Why is the number one not prime?

(from the Prime Pages' list of frequently asked questions)

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The number one is far more special than a prime! It is the unit (the building block) of the positive integers, hence the only integer which merits its own existence axiom in Peano's axioms. It is the only multiplicative identity ($1 \cdot a = a \cdot 1 = a$ for all numbers a). It is the only perfect nth power for all positive integers n. It is the only positive integer with exactly one positive divisor. But it is not a prime. So why not? Below we give four answers, each more technical than its precursor.

Answer One: By definition of prime!

The definition is as follows.

> An integer ***greater than one*** is called a **prime number** if its only positive divisors (factors) are one and itself.

Clearly one is left out, but this does not really address the question "why?"

Answer Two: Because of the purpose of primes.

The formal notion of primes was introduced by Euclid in his study of perfect numbers (in his "geometry" classic *The Elements*). Euclid needed to know when an integer n factored into a product of *smaller* integers (a nontrivially factorization), hence he was interested in those numbers which did not factor. Using the definition above he proved:

The Fundamental Theorem of Arithmetic

> Every positive integer greater than one can be written ***uniquely*** as a product of primes, with the prime factors in the product written in order of nondecreasing size.

Here we find the most important use of primes: they are the unique building blocks of the multiplicative group of integers. In discussion of warfare you often hear the phrase "divide and conquer." The same principle holds in mathematics. Many of the properties of an integer can be traced back to the properties of its prime divisors, allowing us to divide the problem (literally) into smaller problems. The number one is useless in this regard because $a = 1 \cdot a = 1 \cdot 1 \cdot a = \ldots$ That is, divisibility by one fails to provide us any information about a.

Answer Three: Because one is a unit.

Don't go feeling sorry for one, it is part of an important class of numbers call the **units** (or **divisors of unity**). These are the elements (numbers) which have a multiplicative inverse. For example, in the usual integers there are two units {1, -1}. If we expand our purview to include the Gaussian integers {*a*+*bi* | *a*, *b* are integers}, then we have four units {1, -1, *i*, -*i*}. In some number systems there are infinitely many units.

So indeed there was a time that many folks defined one to be a prime, but it is the importance of units in modern mathematics that causes us to be much more careful with with the number one (and with primes).

Answer Four: By the Generalized Definition of Prime.

(See also the technical note in The prime Glossary' definition).

There was a time that many folks defined one to be a prime, but it is the importance of units and primes in modern mathematics that causes us to be much more careful with the number one (and with primes). When we only consider the positive integers, the role of one as a unit is blurred with its role as an identity; however, as we look at other number rings (a technical term for systems in which we can add, subtract and multiply), we see that the class of units is of fundamental importance and they must be found before we can even define the notion of a prime. For example, here is how Borevich and Shafarevich define prime number in their classic text "Number Theory:"

> An element *p* of the ring D, nonzero **and not a unit**, is called *prime* if it can not be decomposed into factors *p*=*ab*, neither of which is a unit in D.

Sometimes numbers with this property are called **irreducible** and then the name prime is reserved for those numbers which when they divide a product *ab*, must divide *a* or *b* (these classes are the same for the ordinary integers--but not always in more general systems). Nevertheless, the units are a necessary precursors to the primes, and one falls in the class of units, not primes.

Prime Pages

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