<table>
<thead>
<tr>
<th>Pyramid number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of cans</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>16</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\((C3)\): Matches are used to build pictures like this:

![Pictures](image1)

The table shows how many matches are used for the different pictures.

Complete the table.

<table>
<thead>
<tr>
<th>Picture number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of matches</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\footnote{Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for reporting on the students.}
Whatever responses the children gave, we asked them to explain their answers by posing questions like: “Can you explain how you got this answer?” or “Convince me that your answer is correct”, or “Show me how you got this answer”. If the students’ explanations were based on the information given in the problem (the database), we accepted it as a satisfactory answer.

SOME RESPONSES

Some general observations
Concerning children’s use of different representations for the problems, it is interesting that all but one of the children worked exclusively in the number context and did not use the structure of the pictures at all. In the problems that were formulated in terms of pictures, children immediately constructed a “table” of values and then used only the table of values in their solutions and explanations.

Concerning children’s strategies, it is interesting that for the simple quadratic problems, nearly all the children recognised the functional rule \( f(n) = n^2 \) from the database and used it to find the values of \( f(5), f(20) \) and \( f(100) \). For example, Thandi explains: “I say 2 times 2 is 4, 3 times 3 is 9, 4 times 4 is 16, 5 times 5 is 25”. However, in all the linear problems all students correctly used recursion to find \( f(5) \) as \( f(4) + d \) where \( d \) is the common difference between successive terms. For example, Thandi explains how she finds \( f(5) = 36 \) in the table for the function \( f(n) = 8n + 4 \): “I say 4 plus 8 is 12, 12 plus 8 is 20, 20 plus 8 is 28, 28 plus 8 is 36”.

It is also interesting that when they had to find \( f(20) \) and \( f(100) \), most children abandoned their successful recursive strategy because they were trying to find a “shortcut” to calculate \( f(20) \) and \( f(100) \). These short methods were mostly not based on the database and were seriously prone to error. None of our students felt the need for any kind of validation. Although they offered some kind of explanation for the method they used in the extended domain, they were not aware of the role of the database in the process of validation.

Some alternative interpretations
In several cases children's answers were far from what we would expect, yet still based on the database. For example, in the cans-problem Roy wrote that \( f(5) = 23, f(20) = 28 \) and \( f(100) = 31 \), because he was using a symmetry structure \((3; 5; 7; 7; 5; 3)\):

\[
\begin{align*}
  f(2) & = f(1) + 3; \quad f(3) = f(2) + 5; \quad f(4) = f(3) + 7 \\
  \therefore \quad f(5) & = f(4) + 7; \quad f(20) = f(5) + 5; \quad f(100) = f(20) + 3.
\end{align*}
\]

He ignored the shaded columns that we intended as representing several “missing” columns in the table.

Also in the cans-problem Sipho wrote: \( f(5) = 20, f(20) = 100 \) and \( f(100) = 600 \). This seemed to us rather arbitrary, but he was in fact using the rule that \( f(n) \) is a multiple of \( n \), without specifying which multiple:

Interviewer: Can you explain how you got 20? [for \( f(5) \)]
Sipho: I took pyramid .... [pause] .... I saw that each doesn’t have a remainder.
In the same problem Vusi wrote \( f(5) = 25 \). We were, of course, sure that he was using the functional rule \( f(n) = n^2 \). However, then he wrote \( f(20) = 120 \) and \( f(100) = 800 \), explaining that “you multiply each number in the upper row by the number of the column”. Closer questioning revealed that he misinterpreted the shaded columns. For Vusi \( n = 20 \) was in the 6th column, so \( f(20) = 20 \times 6 \), \( n = 100 \) was in the 8th column and his rule therefore produced \( f(100) = 100 \times 8 = 800 \). He ignored the first shaded column and then counted the next 3 columns as the 6th-, the 7th- and the 8th column.

Some mistakes
Students made various mistakes, for example, to concentrate only on the relationship between a single pair \((n; f(n))\) and then to use it as a general rule. For example, in the cans-problem Roy saw that \( f(3) = 3 \times 3 = 9 \) and then used the rule \( f(n) = 3n \) to find \( f(5) = 3 \times 5 = 15 \).

However, the most common, nearly universal mistake children made in their efforts to find a manageable method to calculate larger values, was to use the proportionality property that if \( x_2 = k \times x_1 \), then \( f(x_2) = k \times f(x_1) \). For example, in the matches-problem, Mathole, after finding \( f(5) = 11 \), calculates \( f(20) \) as \( 4 \times 11 = 44 \). This mistake was also found by Taplin (1995) and Garcia-Cruz and Martinon (1997).

CREATING CONFLICT
When we were not convinced that the students’ responses reflected awareness of the role of the database in the justification process, we tried to create a cognitive conflict, using three different strategies as described below. (Because children were not using the pictures, we did not use a strategy of drawing pictures to check their answers.)

Strategy 1: The first strategy we used was to confront the answer driven from the recursive approach with the one obtained by the mistaken approach. This strategy was used when the child had in front of him/her a table he/she had formed in order to find some \( f(n) \) through recursion.

