Mathematics - HG - Nov 2001 National Paper 1 [Grade 12 Mathematics - HG]

Ref: M1/1/01	Total pages: 18	Time: 3 hours	Marks: 200

This question paper consists of a cover page, 18 pages and a formula sheet.

INSTRUCTIONS TO CANDIDATES

Read the following instructions carefully before answering the questions:

1. This paper consists of **11** questions. Answer **ALL** the questions.
2. Clearly show **ALL** the calculations, diagrams, graphs, etc. you have used in determining the answers.
3. An approved calculator (non-programmable and/or non-graphical) may be used, unless stated otherwise.
4. If necessary, answers should be rounded off to **TWO** decimal places unless stated otherwise.
5. Graph paper is **NOT** required in this question paper.
6. Number the answers **EXACTLY** as the questions are numbered.
7. It is in your own interest to write legibly and to present the work neatly.
8. **An information sheet containing formulae is provided.**
9. Diagrams provided in this question paper are not necessarily drawn to scale.

QUESTION 1

Determine ALL real solutions of each of the following:

1.1 $27^{x} \times 9^{x-2} = 1$ (4)

1.2 $16x^{4} + 1 = 0$ (2)

1.3 $\sqrt{5-x} - x = 1$ (6)

1.4 $2\,|\,x - 5\,| \geq 7$ (4)

1.5 $2.2^{x} - 8.2^{-x} = 15$ (6)

1.6 $3^{x}.5^{x+1} = 20$ (4)

1.7 $\dfrac{x-2}{3-x} \geq 1$ (6)

[32]

QUESTION 2

2.1 Prove that the roots of $a^2x^2 + abx + b^2 = 0$ are non-real for all real values of a and b, a and $b \neq 0$. (3)

2.2 If m and n are integers such that $m < n < 0$, state whether each of the following is TRUE or FALSE. Write down 'true' or 'false' next to the applicable question number and in each case justify the answer:

2.2.1 $m - n < -n$ (2)

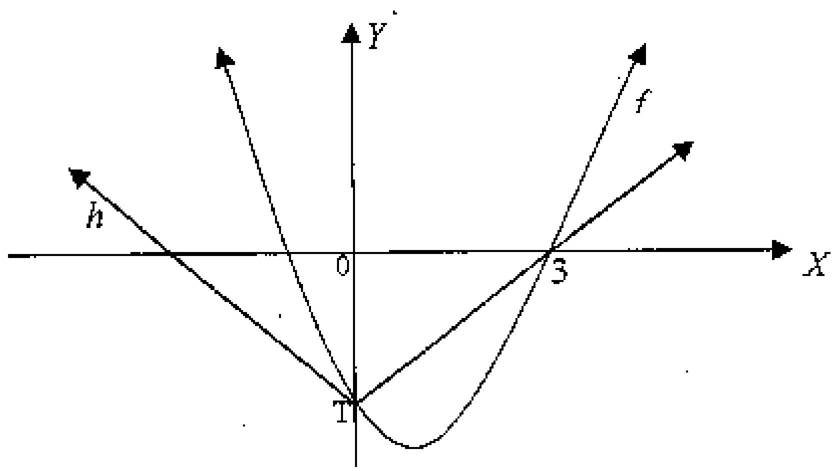
2.2.2 $m^2 < n^2$ (2)

2.2.3 $mn > n^2$ (2)

[9]

QUESTION 3

3.1



In the sketch, the graphs of the functions given by $f(x) = x^2 - 2x - 3$ and h, an absolute value function, are represented. Answer the following questions **with the aid of the sketch**:

3.1.1 For which values of x is f increasing? **(3)**

3.1.2 What is the maximum value of $-x^2 + 2x + 3$? **(4)**

3.1.3 For which value(s) of p will $x^2 - 2x - 3 = p$ have:

(a) Equal roots **(1)**

(b) No real roots **(2)**

3.1.4 For which value(s) of c will the roots of $x^2 - 2x + c = 0$ have the same sign? **(3)**

3.1.5 Determine b if $h(x) = |x| + b$ **(1)**

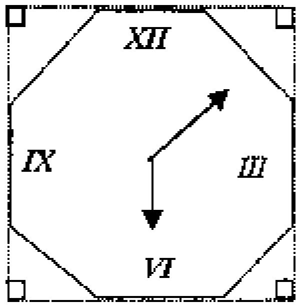
3.1.6 For which values of x is $h(x) > f(x)$? **(3)**

3.2 Determine the points of intersection of the graphs of $f(x) = x^2 - 2x - 3$ and the function defined by $y = -4x + 5$ **(6)**

[23]

QUESTION 4

Carlo manufactures eight-sided wall clocks, all the sides being of equal length. To create the face of the clock, he cuts the corners from a square sheet of glass of sides 16 cm.



