2.5 PROOF: ANALYTICAL AND SYNTHETIC REASONING

Construction of valid arguments or proofs and criticising arguments are essential aspects of doing mathematics. However, constructing proofs (deductive reasoning) is not always so easy. In this section we look at the logic underlying valid proofs and the process of constructing proofs. Let's start with this orientation problem:

PROBLEM 62: CORRECT OR NOT?

Give a mark out of 10 for each of the proofs[1] in A. Motivate!

A

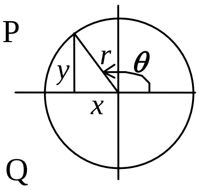
1. Prove that: $\sin^2\theta + \cos^2\theta = 1$

Vusi's proof: $\sin^2\theta + \cos^2\theta = 1$ ----------- P

$\therefore\ (\frac{y}{r})^2 + (\frac{x}{r})^2 = 1$

$\times r^2:\ \therefore\ y^2 + x^2 = r^2$ ----------- Q

Q is true. So P is true. QED



2. Prove that: $\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$

Thandi's proof: $\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$ ------------ P

$\therefore \sin^2\theta = (1-\cos\theta)(1+\cos\theta)$ and $\sin\theta \neq 0, \cos\theta \neq 1$

$\therefore\ \sin^2\theta = 1 - \cos^2\theta$

$\therefore\ \sin^2\theta + \cos^2\theta = 1$ ----------- Q

Q is true. So P is true. QED

3. Sove for x: $\frac{2(x-3)}{5} + \frac{x+3}{3} = 2$

Mary's solution: $\frac{2(x-3)}{5} + \frac{x+3}{3} = 2$ .... P

$\therefore\ 6(x-3) + 5(x+3) = 30$

$\therefore 6x - 18 + 5x + 15 = 30$

$\therefore 11x - 3 = 30$

$\therefore 11x = 33$

$\therefore x = 3$ .... Q

$x = 3$ is the solution of Q, so $x = 3$ is the solution of P

[1] **Q. E. D.** We often joke that QED means "quite easily done"! Euclid (about 300 B.C.) concluded his proofs with hoper edei deiksai, which Medieval geometers translated as *quod erat demonstrandum* ("that which was to be proven"). Isaac Newton used the abbreviation Q. E. D. From Earliest Known Uses of Some of the Words of Mathematics

First comments on A

As they are expressed here, the proofs in 1 and 2 are *invalid!* The well-known procedure for solving equations in 3 is based on exactly the same reasoning structure as the proofs in 1 and 2 and the logic is likewise *invalid!* Learners should get 0 out of 10 for each question!

But w*hy* are the proofs invalid?

Teachers often do not accept the proofs in A, giving as reason that "it is an *identity* and therefore one *must* work with the left-hand and right-hand sides separately". Of course one *can* prove the statements using the LHS-RHS proof scheme (if $a = c$ and $b = c$, then $a = b$), but to insist that it is the *only* proof scheme, *impoverishes* mathematics and *handicaps* us as mathematicians!

Underlying the insistence on the LHS-RHS proof scheme, is a perspective that
"You cannot say they are *equal* when you have to *prove* that they are equal, so you may not use the =-sign".
"You may not *assume* that it is true and then prove that it is true".
"You may only start with something that is *true* (or that is *given)"*.
These perspectives are *not* correct, as you will see in the rest of this section.

We will analyse the proofs in 1 and 2 in two ways: first from the perspective of *logic*, and then from the perspective of *equations*, where we will also come back to discuss the logic of 3. In the process we also reflect on issues about the nature of mathematics and how our views on the nature of mathematics influence the teaching and learning of mathematics.

Logic

Pure mathematics is the class of all propositions of the form 'p implies q' where p and q are propositions ... Bertrand Russell

In deductive reasoning, we argue that if certain *premises* (*antecedent or hypothesis)* (P) are known to be true or assumed, a *conclusion* (Q) necessarily follows from these.

If a conclusion does not follow from its premises, the argument[2] is said to be *invalid* or *non sequitur* (Latin for "it does not follow"). It should be stressed that in an invalid argument the conclusion can be either true or false[3], but the *argument* (the logic of the *reasoning*) is *invalid* because the conclusion does not follow from (is not caused by) the premises.

If the argument is valid but the premises are not true, then the conclusion may or may not be true, but the argument cannot help us decide this[4].

To understand the above statements, let's analyse the logic in the following arguments:

"*If* Graeme arrives late (P), *then* he will miss the bus (Q)."

We call this a *conditional statement*[5] – it establishes the condition, the relationship between the two statements. We say P implies Q, writing it as P $\Rightarrow$ Q.

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[2] An argument is a line of reasoning, a sequence of statements aimed at demonstrating the truth of an assertion.

[3] Note that in logic we say a statement is *true* or *false*, but that an argument (the reasoning structure) is *valid* or *invalid*.

[4] This what A is all about!! Compare: If $3 = 4$, add 1 to both sides, then $4 = 5$ (false). If $3 = 4$, multiply by 0 both sides, then $0 = 0$ (true).

[5] A statement is a declarative sentence which is either true or false, according to the Aristotle *axiom of the excluded middle*. For a *closed sentence* this is immediately obvious, e.g. $2 + 3 = 5$ is true and $2 + 3 = 6$ is false. In the case of an *open sentence*, e.g. $x + 3=5$, whether the statement is true depends on the value of the variable.