For example, Vusi and Thandi, working as a pair, used the recursive method to correctly determine \( f(20) \). In order to determine \( f(100) \), they systematically continued using the recursive method. However, when they reached \( f(50) \) they changed to the multiplication method, claiming that \( f(100) = 2 \times f(50) \). We wanted them to reflect on the incorrect multiplication method. For this purpose we challenged them to apply their multiplication method on the domain between 1 and 50 since they had already obtained these values by the recursive method. Vusi was asked to find \( f(20) \) using \( f(5) \) and the multiplication method.

Vusi: Its 72 [multiplying 18 by 4]. I got 63 [the result he obtained by the recursive method].

Vusi is puzzled but still unconvinced that his method is wrong. He decides to recheck his multiplication method on the database:

Vusi: Lets try this one [looking at \( f(2) \) and \( f(4) \) in the database]. If 2 goes 2 in 4, so I must multiply 9 [the value for \( f(2) \) in the given database] by 2 is 18, but its 15 [the value for \( f(4) \) in the given database].

Interviewer: So what do you say when I ask you about 100?
Vusi: I said 20 times 5 so its 100. So 63 [the value he obtained for \( f(5) \)] times 5.
Vusi is sure that his answer for $f(20)$, 63, he obtained by the recursive method is correct and the other answer for $f(20)$, 72, obtained by $4 \times f(5)$ is wrong. He is sure the method to get $f(4)$ by $2 \times f(2)$ is incorrect but at the same time he is not willing to give up his multiplication method when it comes to $f(100)$.

*Strategy 2:* The second strategy was to create a conflict by choosing a take-off point different from the one the child had used when applying the multiplication method. Choosing different take-off points led to different answers for $f(n)$. For example, Vusi spontaneously evaluates $f(100)$ as $2 \times f(50) = 2 \times 147 = 294$. The interviewer prompts him to use different take-off points. He takes $f(10)$ and $f(20)$ and obtains $f(100) = 10 \times f(10) = 330$ and $f(100) = 5 \times f(20) = 315$.

Interviewer: Oh, so who is right?
Vusi: Now we have three plans.
Interviewer: Ok, I understand three plans, but I also have three answers, 330, 315 and 294.
Are they all right?
Vusi: Yes, they are all right.

Thandi and Vusi are sure about the values they obtained for $f(10)$, $f(20)$ and $f(50)$ since these values were obtained by the recursive method. $f(100)$, however, is an abstract entity for them. The fact that the three different take-off points led to three different answers for $f(100)$ did not lead them to question the *method* they used.

Mathole, when confronted with different answers for different take-off points is also not prepared to abandon the multiplication method, but attempts to give a justification for the different answers:

Interviewer: And now you said that in shape number 20 we have 144 okay? Because you took this 5 ... you divided 20 by 5 and timesed 36 [the value for $f(5)$] by 4. We are sure about it. Okay, let’s say that your friend goes to shape number 4 and he now divides 100 by 4. To divide 100 by 4 gives 5, so he goes and times 28 [the value for $f(4)$ in the given database] by 5, do you follow me?
Mathole: Yes
Interviewer: So he multiplies 28 by 5, how much is it? [works on calculator] 140. So what is the correct one, 144 or 140?
Mathole: 144
Interviewer: Why?
Mathole: Because ... here by the fourth shape you got 28 matches and fifth shape is 36 matches, so if he goes back to the ... to the ... 28 he’ll have to add 4 and if he goes back to the third shape he’ll have to add 8, it’s like you tax a person for going back, you let him pay for going back, so he’ll have to pay 4 for going back, then you’ll have to add a 4 there, then you’ll get the 144.

*Strategy 3:* The third strategy was to implement the method the child used in the extended domain on the domain given in the original table.
For example, Thandi obtained an answer for $f(5)$ by correctly using the recursive method $f(5) = f(4) + 8 = 36$. For $f(20)$, however, she wrote 28, explaining:

Thandi: I count to shape 5, and I count to 20 and then I add this top numbers [refers to the shape number in the table] by 8.
While \( f(5) \) was obtained correctly using the recursive rule \( f(n) = f(n - 1) + 8 \), she now changes her rule to find \( f(20) \) by using the function rule, \( f(n) = n + 8 \). She is then taken back to \( f(5) \) and asked how she obtained 36. She adds 8 to 5 (the shape number) and gets 13, not 36. She now realises that there is a contradiction. Thandi now no longer accepts her answer for \( f(20) \).

**STRUGGLING FOR CONVICTION**

It was clear that conviction about the role of the database in the process of validation develops slowly. Despite our efforts to create conflicts in order for them to reflect on the proportionality multiplication error and on the process of validation in early interviews, the same children repeatedly made the same mistake in later interviews. We follow below Sipho's struggle to come to terms with the proportionality multiplication error.

In the first problem given, Sipho obtained 36 for \( f(5) \) by using recursion correctly. However, for \( f(20) \) he abandoned recursion and used the multiplication method explaining: “5 goes four times in 20 so I multiply 36 [the value he obtained for \( f(5) \)] by 4 to get the number of matches in shape 20.” The interviewer challenged him to apply his multiplication method on the domain 1 to 5, to obtain \( f(4) = 2 \times f(2) \) and \( f(5) = 5 \times f(1) \). Sipho was sure that the answer for \( f(5) \) he obtained by recursion was the correct one and not the answer obtained by the multiplication method. However, when asked again about \( f(20) \) and later on \( f(100) \) he consistently used the multiplication method. This happened again in the next problem.