Calculate:

4.1 The length of a side of the clock (8)

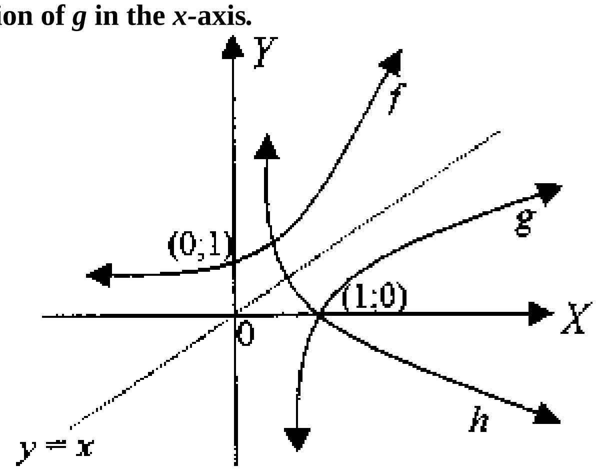
4.2 The area of the face of the clock (3)

[11]

QUESTION 5

5.1 In the sketch, the following functions are represented:

- *f* , with equation $y = x^3$
- *g*, the reflection of *f* in the line $y = x$.
- *h*, **the reflection of *g* in the *x*-axis.**



5.1.1 Determine the defining equations of *g* and *h* in the form.
$y =$ .... (4)

5.1.2 Determine, with the aid of the sketch, the value(s) of *x* for which:

(a) $3^x > 0$ (1)

(b) $\log_{\frac{1}{3}} x \leq 0$ (2)

5.2 Solve for *x*:
$(\log_3 x)^2 - 2 \geq \log_3 x$ (8)

[15]

QUESTION 6

6.1 Given: $f(x) = ax^3 - 5x^2 + bx + 6$
$f(x)$ is exactly divisible by $x - 2$ and leaves a remainder of -3 when divided by $2x$(6) -1. Determine the values of a and b.

6.2 Given: $f(x) = x^n + y^n$
For which value(s) of n is $(x + y)$ a factor of f ? (4)
[10]

QUESTION 7

7.1 Calculate: $\sum_{k=2}^{\infty} 8\left(\frac{1}{2}\right)^{k+2}$ (4)

7.2 In a geometric sequence, the third term is $5m + 1$, the fifth term $m + 1$, and the seventh term $m - 2$. If all the terms are positive, calculate the value of m. (7)

7.3 A man was injured in an accident. He receives a disability grant of R4 800 in the first year. This grant increases by a fixed amount each year.

7.3.1 What is the annual increase if, over 20 years, he would have received R143 500 altogether? (4)

7.3.2 His initial annual expenditure is R2 600 and increases at a rate of R400 per year. In which year will his expenses exceed his income? (6)

[21]

QUESTION 8

8.1 Calculate the derivative of f from first principles, if.
$f(x) = x - x^2$ (5)

8.2 Determine $\frac{dy}{dx}$ if:

8.2.1 $y = (x^3 - 1)^2$ (3)

8.2.2 $y = \frac{x^3 + \sqrt{x^3}}{x}$ (4)

8.3 Given: $f(x) = -x^3 + 6x^2 - 9x + 4$

8.3. 1 Draw a neat sketch graph of f , showing the coordinates of the intercepts with the axes, as well as the coordinates of the turning points. (Show all your calculations.) (17)

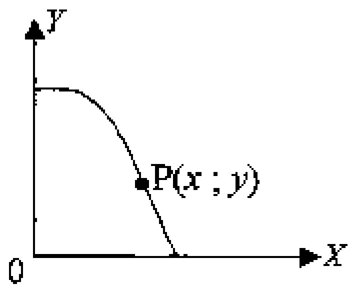
8.3.2 Determine the equation of the tangent to the curve of f at the point (2 ; 2). (5)

[34]