Then we have one further piece of information (fact) about the truth of P or Q (e.g. Graeme was late or did not miss the bus), and then we use this information together with the implication to draw some conclusion. Consider these cases:

Case 1: Fact: Graeme arrives late. Can we conclude that he misses the bus? **Yes! This is a *valid* argument.** The structure is: P $\Rightarrow$ Q is valid, P is true. So Q is true.

Case 2: Fact: Graeme does not miss the bus. Can we conclude that he did not arrive late? **Yes! This is a *valid* argument.** The structure is: P $\Rightarrow$ Q is valid, Q is false. So P is false.

Case 3: Fact: Graeme does not arrive late. Can we conclude that he did not miss the bus? **No! This is an *invalid* argument.** This may *sound* like a good argument, but perhaps he did indeed catch the bus, perhaps he fell asleep and missed it anyway – we have insufficient information to conclude anything.
The structure is: P $\Rightarrow$ Q is valid, P is false. So Q is false.

Case 4: Fact: Graeme misses the bus. Can we conclude that he must have arrived late? **No! This is an *invalid* argument.** This may *sound* like a good argument, perhaps he was indeed late, but maybe he arrived on time but for some reason did not get on the bus, or the bus may have been full – we have insufficient information to conclude anything.
The structure is: P $\Rightarrow$ Q is valid, Q is true. So P is true.

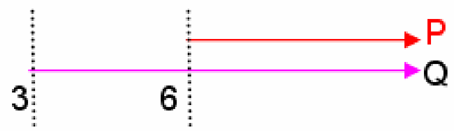
As the two invalid arguments above suggest, the conclusion of an invalid argument does not necessarily have to be false – it's just *unproven* by this particular argument. In everyday language there is often an expectation that if P happens then Q will follow, but if P fails then Q fails also (e.g. if you study hard, you will pass implies that if you do not study hard you will fail – see also case 3 above) and that if Q happened the prerequisite P was fulfilled (e.g. if someone passed, it means that he studied hard – see also case 4 above).

However, in mathematics such assumptions are *not* made. In mathematics a proof in the form ***if P then Q*** **simply requires that *if P is true, then Q must be true* also. If P is false, then no implication as to the truth or falsehood of Q is possible. Likewise if Q is true, we cannot conclude that P is true, as assumed in the "proofs" in A!** When the mathematician says "is true", he means "is always/generally true" or "is necessarily true".

Consider this conditional: *If* $x > 6$ *then* $x > 3$. In mathematics this is considered a valid implication: if x is a number bigger than 6 then it must necessarily also be bigger than 3. However, consider this as separate statements, where P is "$x > 6$" and Q is "$x > 3$". What happens for various values of x? Of course you will expect that if P is true then Q is true, but note that if $x = 5$, P is false and Q is true! Look again: if $x = 5$, Q is true but P is false – does this convince you that the arguments in A are invalid?

Make sure that you agree with the summary of the different possibilities in this table. Maybe you will also find it useful to interpret the relationships on the number line model.

	P: $x > 6$	Q: $x > 3$
$x = 7$	true	true
$x = 5$	false	true
$x = 2$	false	false



Let's try to draw some conclusions about possible P-Q relations from this information:

- What can we say if we know that P is true? You can see in the table that for P there is only one true value, so we know that *if P is true, then Q is true, for certain!*
- What can we say if we know that Q is false? You can see in the table that for Q there is only one false value, so we know that *if Q is false, then P is false, for certain!*
- What can we say if we know that P is false? We cannot conclude anything, because as you can see in the table, *Q can be true or false, so we do not know!*
- What can we say if we know that Q is true? We cannot conclude anything, because as you can see in the table, *P can be true or false, so we do not know!*

So of the above four possibilities, two are valid arguments, and two are invalid.

Mathematicians do not study objects, but relations between objects. Thus, they are free to replace some objects by others so long as the relations remain unchanged. Content to them is irrelevant: they are interested in form only[6]. Jules Henri Poincaré, 1854-1912

Here is a summary/generalisation of the *forms of the logic* (i.e. P and Q are *any* statements):

Valid arguments		**Invalid arguments**	
$P \Rightarrow Q$ P is true So Q is true	$P \Rightarrow Q$ Q is false So P is false	$P \Rightarrow Q$ P is false So Q is false	$P \Rightarrow Q$ Q is true So P is true
Modus Ponens Latin: *mode that affirms* Direct proof	Modus Tollens Latin: *mode that denies* Indirect proof *Reductio ad absurdum*	"Inverse[7] trap" Denying the antecedent	"Converse trap" Affirming the consequent

It should now be clear that all the proofs in A fell into the Converse trap! Maybe this will convince you:

If it rains, it is wet It is raining So it is wet ✓	*If it rains, it is wet* It is not wet So it is not raining ✓	*If it rains, it is wet*[8] It is not raining So it is not wet ×	*If it rains, it is wet*[9] It is wet So it is raining ×

The important thing to understand is that, *if you reason correctly*:

- If you start with a true statement, you must end with a true statement – this is Modus Ponens (Note: we said *if you reason correctly*, i.e. if you use a *valid argument*.)
- If you end with a true statement, you do not know if the original statement is true or false – this would be the Converse trap! These three simple examples illustrate the point:

$2 = 2$, which is true $+3 :\Rightarrow 2+3 = 2+3$, a valid argument $\Rightarrow 5 = 5$, which is true	$2 = 3$, which is false $\times 0 :\Rightarrow 0\times 2 = 0\times 3$, a valid argument $\Rightarrow 0 = 0$, which is true	$3 = -3$, which is false $\Rightarrow 3^2 = (3)^2$, a valid argument $\Rightarrow 9 = 9$, which is true

Check again: The reasoning is valid, and each conclusion is a true statement. But we cannot deduce from this true conclusion whether the original statement was true or not!