In the second interview Sipho was working with David. Both of them used recursion to obtain \( f(5) \). However, for \( f(20) \), David continued systematically with recursion, finding \( f(20) = 63 \), while Sipho used the multiplication method, finding \( f(20) = 4 \times f(5) = 4 \times 18 = 72 \).

Interviewer: I do not follow, shape number 20 is 63 or 72? [strategy 1 as above]
David: I go my way, adding 3 and 3 and 3
Sipho: [to David] I see the method is right, but can you tell me what I have done wrong to get the wrong answer?

At this point Sipho confronts the two methods which is significant since he realises that there is a conflict. He is sure the recursion method gives the correct answer and realises that his multiplication answer gives an incorrect answer. He is interested in why the multiplication method is wrong. However, in the very next moment Sipho again succumbs to the multiplication error:

Interviewer: What about shape 100?
Sipho: I times because you know that I get 5 20’s .... I think I’ll times 63 by 5 to get it.
David: That’s the wrong way.
Sipho: If I didn’t times, I added 3,3,3, .... I would get the same answer.

Interviewer: Where do you see multiplication? [Strategy 3] I can understand where the 3 came from. I saw that it’s given here [points at the table and the differences between the number of matches]. Where did you get the multiplication? Can we check?
Sipho: Shape 3. I just go to shape 4.
Interviewer: If you want to get to shape 4 with your method what would you times?
Sipho: I would times the number of matches here [points at shape 2] by 2.
Interviewer: And what number of matches will you get?
Sipho: 18
Interviewer: And what is written here? [points at f(4) in the given data base]
Sipho: 15
Interviewer: So?
Sipho: Yes, eh

At the end of the interview we are left with the impression that Sipho is convinced that his multiplication method is incorrect, because the given database does not reflect the multiplication method.

In the third interview it appears as if the previous discussions with Sipho had not taken place. He still uses the incorrect multiplication method to obtain f(20). He is taken back to the given database to reflect on how he obtained f(5):

Sipho: Because shape 1 is 3 and shape 2 is 5 and the difference is 2.[referring to C3]

Sipho is now pushed to reflect on the given database and his method for obtaining f(20). He realises that if he uses the multiplication method on f(1) to obtain f(4), it will not be the same as the value for f(4) in the given database. This conflict leads him to use recursion to find f(20) = 41. Yet he reverts to the multiplication method to obtain the value for f(100). He is challenged by the interviewer:

Interviewer: It does not work for f(20) but you think it might work if you go from 20 to 100?
Sipho: Yes, because I think the number of matches in shape 20 is now right.

This remark sheds some light on Sipho’s line of thought. He thinks since he now has the correct value of f(20) he can use it for f(100). For him, the problem was not the method but the wrong value of the take off point. It seems that Sipho is sure about the value of f(20) which was obtained by recursion. He is convinced that the multiplication method does not work for f(20), but nevertheless, from his perspective, it still works for f(100), provided that the value of f(20) is correct. He is now challenged to use the multiplication method on f(5) to obtain f(20). This yields a value of 44, which he knows is wrong because he obtained f(20) = 41 by recursion. He is puzzled:

Interviewer: Now, you think 5 times 41, 205, you say it’s right for f(100).
Sipho: I think it’s wrong.

Interviewer: Why
Sipho: Because I did the same thing when I multiplied. I tried to multiply the number of matches by 5 … I saw that I was wrong.

Interviewer: So how will you then do 100 [shape number 100]?
Sipho: I think I have to do it like this [points at the list for f(20)] but it will take a long time.
Sipho: [Long pause] … I’m trying to think if I can do another method to get the answer of eleven [the answer for f(5) - he is trying to look for a functional rule]. I’m trying to multiply the number of … number of matches in shape 5.

Yet, albeit a slow development, there were successes: in the final written test six out of the ten students avoided making the multiplication error.
DISCUSSION

Our study shows that our interviewees did not view their answers as hypotheses that should be validated. They were not aware of the role of the database in the process of generalisation and of validation.

Although we were aware that students frequently succumb to the proportional multiplication error, its persistence and obstinacy to change surprised us. On the one hand students easily convinced themselves that when a value for $f(n)$ they obtained using recursion differed from the value they obtained using their multiplication method (our validation strategy 1 described above), the result obtained by multiplication was incorrect. On the other hand, the knowledge that their multiplication method produced incorrect answers did not prevent them from making the mistake again (and again). Indeed, when they were asked for the value of $f(m)$ for $m > n$ in the same problem, all the students again resorted to the multiplication method and were sure that their answers are correct. Our efforts to create cognitive conflict by leading students to apply the multiplication method to different take-off points and getting different values for $f(n)$ (our strategy 2 above), or drawing their attention to the fact that the multiplication method is not applicable in the domain given in the table (our strategy 3), did not easily eliminate the error. Most students continued using the multiplication method throughout the interviews. Six out of the ten students eventually avoided the error in the fourth session.