QUESTION 9

9.1 The graph of $x^3 + y^2 = 8$, not drawn to scale, is represented alongside for the interval 0

$\leq x \leq 2$
P(***x* ; *y*) is any point on the graph.**



9.1.1	Determine OP^2 in terms of x and y.	(1)
9.1.2	Show that $OP^2 = -x^3 + x^2 + 8$	(1)
9.1.3	Determine the maximum value of OP^2.	(5)
9.1.4	Calculate the shortest distance from the origin to the graph.	(3)

9.2 The equation of the tangent to $y = f(x)$ at $x = -1$ is $y = -3x + 4$.
Determine $f(-1)$ and $f'(-1)$ (2)

9.3 The accompanying sketch represents the curve of $f'(x)$.

9.3.1	what is the gradient of f at $x = 0$?	(1)
9.3.2	f has a maximum at a. Determine the value of a.	(2)
9.3.3	For which values of x is f increasing?	(2)

[17]

QUESTION 10

An entrepreneur manufactures two types of furniture pieces: chairs and tables.
The costs are R250 per chair and R200 per table. He sells each chair for R300 and each table for R400. He makes x chairs and y tables each month, so that the points $(x ; y)$ he only in the shaded (feasible) region below.

10.1 Write down the inequalities which describe the feasible region. (6)

10.2 Determine the coordinates of P and T. (6)

10.3 Determine the minimum total cost. (3)

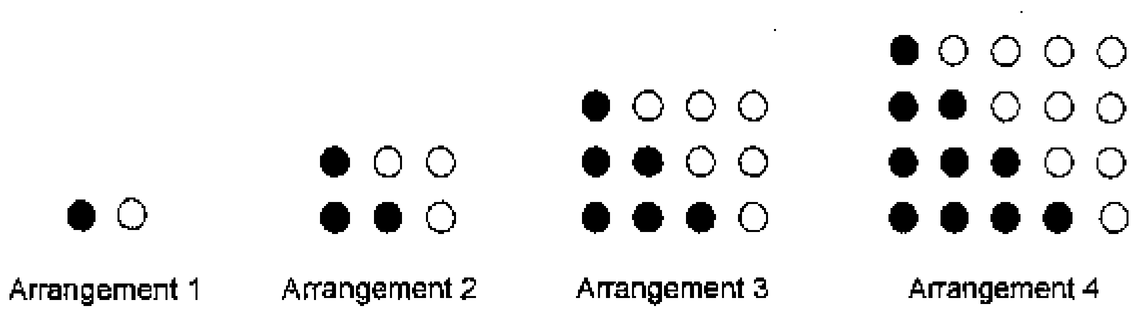
10.4 Determine the maximum profit. (3)

10.5 If the production cost for a table increases to R500, what would the minimum cost be? (2)

[20]

QUESTION 11

Black and white dots are packed as shown in the arrangements below:



11.1 If T_n is the total number of dots in the n^{th} arrangement, determine:

11.1.1 T_5 (1)

11.1.2 T_{10} (2)

11.1.3 T_n (3)

11.2 **Use the answer in QUESTION 11.1.3** to write down a formula for the sum of the first *n* natural numbers. (2)

[8]

TOTAL: 200

Mathematics Formula Sheet (HG and SG)

Wiskunde Formuleblad (HG en SG)

NATIONAL DEPARTMENT OF EDUCATION

SENIOR CERTIFICATE EXAMINATION

Mathematics - HG - Nov 2001 National Paper 1 Memorandum
[Grade 12 Mathematics - HG]

1.1

$$27^x \times 9^{2x} = 1$$
$$3^{3x}3^{2x-4} = 3^0$$
$$\therefore\ 3x + 2x - 4 = 0$$
$$\therefore\ x = 4/5$$

1.2

$$16x4 + 1 = 0$$
$$\therefore\ x4 = -1/16$$
No solution in R

1.3

$$\sqrt{5-x} - x = 1$$
$$\sqrt{5-x} = x + 1$$
$$\therefore\ 5 - x = (x+1)^2$$
$$\therefore\ x^2 + 3x - 4 = 0$$
$$(+4)(x-1) = 0$$
$$x = -1 \text{ OF } x = 1$$

Check both answers using first eq.
Solution: x = 1
[x = -4 is not a solution]

1.4

$$2(x-5) \geq 7$$
$$(x-5) \geq 3{,}5$$

If $x - 5 \geq 0,$ OR as $x - 5 < 0$
$x \geq 5$ $x < 5$
then $x - 5 \geq 3{,}5$ then $-(x-5) \geq 3{,}5$
$\therefore\ x \geq 8{,}5$ $\therefore\ -x \geq -1{,}5$
$\therefore\ x \leq 1{,}5$