Let's return to Ame's famous $x + x = x^2$:

To check if the manipulation $x + x = x^2$ is correct, if we check for $x = 1$:

$x + x = x^2$ (P) $\Rightarrow$ 1 + 1 = 1^2, i.e. 2 = 1 (Q). Q is false, so P is false by Modus Ponens[10].

However, if we check for $x = 2$:

$x + x = x^2$ (P) $\Rightarrow$ 2 + 2 = 2^2, i.e. 4 = 4 (Q). Q is true. But we cannot deduce that P is true – *it is the Converse trap.*[11]

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6 Convince yourself that the *content* of our bus and mathematics arguments above is different, but their logical *form* is similar.

7 The Converse of *If P then Q* is *If Q then P;* Inverse: *If not P then not Q*; Contrapositive: *If not Q then not P*

8 *Compare:* If I am in Stellebosch, I am in South Africa. I am not in Stellenbosch. Therefore, I am not in South Africa?

9 *Compare:* If I am a cat, I am a mammal. I am a mammal.Therefore, I am a cat?

10 Note that *disproving* a statement using a *counter example*, is therefore proof by *Modus Ponens!*

11 This is also the *logical explanation* why partial *induction* can never be a proof! For example: F(n) = n^2 – n+11 is prime (P) $\Rightarrow$ F(1) = 11 is prime, indeed …. $\Rightarrow$ F(10) = 101 is prime (Q), is the Converse trap! P $\Rightarrow$ Q is valid, Q is true.

PROBLEM 63: ILLOGICAL LOGIC

If we use valid reasoning (each step in the argument is valid), a true statement cannot lead to a false statement. If the conclusion is false, there must be an error in the reasoning somewhere! Where is the error in the logic in these arguments?

$$\begin{aligned} -20 &= -20 \\ \Rightarrow 25-45 &= 16-36 \\ \Rightarrow 25-45+\frac{81}{4} &= 16-36+\frac{81}{4} \\ \Rightarrow (5-\frac{9}{2})^2 &= (4-\frac{9}{2})^2 \\ \Rightarrow 5-\frac{9}{2} &= 4-\frac{9}{2} \\ \Rightarrow 5 &= 4 \end{aligned}$$

$$\begin{aligned} & x = 2 \text{ .... } (1) \\ & \Rightarrow x-1 = 1 \\ & \Rightarrow (x-1)^2 = 1 \\ & \Rightarrow x^2 - 2x + 1 = 1 \\ & \Rightarrow x^2 - 2x = 0 \\ & \Rightarrow x(x-2) = 0 \\ & \Rightarrow x = 0 \text{ or } x = 2 \\ & \text{Put } x = 0 \text{ in } (1):\ 0 = 2 \end{aligned}$$

$$\begin{aligned} & x = 2 \text{ .... } (1) \\ & \Rightarrow x-1 = 1 \\ & \Rightarrow (x-1)^2 = 1 = x-1 \\ & \Rightarrow x^2 - 2x + 1 = x-1 \\ & \Rightarrow x^2 - 2x = x-2 \\ & \Rightarrow x(x-2) = 1(x-2) \\ & \Rightarrow x = 1 \\ & \text{Put } x = 1 \text{ in } (1):\ \text{So } 1 = 2 \end{aligned}$$

PROBLEM 64: ILLOGICAL LOGIC TOO?

If we use valid reasoning (each step in the argument is valid), a false statement can lead to a true statement. Or can you find an error in the reasoning in the following argument?

$$\begin{aligned} 5 &= 4 \\ \Rightarrow 5-\frac{9}{2} &= 4-\frac{9}{2} \\ \Rightarrow (5-\frac{9}{2})^2 &= (4-\frac{9}{2})^2 \\ \Rightarrow 25-45+\frac{81}{4} &= 16-36+\frac{81}{4} \\ \Rightarrow 25-45 &= 16-36 \\ \Rightarrow -20 &= -20 \end{aligned}$$

Remark:

"Logic" is not formally included as content in the school mathematics curriculum. But it should be, and teachers should make sure that their learners understand the basic logical principles. Erroneous logical generalisations are not only apparent in proofs as illustrated in A, but it effects children's everyday mathematical activity.

Example from Geometry[12]:

Learners often assume that the c*onverse* of a statement is true. For example, the theorem "If two triangles are congruent, then corresponding angles are equal" is true, but the converse is not: "If corresponding angles of two triangles are equal, then the triangles are congruent."

Example from Algebra.

Most school children have the misconception that if $x + y = 4$, then x and y cannot both be 2. (If you do not believe it, you should check your learners!) Children often "deduce" this idea through erroneous logic (Olivier, 1988), namely:

[12] Actually there is a *serious* problem here: Learners mostly assume converses are true, while, in general, converses are not (the converse trap)! We should rather develop the attitude to view each true converse as an exception, as an unexpected surprise, which it is! What is the converse of "If a figure is a triangle, then it is a polygon"? What is the converse of "If a figure is a rectangle, then it is a square"? And "If triangles are congruent, then they have equal areas"?