One could argue that our choice of numbers triggered the proportional multiplication error, i.e. that our use of “seductive numbers” like $n = 5, 20$ and $100$ stimulated the error. One could also argue that if we used non-seductive numbers like $n = 17, 27$ and 83 children would not use the erroneous multiplication method. However, we believe that our evidence shows that children, in their quest for a manageable short method, create “seductive numbers” themselves. For example, Thandi was busy using a laborious recursive strategy on her way to calculate $f(100)$ – she continued to $f(50)$ and then suddenly stopped and calculated $f(100)$ as $2 \times f(50)$, probably because she immediately recognised the multiplicative relationship between 50 and 100. Nevertheless, it remains a question for further research to establish whether an approach with non-seductive numbers will prevent children from making the multiplication error, also when they encounter seductive numbers in other problems.

REFERENCES


THE INFLUENCE OF DIFFERENT REPRESENTATIONS ON CHILDREN'S
GENERALISATION THINKING PROCESSES

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Mathematics Learning and Teaching Initiative, South Africa

This is a second report on our ongoing research into students' thinking processes in generalisation situations. In this study we varied the representation of the activities we presented to children along several dimensions, namely in the type of function, the nature of the numbers, the format of the tables, and the structure of the pictures. Our results show that varying these dimensions has little effect on children's thinking – as before, few children tried to find a functional relationship between the variables, except in two simple cases, but persisted with using the recursive relationship between function values. While using recursion was successful in extending number patterns to nearby values, students find it tedious for finding larger function values. They then mostly attempted to adapt their recursion strategy in some way, but made many logical errors in the process. Our biggest concern is not so much the fact that students make many errors, but that they do not feel the need, or do not have the know-how, to verify their methods or answers against the given data.

INTRODUCTION

Number patterns, the relationship between variables and generalisation are considered important components of algebra curricula reform in many countries. It is evident that South Africa shares this view as can be seen in the curricular changes to the Junior Secondary syllabus (Syllabus of Western Cape Education Department, 1996). South Africa’s new curriculum plans (Curriculum 2005) also emphasises the importance of generalisation as is evident from the following specific outcomes for Mathematical Literacy, Mathematics and Mathematical Sciences:

“Use mathematical language to communicate mathematical ideas, concepts, generalisations and thought processes.
Use various logical processes to formulate, test and justify conjectures”.

There have also been suggestions to use generalised number patterns as an introduction to algebra. However, there is insufficient research that deals with the cognitive difficulties children encounter and the feasibility of such an approach. Much of the available research on children’s thinking processes in generalisation reports on children’s strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g. Garcia-Cruz and Martinon, 1997; Orton and Orton, 1994; Taplin, 1995).

In a previous study (Linchevski, Olivier, Sasman & Liebenberg, 1998) we presented grade 7 students with problems like the following:

(C3): Matches are used to build pictures like this:

![Picture 1](Picture 1) ![Picture 2](Picture 2) ![Picture 3](Picture 3) ![Picture 4](Picture 4)

The table shows how many matches are used for the different pictures. Complete the table.

<table>
<thead>
<tr>
<th>Picture number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>20</th>
<th>100</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of matches</td>
<td>3</td>
<td>5</td>
<td>7</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We found that most students’ generalisations and justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation. We also found that children worked nearly exclusively in the number context and did not use the structure of the pictures at all. Also our interviewees did not view their answers as hypotheses that should be validated. For example, they did not, and seemed unable, to verify their justification against the given data pairs \((1; 3), (2; 5), (3; 7), (4; 9)\).

Few children managed to construct a function rule to find function values. Rather, they focussed on recursion (e.g. \(f(n + 1) = f(n) + 2\) in problem \(C_3\) above), which led to many mistakes as they tried to find a manageable method to calculate larger function values. The most common, nearly universal mistake was to use the proportionality property that if \(n_2 = k \times n_1\), then \(f(n_2) = k \times f(n_1)\). For example, in problem \(C_3\) above, from \(f(5) = 11\) they deduced that \(f(20) = 4 \times 11 = 44\). Although this property applies only to functions of the type \(f(n) = an\), children erroneously applied it to any function. It is possible that our choice of numbers might have triggered the proportional multiplication error, i.e. that our use of “seductive numbers” in a sequence like \(n = 5, 20\) and 100 stimulated the error (we regarded these numbers as seductive from a multiplicative point of view).

Based on the above we viewed the following as questions for further research:

- whether the use of non-seductive numbers will prevent children from making the multiplication error, also when they encounter seductive numbers in other problems
- whether the visual impact of the table, as for example shown in problem \(C_3\) above, also contributed to the persistence of the proportional multiplication error
- whether pictorial representations in which the function rule is “transparent” will encourage children to use the structure of the pictures to more easily find function rules.

In this paper we report on some first findings on these three questions.

**RESEARCH SETTING**

**The activities**

We designed a series of eight generalisation activities in which we varied the representation of the activities. Four activities were formulated in terms of numbers only (in the form of a table of values), and four were formulated in terms of pictures only (in the form of a drawing of the situation). Each pictorial representation had a corresponding numerical representation.