Solution : $x \leq 1{,}5$ OR $x \geq 8{,}5$

1.5

$$2.2^x - 8.2^x = 15$$
[Multiply by 2^x]

$$\therefore\ 2.2^{2x} - 8 - 15.2^x = 0$$
$$\therefore\ 2a^2 - 15a - 8 = 0$$
$$(2a+1)(a-8) = 0$$
$$\therefore\ a = -½ \text{ or } a = 8$$
$$\therefore\ 2^x = -½ \text{ or } 2^x = 8$$
onm. -2^3
$$\therefore\ x = 3$$

1.6

$$3^x.5^{x-1} = 20$$
$$3^x.5^x.5 = 20$$
$$\therefore\ 15^x = 4 \text{ [Now take logs]}$$
$$x\log 15 - \log 4$$
$$\therefore = (\log 4)/(\log 15)$$
$$= 0{,}51$$

1.7 If $3 - x > 0$ OR If $3 - x < 0$

$x < 3$ $x > 3$

then $x - 2 \geq 3 - x$ then $x - 2 \leq 3 - x$

$2x \geq 5$ $2x \leq 5$

$\therefore\ x \geq 2{,}5$ $\therefore\ x \leq 2{,}5$

no solution

Solution : $2{,}5 \leq x < 3$

2.1 $a^2x^2 = abx + b^2 = 0$

$= a^2b^2 - 4.a^2 - 4a^2.b^2$

$= -3a^2b^2$

a^2 en b^2 is positive and multiplied by -3.

Thus) < 0

∴Roots non-real.

2.2.1 TRUE [Add n to both sides]

2.2.2 FALSE [* m * n *]

2.2.3 TRUE [Divide by n; negative]

3.1.1 turning point: x = -b/2a = -(-2)/2 = 1

f increasing for $x > 1$

3.1.2 Turning point of f: (1; -4)

The maximum of the mirror image of f in the X-axis will be 4.

3.1.3 (a)

Move the graph upwards 4 units. The y-

intercept is then 1.

∴p = -4

(b)

p < -4

[Hint: Check by taking p to LHS.]

3.1.4 Move the graph upwards slowly.

If c = 0, the roots will be 0 and a positive number.

If c = 1, the graph touches the X-axis.

Solution: $0 < c \leq 1$

3.1.5 b = -3

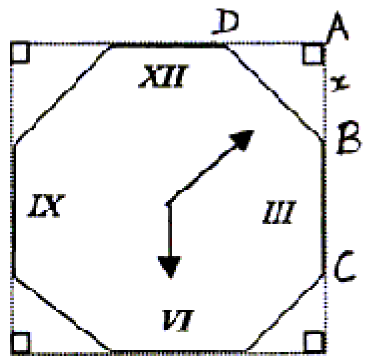
3.1.6 $0 \leq x \leq 3$

$\therefore\ x = -4$ OF $x = 2$

en $y = 21$ $y - -3$ [replace]

Points of intersection: (-4;21) en (2;-3)

4.1 Let AB = AD = x
then BC = 16 – 2x
and BD = $\sqrt{2}$ x [Pythagoras]



Eight equal sides:

$$\therefore \sqrt{2}x = 16 - 2x$$

$$2x^2 = 256 - 64x + 4x^2$$

$$x2 - 32x - 128 = 0$$

$$x = \frac{32 \pm \sqrt{32^2 - 4(128)}}{2}$$

$$= 4{,}686$$

[The other answer not acceptable]
Length of side: 16 – 2x = 6.63

4.2 Area of face: 16 - 4) ABD = 16 – 4 (0,5x^2) = 212,1

5.1.1

$$g : y = \log_3 x$$

$$h : y = -\log_3 x \text{ of } y = \log_{\frac{1}{3}} x$$

5.1.2 (a) All real values of x.
(b) x ≥ 1

5.2

$$(\log 3x)^2 - 2 \geq \log 3x$$

$$a2 - a - 2 \geq 0$$

$$\therefore (a-2)(a+1) \geq 0$$

$$a \leq -1 \text{ OR } a \geq 2$$

$$\therefore \log 3x \leq -1 \text{ OR } \log 3x \geq 2$$

$$x \leq 3^{-1} x \geq 3^2$$

$$x \leq 1/3 x \geq 9$$

but x > 0, definition
Solution: 0 < x ≤ 1/3 OF x ≥ 9

6.1

$$f(x) = ax^3 - 5x^2 + bx + 6$$
$$f(2) = 8a - 20 + 2b + 6 = 0$$
$$8a + 2b = 14$$
$$4a + b = 7 \quad .....(\text{i})$$