If the symbols are the same, then their values are the same (P $\Rightarrow$ Q)
So: *If the symbols are different, then their values are different* (Inverse trap: not P $\Rightarrow$ not Q)
or
If the symbols are the same, then their values are the same (P $\Rightarrow$ Q)
So: *If the values are the same, then the symbols are the same* (Converse trap: P $\Leftarrow$ Q)

Analytical reasoning

Let us now look at the "proofs" in A with different lenses. Although the proofs are logically invalid, the kind of *reasoning* is essential in proving statements.

When we reason *backwards* to find out how we could prove it, we call the reasoning *analysis*:

(B) ANALYSIS

We can prove that $\sin^2\theta + \cos^2\theta = 1$ ------- P
if we can prove that $(\frac{y}{r})^2 + (\frac{x}{r})^2 = 1$
i.e. if we can prove that $y^2 + x^2 = r^2$ ------- Q
But we know that Q is true, we do not have to prove it!

We can prove that $\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$ ------ P
if we can prove that $\sin^2\theta = (1-\cos\theta)(1+\cos\theta)$
i.e. if we can prove that $\sin^2\theta = 1-\cos^2\theta$
i.e. if we can prove that $\sin^2\theta + \cos^2\theta = 1$ ---- Q
But we know that Q is true, we do not have to prove it!

Now we can easily prove the two original statements in A through Modus Ponens by simply reversing the reasoning in B, i.e. reasoning back from the bottom to the top:

(C) SYNTHESIS

$y^2 + x^2 = r^2$ (Pythagoras) --------- Q
$\div r^2: \Rightarrow (\frac{y}{r})^2 + (\frac{x}{r})^2 = 1$ if $r \neq 0$
$\Rightarrow \sin^2\theta + \cos^2\theta = 1$ ----------------- P
So Q $\Rightarrow$ P is valid. Q is true. So P is true.

$\sin^2\theta + \cos^2\theta = 1$ ------------------- Q
$\Rightarrow \sin^2\theta = 1-\cos^2\theta$
$\Rightarrow \sin^2\theta = (1-\cos\theta)(1+\cos\theta)$
$\div \sin\theta(1-\cos\theta): \Rightarrow \frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$ if $\sin\theta \neq 0$, $\cos\theta \neq 1$ --- P
So Q $\Rightarrow$ P is valid. Q is true. So P is true.

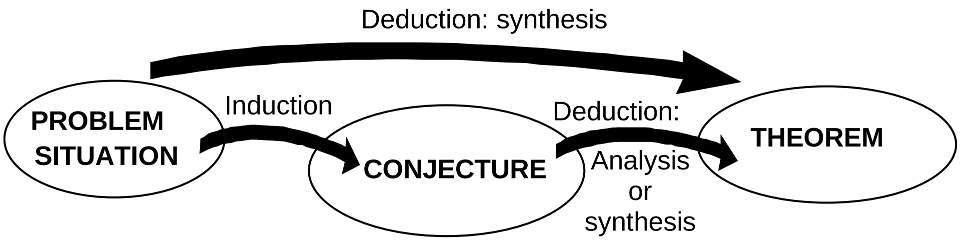
The proofs in C are completely valid – both use the Modus Ponens proof structure, i.e. $Q \Rightarrow P$ is valid, Q is true, so P is true. Do not be confused because our Ps and Qs are now reversed from the original formulation of the Modus Ponens structure – we used the same notation as in A, but reasoned backwards, that is why they are reversed! Both proofs are extremely beautiful and elegant. The point is that we "discovered" these elegant proofs through *analysis*, i.e. by reasoning *backwards* from what we had to prove, not by starting with what we had.

Using the Modus Ponens proof structure, i.e. logically reasoning from facts that we accept as true (or given information) towards conclusions, is called a *synthesis* or *synthetic reasoning*.

About the nature of mathematics

Our views on the nature and place of analysis and synthesis influences our view of the nature of mathematics. Or probably it is the other way around – our views on the nature of mathematics determines our view of the nature and place of analysis and synthesis in doing mathematics and in teaching and learning mathematics.

More generally, it is a question about the place of *informal* and *formal* mathematics in mathematics. Griffiths (1978) talks of *untidy and tidy mathematics*. The exposition in B – analysis – and induction are important aspects of "untidy mathematics". The exposition in C – synthesis – is an aspect of "tidy mathematics". The role of induction and deduction and the two faces of deduction (analysis and synthesis) are illustrated in the diagram below. Note that from the nature of analysis – to reason *backwards* from the result we want to prove – that analysis is only possible if the result is already known. To start from a problem and deduce a *new* result by deduction, we must necessarily use *synthetic* reasoning without the help of *analysis* – and this is often difficult, because we are working in the "dark" without direction.



The point we are making is that when doing mathematics, mathematicians use "untidy mathematics" such as induction and analysis to discover their results. But when it comes to *writing up* the results of a piece of mathematical work in books and journal articles, the *culture* in the community of mathematicians is to show only the tidy *product* C, while omitting both the untidy *process* of analysis in B underlying the method of the proof and the inductive reasoning through which the conjecture may have been discovered. This is well expressed by Richard Feynman in his Nobel Lecture in 1966:

We have a habit in writing articles published in scientific journals to make the work as finished as possible, to cover up all the tracks, to not worry about the blind alleys or describe how you had the wrong idea first, and so on. So there isn't any place to publish, in a dignified manner, what you actually did in order to get to do the work.