The numerical tables of values were presented in different formats: “continuous” (e.g. \(I_T\) below, in which input values for which the corresponding function values had to be calculated were included) and “non-continuous” (e.g. \(II_T\), where the input values were not given, but were presented verbally by the interviewer). The tables were presented in both vertical and horizontal format.

The pictorial representations of the activities were chosen to be either “transparent”, i.e. the function rule is embodied in the structure of the pictures (e.g. in \(I_P\) below), or “non-transparent”, i.e. the function rule is not easily seen in the structure of the pictures (e.g. in \(III_P\)). As with tables, pictures were presented in both “continuous” and “non-continuous” format.

---

\(^1\) Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for reporting on the students.
The questions in each activity were basically the same, namely given the values of \( f(1), f(2), f(3), f(4), f(5) \) and \( f(6) \), we asked students to first find \( f(7) \) and \( f(8) \), and then the function values of certain further input values and to explain and justify their answers and strategies. These input values were both “seductive” (e.g. 20, 60) and “non-seductive” (e.g. 19, 59). Two of the functions were linear functions of the form \( f(n) = an + b \), and two functions were simple quadratic functions. We supply below a selection of the activities.

**I\textsubscript{P}**

Blocks are packed to form pictures that form a pattern as shown below:

![Picture 1](image1)

**II\textsubscript{T}**

Tiles are used to build pictures to form a pattern. The table below shows the number of tiles in a particular picture.

<table>
<thead>
<tr>
<th>Picture number</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>...</th>
<th>20</th>
<th>...</th>
<th>60</th>
<th>...</th>
<th>( n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of tiles</td>
<td>2</td>
<td>5</td>
<td>10</td>
<td>17</td>
<td>26</td>
<td>37</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**II\textsubscript{F}**

Matches are used to build shapes to form a pattern. The table shows the number of matches used to build a particular shape:

<table>
<thead>
<tr>
<th>Shape number</th>
<th>Number of matches</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>28</td>
</tr>
<tr>
<td>5</td>
<td>36</td>
</tr>
<tr>
<td>6</td>
<td>44</td>
</tr>
</tbody>
</table>

**II\textsubscript{P}**

Matches are used to build shapes. A different number of matches is used to build each shape.

![Shape 1](image2)

![Shape 2](image3)

![Shape 3](image4)

![Shape 4](image5)

![Shape 5](image6)

![Shape 6](image7)

**III\textsubscript{P}**

![Pyramid 1](image8)

![Pyramid 2](image9)

![Pyramid 3](image10)

![Pyramid 4](image11)

![Pyramid 5](image12)

![Pyramid 6](image13)

\(^2\) The subscript \( \text{P} \) indicates that the problem was presented in a spatial context in the form of a pictorial representation of the situation and the subscript \( \text{T} \) indicates the problem was presented in a numerical context in the form of a table of values.

\(^3\) All the drawings were presented to students in vertical format, but is here given horizontally due to space considerations.
Methodology
We interviewed ten grade 8 students at one of our project schools in a historically disadvantaged area of Cape Town before they had received any instruction on patterns, sequences or algebra. The students were selected by the teacher so that they were representative of the grade 8 class. Each student was interviewed three times in 45-minute sessions by either two or one of the researchers. The interviews of each student took place 5 to 7 days apart. In the first two interviews each student was presented with three activities and they were asked to explain or clarify their answers or strategies, but were not challenged in any way. We wanted to ascertain what they did spontaneously. The pictorial and numerical activities were presented to the children on different days. In the third interview they were given two further activities and then asked to reflect on some of their previous solutions and to justify their answers. Based on their responses the researcher asked questions to create cognitive conflict. All interviews were videotaped. In addition to the video protocols, written transcripts of the subjects’ verbal responses as well as their paper-and-pencil activities will be used in the analysis. The analysis will be used to design a teaching intervention aimed at addressing the cognitive difficulties children have in the processes of generalisation.

RESULTS AND ANALYSIS
Most students had no difficulty finding f(7) and f(8) in any of the activities – they either found and used the function rule correctly, or used recursion correctly for these nearby values. However, in trying to find a manageable strategy for finding further-lying function values, children invented a variety of different strategies, both correct and incorrect. These strategies and their frequency are summarised in Table 1. We will refer back to the table in our analysis.

The nature of the function
Finding function rules
It is interesting to note from the data in Table 1 that more than half of the students found and used the function rules in activities IP and IT. These both represent simple quadratic functions. One could be tempted to conclude that students easily recognise such simple quadratic function rules.

### III_T

Peter uses blocks to build pictures that form a pattern. The table shows the number of blocks he needs to build a particular picture.