$$\text{en } f\left(\frac{1}{2}\right) = \frac{a}{8} - \frac{5}{4} + \frac{b}{2} + 6 = -3$$
$$a - 10 + 4b + 48 = -24$$
$$\therefore a + 4b - -62 \quad .....(\text{ii})$$

$$(\text{i}) \; x4: 16a + 4b = 28$$
$$(\text{ii}) \qquad a + 4b = -62$$
$$\text{subtract}: \quad 15a \quad = 90$$
$$a = 6$$
$$\text{en } b = -17$$

6.2 (x + y) is a factor of f if f(-y) = 0

$(-y)^n + y^n = 0$ if n is an odd number

[We know it is true for n = 3].

7.1

$$\sum_{k=2}^{\infty} 8\left(\frac{1}{2}\right)^{k+2}$$

$T_1 = ½$

$T_2 = ¼$

T3 = 1/8

$$S\infty = \frac{a}{1-r} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1$$

7.2

$$\frac{T_7}{T_5} = \frac{T_5}{T_3}$$
$$\therefore \frac{m-2}{m+1} = \frac{m+1}{5m+1}$$

$$\therefore m^2 + 2m + 1 = 5m^2 - 9m - 2$$
$$4m^2 - 11m - 3 = 0$$
$$(4m+1)(m-3) = 0$$
$$\therefore m = -1/4 \text{ OF } m = 3$$

NO

Solution : m = 3

7.3.1 Arithmetic series with
a = 4800 and S_{20} = 143 500.
d is required.

$$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$$

$$143500 = \frac{20}{2}\left[9600+19d\right]$$

$$14350 = 9600+19d$$

$$d = 250$$

.

7.3.2 Income : a = 4800 and d = 250
Expenses : a = 2600 and d = 400
When will T_n be equal?

$$Tn - a+(n-1)d$$

$$\therefore\ 4800+(n-1)\,250 = 2600+(n-1)\,400$$

$$4800+250n-250 = 2600+400n-400$$

$$150n = 2350$$

$$\therefore\ n = 15{,}7$$

The expenses will exceed the income in the 16th year.

8.1 f(x) = x – x^2

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)-(x^2+2hx+h^2)-x+x^2}{h}$$

$$= \lim_{h\to 0}\frac{x+h-x^2-2hx-h^2-x+x^2}{h} = \lim_{h\to 0}\frac{h-2hx-h^2}{h} = \lim_{h\to 0}\frac{h(1-2x-h)}{h}$$

$$= \lim_{h\to 0}(1-2x-h)$$

$$= 1-2x$$

8.2.1

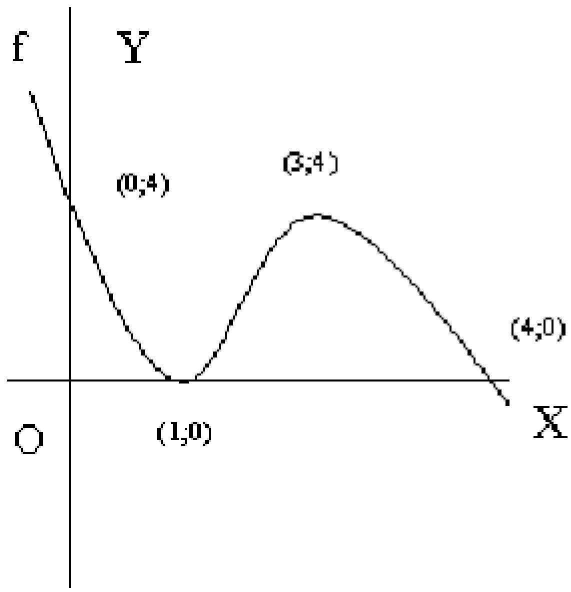
$$y = (x^3-1)^2 = x^6-2x^3+1$$

$$\frac{dy}{dx} = 6x^5-6x^2$$

8.2.2

$$y = \frac{x^3+\sqrt{x^3}}{x} = \frac{x^3+x^{\frac{3}{2}}}{x} = x^2+x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = 2x+\frac{1}{2}x^{\frac{-1}{2}} = 2x+\frac{1}{2x^{\frac{1}{2}}}$$