In this way many users of mathematics – people reading books and journals, but who have never really done any independent *creative* mathematics themselves, and this includes most mathematics teachers – develop the total misconception that C is the Mathematics, *and that the results were developed or discovered through the process of C (synthesis).* They are confusing the formal description of mathematics as a deductive system with the living, creative activity of mathematicians! Unfortunately school mathematics textbooks and many mathematics teachers, well-meaning as they may be, convey and perpetuate a distorted view of the true nature of mathematical activity. Their expectation that learners must then try to mimic this "sterile" formalist style in doing mathematics, without explicit knowledge of inductive and deductive reasoning (in particular analysis and logic) is a main contributory factor that mathematics continue to mystify most learners.[13]

Lakatos (1976, 142) uses the term "deductivist style" for synthesis and says:
In deductivist style, all propositions are true and all inferences valid. Mathematics is presented as an ever-increasing set of eternal, immutable truths. … Deductivist style hides the struggle, hides the adventure. The whole story vanishes, the successive tentative formulations of the theorem in the course of the proof-procedure are doomed to oblivion while the end result is exalted into sacred infallibility.

Schultze describes synthesis and analysis quite elegantly:
A synthesis shows that every step is true, but does not explain why this step was taken (and not another). A synthetic proof convinces the reader that the fact to be demonstrated is true, but it does not reveal to him the real plan of the demonstration, does not tell him why this sequence of arguments was selected. Proofs are not discovered by synthetic methods, and if forgotten, synthetic demonstrations are most difficult to reconstruct. But synthetic proofs are usually short and elegant, and are in place when no pedagogical conditions need to be considered.

An analysis, on the other hand, is lengthy and not elegant, but is the only method that accounts fully for each step in the demonstration. It is the only method by which students can hope to discover proof, or to re-discover then after forgotten. Analysis is the method of discovery, synthesis the method of concise and elegant presentation.

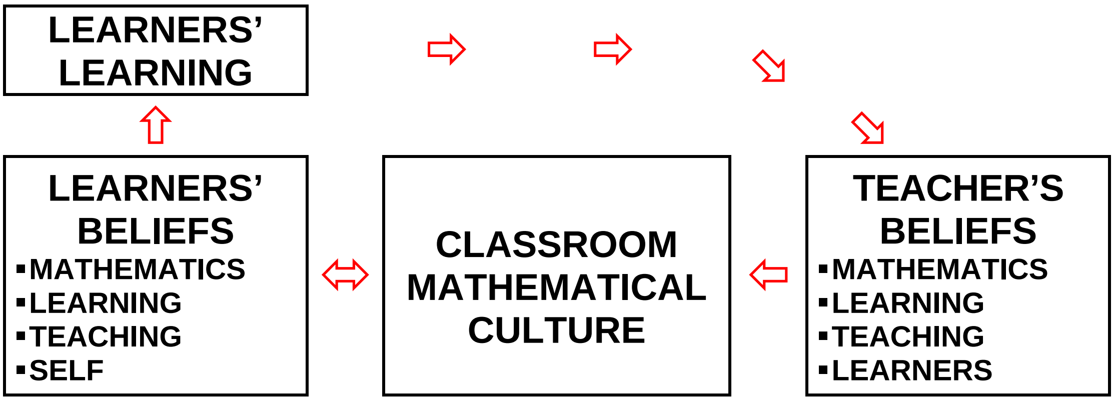
However, this description maybe gives the impression that analysis is "not the real thing", not real Mathematics, but that it could be used as a *teaching strategy*. Nothing could be further from the truth: Analysis, and in general "untidy mathematics" *is* Mathematics with a capital M, as Davis and Hersch (1981) emphasise:

The criticism of formalism in the high school has been primarily on pedagogic grounds: this is the wrong thing to teach, or the wrong way to teach. But all such arguments are inconclusive if they leave unquestioned the dogma that real mathematics is precisely formal derivations from stated axioms. If this philosophical dogma goes unchallenged, the critic of formalism in the school appears to be advocating a compromise in quality: he is a sort of pedagogic opportunist, who wants to offer the students less than the real thing. The issue then, is not, what is the best way to teach, but what is mathematics really all about. . . . Controversies about high school teaching cannot be resolved without confronting problems about the nature of mathematics.

[13] We emphasise again that the main objective of this course is exactly to try to give teachers the kind of creative experiences that will influence their perspective on the true nature of mathematical activity, and that this will change the atmosphere of their mathematics classroom away from repetitive routine reproduction towards mathematics as a problem solving process.

On teaching and learning

If teachers have an impoverished perception that mathematics is only the formal/tidy aspects of mathematics, they carry that false perception into the classroom and create an impoverished mathematics culture, that research show radically influences learners' beliefs and attitudes and determines *how* and *what* they learn. See Constructivism.



To give the impression that problems are (only) solved using synthetic reasoning like in C, puts learners in an impossible position: Learners can only try to fulfil the requirement through *memorisation*, *re-production*, *imitation* and *rote learning*. Then the classroom mathematical culture is characterised by *memorisation*, *re-production*, *imitation* and *rote learning*, making an *inquiry* attitude impossible and then it is impossible to achieve *any* of the wonderful Curriculum 2005 outcomes described in the introduction!