<table>
<thead>
<tr>
<th>Picture number</th>
<th>Number of blocks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
<tr>
<td>6</td>
<td>36</td>
</tr>
</tbody>
</table>

### IV_P
Tiles are arranged to form pictures like this:

![Picture of tiles arranged to form pictures](image_url)

---

III_T

IV_P

Methodology
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<table>
<thead>
<tr>
<th>Activity number and representation format</th>
<th>Recursion, counting-on</th>
<th>Proportional multiplication error</th>
<th>Decomposition of input value, e.g. ( f(n) = f(a) + f(b) + f(c) ) where ( a + b + c = n )</th>
<th>Difference method ( f(n) = n \times d )</th>
<th>Extended recursion: ( f(n) = (n - k)d + f(k) ) (( d ) is common difference)</th>
<th>Function rule</th>
<th>Other</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>I_P</strong> Transparent picture, continuous</td>
<td>2</td>
<td>1</td>
<td>2 wrong variations</td>
<td>5</td>
<td>Non-seductive input values (20, 60, ( n )) Quadratic function ( f(n) = n^2 + 1 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Horizontal continuous table</td>
<td>4</td>
<td>1</td>
<td>1 wrong variation</td>
<td>3</td>
<td>Non-seductive input values (20, 60, ( n )) Quadratic function ( f(n) = n^2 + 1 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td><strong>II_P</strong> Transparent picture, continuous</td>
<td>1</td>
<td>1</td>
<td>2 wrong variations</td>
<td>2</td>
<td>Non-seductive input values (19, 59, ( n )) Linear function ( f(n) = 8n - 4 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Vertical non-continuous table</td>
<td>3</td>
<td>1</td>
<td>1 wrong variation</td>
<td>3</td>
<td>Non-seductive input values (20, 60, ( n )) Linear function ( f(n) = 8n - 4 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td><strong>III_P</strong> Non-transparent picture, non-continuous</td>
<td>3</td>
<td>1</td>
<td>2 wrong variations</td>
<td>2</td>
<td>Non-seductive input values (23, 79, ( n )) Quadratic function ( f(n) = n^2 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Vertical non-continuous table</td>
<td>1</td>
<td>1</td>
<td>1 wrong variation</td>
<td>6</td>
<td>Non-seductive input values (29, 87, ( n )) Quadratic function ( f(n) = n^2 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td><strong>IV_P</strong> Transparent picture, non-continuous</td>
<td>1</td>
<td>2</td>
<td>2 wrong variations</td>
<td>1</td>
<td>Non-seductive input values (20, 60, ( n )) Linear function ( f(n) = 4n + 1 )</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>Horizontal continuous table</td>
<td>1</td>
<td>3</td>
<td>2 wrong variations</td>
<td>2</td>
<td>Non-seductive input values (23, 117, ( n )) Linear function ( f(n) = 4n + 1 )</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>
One immediately, however, also notices the marked differences in students’ responses for the same functions in the picture and the table contexts. In activity I the picture is transparent, but students find it much more difficult to recognise the same function rule from the equivalent table in activity I. In activity III on the other hand, children easily find the rule in the table, but not in the non-transparent picture.

It is clear that students found it much more difficult to formulate function rules for linear functions. From our interviews it seems that children try to construct simple multiplication (proportional) structures, but when it does not fit the database, they quickly give up and then invent all kinds of error-prone recursion strategies.

Recursion

When students focus on recursion patterns, however, they find the constant difference between consecutive terms in linear functions much easier to handle than the changing (increasing) difference in quadratic functions, leading to many errors. We describe these strategies and errors in the following sections.

Seductive vs. non-seductive numbers

The proportional multiplication error

In our earlier work with grade 7 students we found a persistence with the erroneous proportional multiplication error. In this study six of the ten students interviewed used it at least once in the series of activities. For example:

Interviewer: How many tiles in Picture 20? (in IV_p)
Peter: OK, I am using 5 (meaning n = 5; f(5) = 21 in the picture) to get to 20. So 21 times 4 is 84, because 5 times 4 is 20.

Interviewer: How many tiles do you think we’ll use for Picture 60?
Peter: (Pause ... looking at the numerical “table” he had prepared from the given database) ... Picture 10 is 41 tiles, so 41 times 6 ... the answer is 246.

This erroneous strategy was used only with what we call “seductive numbers”. Vergnaud (1983) argues that this is an over-generalisation of the many direct proportional relationships that students are intuitively aware of from an early age. Fischbein et al (1985) posit that children generalise the way they were initially taught in school before they develop a critical attitude and that some mental behaviours tend to act beyond any formal control because these behaviours shape the ideas and the facts at hand in a meaningful way.

When students could easily find the function rule the nature of the input values was immaterial, i.e. they did not make the multiplication error, even for seductive numbers.

Extending recursion

A few students managed to adapt their focus on recursion to a manageable strategy for finding further-lying function values. This extended recursion method is symbolised by f(n) = (n – k)d + f(k), where (d is the common difference between consecutive terms. Here is an example:

Interviewer: OK, Shape 59? (How many matches in Shape 59 in II_p?)
Hamid: So first I subtract 19 (he had previously calculated f(19) = 148) by 59 and then you get your answer of 40 and then I times it by 8 (the common difference between terms) and then I get my answer and then I add it by 148, that is Shape 19’s answer.