8.3.1 $f(x) = -x^3 + 6x^2 9x + 4$

$f(1) = -1 + 6 - 9 + 4 = 0$, Thus : $(x-1)$ is 'n factor of $f(x)$

$\therefore f(x) = (x-1)(-x+4)$

X − intercepts : $f(x) = 0$

Dus : $x = 1$ OR $x = 4$

Turning points : $f(x) = 0$

$\therefore -3x^2 + 12x - 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$\therefore x - 3$ OR $x = -1$

$y = 4 y = 0$

8.3.2 Tangent : m = f(2)

$f'(x) = -3x^2 + 12x - 9$

$f'(2) = -3(4) + 12(2) - 9$

$= 3$

Eq : $y = 3x + c$

$\therefore\ 2 = 3(2) + c$

$\therefore\ c = -4$

Tangent : $y = 3x + c$ deur $(2 : 2)$

$\therefore\ 2 = 3(2) + c$

$\therefore\ c = -4$

Tangent : y = 3x - 4

9.1.1 $OP^2 = x^2 + y^2$

9.1.2 $OP^2 = x^2 + 8 - x^3$

$= -x^3 + x2 + 8$

9.1.3 For max/min values $f'(x) = 0$

$\therefore\ -3x^2 + 2x = 0$

$x(-3x + 2) = 0$ Thus : $x = 0$ OR x = 2/3

Substitute : x = 0: y = 0 + 0 + 8 = 8

Substitute : x = 2/3

$$y = -\left(\frac{2}{3}\right)^3 + \left(\frac{2}{3}\right)^2 + 8$$

$$= \frac{-8}{27} + \frac{4}{9} + 8 = 8\frac{4}{27} = 8{,}15$$

Maximum of OP^2 is 8,15.

9.1.4 If the question refers to the graph in 9.1:

The Y-intercept of the graph is $2\sqrt{2}$ and the X-intercept is x = 2.
Calculate different lengths of OP by substituting values for x and y.
The minimum value is found where y = 0 and x = 2.
The minimum length of OP is 2.
[The graph of OP^2 can also be considered but is more complicated.]

9.2 Die raakpunt lê op die raaklyn sowel as op f(x).

$f'(-1) = -3$ [gradient of tangent]

f(-1) is die y- value of the point of tangency.

$\therefore y = -3(-1) + 4 = 7$

$\therefore f(-1) = 7$

9.3.1 m = 40

9.3.2 a = 30

9.3.3 f increases if f ´(x) is positive.
$0 \leq x < 30$

10.1
$$2y \leq x + 1000$$
$$2y \geq -x + 2000$$
$$500 \leq y < 1500$$
$$x \leq 2000$$
x and y positive integers.

10.2 Coordinate s P : Solve equations simultaneo usly.

$$2y - x = 1000$$
$$2y + x = 1000$$
$$\therefore -2x = -1000$$
$$x = 500$$
$$\text{en } y = 750 \qquad P(500,750)$$

Coordinate s of K :

$$x + 2y = 2000 \qquad [\text{subst. } y]$$
$$x + 1000 = 2000$$
$$\therefore x = 1000$$
$$\text{and } y = 500 \qquad T(1000 : 500)$$

10.3 Cost : $K - 250x + 200y$

$$\therefore y = \frac{-5}{4}x + \frac{K}{200}$$

Min. by P : $K = 250(500) + 200(750)$

$= 275\ 000$

10.4 Profit : $W = 50x + 200y$

$$\therefore y = \frac{-1}{4}x + \frac{W}{200}$$

Maks. at Q(2000;1500)

W = 50(2000) + 200(1500)

= 400

[The dotted line at Q may indicate that the number of tables should be less than 1500, i.e. 1499. The Profit will then be reduced by R200.]

10.5 Cost : $K = 250x + 500y$

$\therefore\ y = \frac{-1}{2}x + \frac{K}{500}$

Gradient of this line is -½.

Gradient of PT is also -½.

Any point on PT (integers) will give a minimum cost.

At P : K = 250(500) + 500(750) = 500 000

At T : K = 250(1000) + 500(500) = 500 0

11.1.1 T_5 = 30 of (5 x 6); Each number is the product of two consecutive numbers.

11.1.2 T_{10} = 10 x 11 = 110

11.1.3. T_n = n(n + 1)

11.2 $S_n \frac{n(n+1)}{2}$