Take the following example of a problem and its synthetic proof. Make sure that you understand it before continuing!

$$\text{Prove that } \quad a^2 + b^2 \geq 2ab \ \text{ for } a, b \in R$$

$$\begin{aligned} \text{Proof : } & (a-b)^2 \geq 0 \quad \text{for } a, b \in R \text{ -------P} \\ & \Rightarrow a^2 - 2ab + b^2 \geq 0 \\ & \Rightarrow a^2 + b^2 \geq 2ab \text{ -------------Q} \end{aligned}$$

This is a beautiful, elegant and genial proof (do you agree?) using Modus Ponens: P $\Rightarrow$ Q is valid, P is true, so Q is true. However, the teacher "forgot" to show learners how she knew to begin with $(a - b)^2 \geq 0$!! Where did this come from?

Now think of your own understanding of the proof – did you maybe also feel, as Schultze says (see previous page), that you understand every step, but not "*why this step was taken, and not another*"? Were you able to construct this proof on your own? Or do you feel like many learners that you will never be able to ever know how to begin with something like $(a - b)^2 \geq 0$?

The only way for the learners and for the teacher to understand this genial proof, or to later re-construct it if one has "forgotten", or to produce such a proof yourself, is to first understand the *analysis* that underlies the synthesis:

Analysis (↓)

$$a^2 + b^2 \geq 2ab$$
$$\Rightarrow\ a^2 - 2ab + b^2 \geq 0$$
$$\Rightarrow\ (a-b)^2 \geq 0 \quad \text{provided } a, b \in R \quad (\text{remember, } (\sqrt{-1})^2 = -1)$$

(↑) **Synthesis**

Of course, everyone should know that the analysis is not an acceptable proof (beware of the Converse trap!). But now learners and the teacher can easily write down the genial synthetic proof *from bottom to top*. Such an "open" approach should contribute to demystifying the mystique of mathematics and mathematicians and remove feelings of incompetence. *The good teacher therefore shows learners the analysis! Then B, just as much as C, becomes part of the mathematical culture in the classroom!*

PROBLEM 65:

Prove: If $\frac{a}{b} = \frac{c}{d}$ then $\frac{ac+2b^2}{bc} = \frac{c^2+2bd}{dc}$

> You should *really* not read the solution until you have spent some time doing it *yourself!*
>
> *The idea is that you should try to solve it using analysis. Of course you can solve it using the LHS-RHS scheme. Maybe you should (also) solve it like that and reflect on your thinking and compare it with analysis …*

Now try to understand this beautiful *synthetic proof*:

$$\frac{a}{b} = \frac{c}{d}$$
$$+\frac{2b}{c}: \Rightarrow \frac{a}{b} + \frac{2b}{c} = \frac{c}{d} + \frac{2b}{c}$$
$$\Rightarrow \frac{ac+2b^2}{bc} = \frac{c^2+2bd}{dc}$$

The *analysis* underlying the proof is:

We can prove that $\frac{ac+2b^2}{bc} = \frac{c^2+2bd}{dc}$

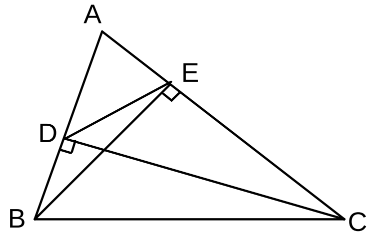
if we can prove that $\frac{a}{b} + \frac{2b}{c} = \frac{c}{d} + \frac{2b}{c}$ (division is right - distributive over addition)

i.e. if we can prove that $\frac{a}{b} = \frac{c}{d}$ $(-\frac{2b}{c}$ both sides)

But $\frac{a}{b} = \frac{c}{d}$ is given!

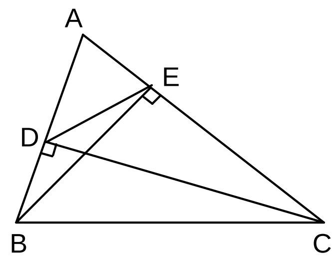
PROBLEM 66:

Prove in the sketch that DE.AC = BC.AD



Try to solve the problem before continuing reading!
Reflect on the reasoning *process*.

Here is a "performance" (the *synthesis*):



$\angle$ BDC = $\angle$ BEC = 90^0 (given)

$\Rightarrow$ DBCE is a cyclic quadrilateral (angles on the same chord)

(1) $\angle$ A = $\angle$ A

(2) $\angle$ AED = $\angle$ DBC (external angle of cyclic quadrilateral DBCE)

$\Rightarrow$ Δ ADE is equi-angular to ΔACB (sum of internal angles is 180^0)

$\Rightarrow$ Δ ADE /// Δ ACB (equi-angular Δ*s* are similar)

$$\Rightarrow \frac{DE}{CB} = \frac{AD}{AC}$$

$\Rightarrow$ DE.AC = BC.AD

This is a beautiful synthetic proof. But it omits *the thinking behind the proof*, the *analysis.* It also says nothing about wrong and aborted attempts that the teacher tried last night when preparing for the lesson. If the teacher then tomorrow impress the learners with a genial synthetic proof, he is mathematically and pedagogically dishonest! One can make hundreds of valid statements about the figure (Interior angles of a triangle is 180^0, vertically opposite angles are equal, external angle of a triangle ...). But the poor learners do not understand how the teacher knows to use *these specific statements* (theorems) and not others. They may understand each "step" in the proof, but they do not understand *why* the teacher chose this step and not another, and they do not understand where this is all going – they do not understand the *plan!* The famous mathematician Poincaré also emphasises that if one does not see the total *plan*, it leads to a feeling of not understanding:

To understand the demonstration of a theorem, is that to examine successively each of the syllogisms[14] composing it and to ascertain its correctness, its conformity to the rules of the game? For the majority [of people], no. Almost all are more exacting; they wish to know not merely whether all the syllogisms of a demonstration are correct, but why they link together in this order rather than another. In so far as to them they seem engendered by caprice and not by an intelligence always conscious of the end to be attained, they do not believe they understand.