Some children used this method also in the case of seductive numbers.
Some children also worked with this method, but they often seemed to lose track of all the details. This was mostly because they worked verbally, and did not write down information or their strategy. In this example Voda correctly calculates \((n - k) \times d\), but then does not add \(f(k)\):

**Interviewer:** Shape 60? *(how many matches in Shape 60 in \(I_T\))*

**Voda:** *(works on calculator)* 320

**Interviewer:** Please explain to us

**Voda:** I subtracted 20 by 60 *(he means 60 − 20)* and then I times 40 by 8.

While the extended recursion method is correct for linear functions, many students also erroneously applied it to or adapted it for the quadratic functions. For example:

**Interviewer:** Ok, and then Picture 20? *(How many matches in Picture 20 in \(I_T\))*

**Harold:** I subtract 20 by 8 *(he had previously calculated \(f(8) = 65\)) \ldots I subtract 8 by 20, then I get 12 \ldots with that 12 I times by 2 is equal to 24 \ldots then I add 24 by 15, is equal to 39 then I add 39 to 65, is equal to 104.

**Interviewer:** Just explain the 15 please

**Harold:** That's the 15 I added by 50 *(\(f(7)\))* to get 65 *(\(f(8)\))*.

**Decomposition of input value**

The introduction of “non-seductive numbers” gave rise to other inappropriate strategies when students could not find a multiplicative relationship between the non-seductive numbers. For example:

**Interviewer:** How many cans do we need to build Pyramid 23? *(activity \(III_P\))*

**Errol:** *[Long pause ... staring at the database. In the previous problem (activity \(I_P\)) he prepared a “table” of values using recursion up to \(f(20)\) and then used the proportional multiplication method to predict \(f(60)\). He presses on the calculator \(64 + 64 + 49 + 30\)…ya .. it’s ..207.]*

**Interviewer:** Can you explain how you got your answer please?

**Errol:** Ya, \ldots 8 *(referring to Pyramid 8) is 64 and 7*(referring to Pyramid 7) is 49 \ldots so I add 64 + 64 + 49 and then another 30.

**Interviewer:** Why did you add those numbers?

**Errol:** \ldots Uhmm \ldots because if I take \ldots 8 + 8 + 7 = 23 \ldots so I take the number by 8 *(referring to \(f(8)\))* , then I add it to itself \ldots and then I add the number by 7 to it.

**Interviewer:** Okay I understand, but where does the 30 come from?

**Errol:** I minus the 8 by 23 *(he means 23 minus 8)*, so I get 15 \ldots so I multiply by 2, then I get 30.

**Interviewer:** Can you explain to me why you did that?

**Errol:** \ldots the difference in between is growing by 2 every time \ldots

This method, generally symbolized by \(f(n) = f(a) + f(b) + f(c)\) where \(a + b + c = n\), was also used by other students. It seems that this strategy is born out of students’ inability to find factors for numbers such as 19, 23, 59, 117. One student calculated \(f(60) = f(20) + f(20) + f(20)\). This method could be seen as a variation of the proportional multiplication error, rooted in the close relationship between repeated addition and multiplication.

**The difference method**

The erroneous difference method, symbolised by \(f(n) = n \times d\) was invoked with both “seductive” and “non-seductive” numbers:

**Interviewer:** Ok, how many in Picture 23? *(How many tiles in Picture 23 in \(IV_T\))*

**Linda:** *(works on calculator)* \ldots 92

**Interviewer:** Just explain please?

**Linda:** It will take too long to add 4 every time *(she previously found a constant difference of 4 between the terms of the sequence)*. So I just said 23 times 4.
It could be argued that the inclusion of a direct proportional example where \( f(n) = an \) would have presented children with the conflict situation which would then perhaps limit both the proportional multiplication and difference method errors. However, Orton and Orton (1994) included such an example, but these errors still persisted.

**The visual impact of tables**
From Table 1 it is clear that the visual presentation of the numbers in a table format for the function did not impact on the errors children made. The table in activity I\(_T\) was horizontal and “continuous” whereas the table in II\(_T\) was vertical and “non-continuous”. Four children made the proportional multiplication error in both these examples. One student committed the difference method error in I\(_T\) whilst 3 students committed the error in II\(_T\). The way we presented the questions as “continuous” or “non-continuous” in the picture activities also did not effect children’s strategies. This can probably be explained by the fact that when the input numbers were not presented in writing, children made their own “continuous” “tables”, so the visual distraction remained.

**“Transparent” vs. “non-transparent” pictures**
In I\(_P\) five students successfully recognised the function rule from the structure of the picture. Two other children, when counting the number of tiles in each picture aloud, used the structure of the picture: “\( \ldots 4 \times 4 + 1 = 17, \ldots 5 \times 5 + 1 = 26, \ldots 6 \times 6 + 1 = 37 \ldots \)” but did not reflect on the structure they had verbalised and thus could not find the function rule.

In II\(_P\) most students recognised that 2 squares (8 matches) were being added but then converted to numerical mode, constructing their own “table” of values, e.g. “\( \ldots 1 = 4, 2 = 12, 3 = 20, \ldots \)” . Only two children described the function rule from the structure of the pictures, namely as \((n + n − 1) \times 4\) and \(n \times 4 + (n − 1) \times 4\) respectively.

No child could recognise the function rule of using the picture as the database in III\(_P\). Two students found the function rule once they reverted to the number context.

Only one student used the structure of the picture in IV\(_P\) to identify the function rule.