Learners cannot construct such synthetic proofs if we do not also teach them the underlying *analytical reasoning*:

How can we prove that DE.AC = BC.AD?

If we can prove that $\frac{DE}{CB} = \frac{AD}{AC}$

and we can prove this if we can prove that ΔADE///Δ ACB

and we can prove this if we can prove that ΔADE is equi-angular to ΔACB

and we can prove this if we can prove that two angles in the triangles are equal

We know that (1) $\angle$A = $\angle$A

We can say (2) $\angle$AED = $\angle$DBC

if we can prove that DBCE is a cyclic quadrilateral (AED is external and DBC an internal angle)

We can prove that DBCE is a cyclic quadrilateral if we can prove angles on the same chord equal

But we know $\angle$BDC = $\angle$BEC = 90^0 (given)

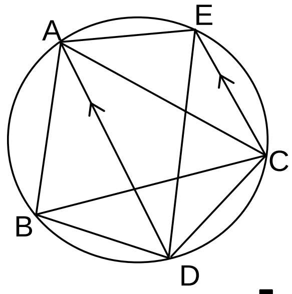
---

[14] A syllogism, also known as a rule of inference, is a formal logical scheme used to draw a conclusion from a set of premises. An example of a syllogism is Modus Ponens. The Greek "sullogismos" means "deduction".

Through analysis the *plan* and the necessary individual *steps* is now clear and transparent for everyone to see and understand. We can now write down the synthetic proof by reasoning backwards, from the bottom to the top. Even if the teacher can write down a synthetic proof directly, the good teacher will *act* and *pretend* that he cannot, and take learners with him through the whole analysis and the synthesis following it.

One of the best teaching strategies to develop analytical thinking is to let learners write down proofs *from the bottom to the top*: when one has finished the analysis (writing from bottom to top), then the (from top to bottom) is automatically finished! Learners should therefore be able to write down the last step first, then the second last step, etc. Let's illustrate with the following example:

PROBLEM 67:
In the sketch AD // EC and $\angle BAD = \angle DAC$.
Prove that AB // ED



The last step is therefore AB//ED:

??

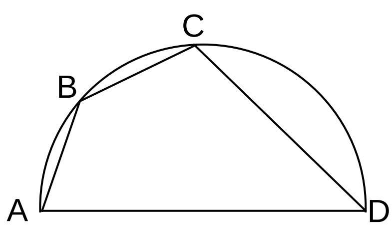
ANALYSIS ↑ $\Rightarrow \angle BAD = \angle ADE$ ↓ ***SYNTHESIS***

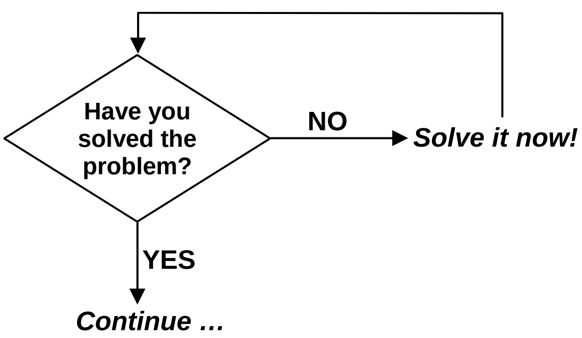
$\Rightarrow$ AB // ED (Alternate angles are equal)

The previous step – the premise for the conclusion that AB//ED – can be that alternate angles are equal, or corresponding angles are equal, or co-interior angles are supplementary, or that ABDE is a trapezium, or … Choose one, e.g. that alternate angles are equal. Then the previous step is to find a premise leading to the conclusion that these (alternate) angles are equal, etc. Complete the proof …

Whether you try to physically write proofs "bottom-up" or not, the fact is that we can only conceive the *plan* for a proof through analysis, and that means "reasoning backwards"! For each statement Q we need to prove, we need to find a premise P so that we can say P $\Rightarrow$ Q. But we must first prove P, so we need a premise M so that we can say M $\Rightarrow$ P, etc.

PROBLEM 68:
AD = 4 cm is the diameter of a circle. AB = BC = 1 cm.
Find the length of CD.