It seems that these students do not have the necessary know-how of how to use the structure of a picture to find a functional relationship. If one wants to find a function rule in a table, one necessarily takes some specific value of the independent variable (input number) and tries to construct a relationship between this input-output pair. In the case of pictures, few children seem to intentionally take a specific input number and try to see this number in the structure of the picture, as illustrated in the following diagram:

Of course, it further requires a rich number sense, e.g. in II to see a further relationship in the numbers (2 is one less than 3, and 3 is one less than 4) before one can formulate the function rule \([n + (n − 1)] \times 4\). In IV one must see the multiplication or equal addition structure before one can formulate the rule \(4 \times n + 1\). A weak number sense will therefore also contribute to students’ difficulties in using the structure of pictures to see the general in the particular required to formulate function rules.
Most students could see and use the structure of the pictures in a recursive way, e.g. in II students used the structure that 2 squares (8 matches) are added each time, and in IV they used the structure that 4 tiles are added to each successive picture. However, this did not help them to find the function rule, and students mostly then constructed a table of these values and then used the numbers in the table inductively. Of course, one could use the extended recursion method to use this recursive structure to formulate the function rules as \(4 + 8(n - 1)\) and \(5 + 4(n - 1)\) respectively. It is interesting that all cases of the use of the extended recursion method were in the context of tables and none in the context of pictures.

In all activities where students identified a function rule, most of them described their rule in words rather than using symbols. A distinction can also be made between those who could verbalise the rule using general language (e.g. in IP “..times the number by itself and add 1”) and those who could only verbalise the rule in terms of a specific number (e.g. “…if I know the number, say if it is 100, then I times 100 by itself and add 1).”

**Verification of strategies**

Consider the following protocol:

Interviewer: Ok, Shape 19? (How many matches in Shape 19 in IIp?)

Peter: (Peter successfully found \(f(7)\) and \(f(8)\) by counting the number of squares and then multiplying by 4 to get the number of matches. Now he starts making a systematic table of the number of squares in each Shape, using a recursive pattern:

<table>
<thead>
<tr>
<th></th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
<th>15</th>
<th>16</th>
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</thead>
<tbody>
<tr>
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<td>15</td>
<td>17</td>
<td>19</td>
<td>21</td>
<td>23</td>
<td>25</td>
<td>27</td>
<td>29</td>
<td>31</td>
</tr>
</tbody>
</table>

He then stops and goes back to looking at the pictures again.)

OK, I realised if I do this it is a bit of a hassle, so I looked at the pattern (in his database) and I figured the difference (between \(f(n)\) and \(n\))

I took here (pointing at \(f(5)\)) . . . the difference between 9 (\(f(5)\)) and 5 (\(n\)) is 4 and by number 6 it is 5 . . . yes (he checks again) . . .5. And by number 7 it is 6 and by number 8 it is 7. So I just tried it out. So I said to myself OK it is right and it will take too long to do it like this (referring to his table of values). So Shape 19 is 19 + 18, is 37, so 37 blocks times 4 gives you . . . 148 (using the calculator).

Interviewer: OK, and in Shape 59?

Peter: OK, its 59 + 58 is equal to 117, that is the number of blocks and then I take 117 (enters on the calculator) times 4 which gives the number 468, that is the number of matches in Shape 59.

Clearly, Peter has constructed an efficient rule, which we can symbolise as \([n + (n - 1)] \times 4\), based on a sound analysis of patterns in the given and extended database, and he verified that his pattern holds against the database several times. He was convinced and he could use the method with assurance.

However, this style of working stands in stark contrast to most students' approach to such generalisations. While the students who used a function rule necessarily deduced the rule from the database, the other strategies reported in this paper are mostly not based on the database – students did not find the methods in the database, nor did they check it against the database. This applies to correct as well as to incorrect strategies. Students seem not to realise the need to validate their generalisations, and seem not to have the know-how of how to validate a generalisation against the database.
DISCUSSION
As in our previous study, students worked nearly exclusively in the number context and not with the pictures, favoured recursion methods, had difficulty in finding function rules and made many errors, including the proportional multiplication error. There is, however, one marked difference, namely the variety of strategies used by the students in the present study in comparison to the previous study. Of course this could be attributable to the differences in the subjects, who are at a different grade level, and from a different socio-economic background. We would argue, however, that the difference is mainly attributable to the introduction of non-seductive numbers in our activities.

It is for this reason (the greater variety of strategies), that we plan to extensively use non-seductive numbers in our planned intervention. However, as is evident from the examples in this paper, the use of non-seductive numbers will probably not prevent the ubiquitous proportional multiplication error when students encounter seductive numbers, nor will it prevent the other erroneous strategies reported here. For that we believe we should address two more fundamental issues, viz.

1. The development of an awareness of the need to view any strategy as an hypothesis that should be validated against the database, and a focus on skills of how to do it.
   When one looks at the variety of strategies used by children, one can probably safely say that they have the ability and flexibility to find many patterns and relationships between numbers. The problem, however, is that most children are finding random relationships between the numbers without reference to the given database. We are struck not so much by the frequency and persistence of children's errors, but by their lack of an essential aspect of "mathematical culture", namely to view any strategy as an hypotheses that should be justified or verified against the given database. It seems that students lack simple strategic knowledge, e.g. to test an hypothesis against special cases.

2. A more explicit study of the properties of different function types, and a comparison of such properties to become aware which properties apply to which function types.

REFERENCES