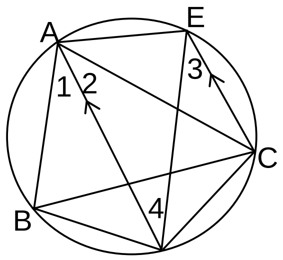


PROBLEM 67 Solution:

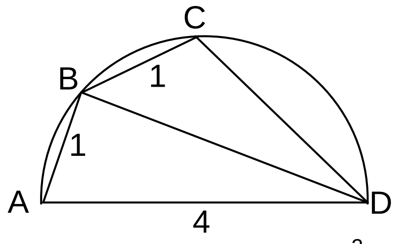
$\angle 1 = \angle 2 = \angle 3 = \angle 4$ ↑ **Analysis** **Synthesis** ↓

$\Rightarrow \angle 1 = \angle 4$

$\Rightarrow$ AB // ED (Alternate angles are equal)



PROBLEM 68 Solution:



What resources (knowledge) is applicable? There can be many approaches. We will use the cosine formula in ΔBCD. Substituting the given value BC = 1, we have:

$$BD^2 = BC^2 + CD^2 - 2BC.CD.\cos C$$

$$\Rightarrow BD^2 = 1 + CD^2 - 2CD\cos C \text{ .... } (1)$$

It is clear that to solve for CD, we first need to find the values of the other unknowns, i.e. BD^2 and cos C. These then become *sub-goals* and give direction to our activity. Without such direction, we can try all kinds of directions in a wild goose chase that may lead nowhere!

Sub-goal 1: To find the length of BD:
What resources can we bring to bear on the problem?
We know that AD is a diameter of the circle, so we can use the general theorem *If we have a diameter of a circle, then the diameter subtends a right angle.* But we know AD is the diameter, therefore we can conclude that $\angle ABD = 90^0$.

This means that we can apply the Theorem of Pythagoras: If $\angle ABD = 90^0$, then $AD^2 = AB^2 + BD^2$. But we know that $\angle ABD = 90^0$. Therefore:

$$AD^2 = AB^2 + BD^2$$

$$\Rightarrow 4^2 = 1^2 + BD^2$$

$$\Rightarrow BD^2 = 15\text{ cm} \text{ .... } (2)$$

Sub-goal 2: To find the value of cos C:
If you have a cyclic quadrilateral, then the sum of opposite interior angles is 180^0.
But ABCD is a cyclic quadrilateral by definition (the four corners lie on the circle).

$$\Rightarrow \angle A + \angle C = 180° \quad \Rightarrow \angle C = 180° - \angle A$$

$$\Rightarrow \cos C = \cos(180 - A) = -\cos A = -\frac{1}{4} \text{ …...... } (3)$$

Now we simply have to substitute equations 2 and 3 into 1, and we are done:

$$15 = 1 + CD^2 + 2CD.\frac{1}{4}$$

Well almost! Again, it is now recognising the *structure* or form of the equation to recall a relevant knowledge structure. The CD^2 should *trigger* that it is a *quadratic equation*. If we now recall our “quadratic equation schema”, we know we must write the equation in the standard form, factorise, etc.:

$$2CD^2 + CD - 28 = 0$$

$$\Rightarrow (2CD - 7)(CD + 4) = 0$$

$$\Rightarrow CD = -4 \text{ or } CD = 3,5$$

$$\Rightarrow CD = 3,5\text{ cm}$$

Reflect for a moment! Does the answer make sense in the situation? Well yes, it is shorter than 4 cm, as it should be, because the diameter is the longest chord in a circle!

PROBLEM 69:

Prove that $\dfrac{-b-\sqrt{b^2-4ac}}{2a} = \dfrac{2c}{-b+\sqrt{b^2-4ac}}$

PROBLEM 70:

(a) Prove:

$m^3 - m$ is divisible by 3 for all $m \in$ N

$m^5 - m$ is divisible by 5 for all $m \in$ N

(b) For which $n \in$ N is $m^n - m$ divisible by n?

Be sure to try to solve it – several times – before continuing reading!

To make sure that we understand the problem, and believe that it is true, we can *specialise.* In the case of (a) we have

m	$m^3 - m$	Divisible by 3?
1	1 – 1 = 0	Yes
2	8 – 2 = 6	Yes
3	27 – 3 = 24	Yes
4	64 – 4 = 60	Yes

It certainly looks like the answer is divisible by 3. In fact, one could probably make a stronger conjecture: the result is always divisible by 6!

But of course we are aware of the dangers of generalising through incomplete inductive reasoning – to *prove* that the answer is *always* divisible by 3 (or 6) for *all* values of *m*, we will have to construct a general deductive argument. Also, our numerical examples give us no indication of *why* the answer should be a multiple of 3 – only an analysis of the *structure* of the situation, i.e. general deductive reasoning will explain the nature of the result.

Now that we have made sure that we understand the problem (Polya's phase 1), we must make a plan (Polya's phase 2)! How will we know if a number is divisible by 3? We need an implication of the type P $\Rightarrow$ Q, where Q is the statement "the number is divisible by 3". What can P (the premise) be? We need to recall some relevant knowledge. Perhaps you know the divisibility test for 3:

If the sum of the digits of a number is divisible by 3, then the number is divisible by 3.

PROBLEM 71:

Prove the divisibility test for 3, i.e. a number is divisible by 3 iff[15] the sum of the digits is divisible by 3.

[15] "iff" is mathematical notation for "if and only if", which is a two-way implication P $\Leftrightarrow$ Q

So if we can prove that in any number of the form $m^3 - m$ the sum of the digits is divisible by 3, then we can conclude that these numbers are divisible by 3. It may be possible to prove it in this way, but I see no entry, so I abort the approach. *Control!* ☺

What other P (premise) could be useful? Well, if all else fails, let's manipulate the expression and see what happens!

$$\begin{aligned} m^3 - m &= m(m^2 - 1) \\ &= m(m+1)(m-1) \\ &= (m-1)m(m+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1) \end{aligned}$$

Can we conclude anything from this *structure*? Well, we can recognise that (m – 1), m and (m+1) are three *consecutive* natural numbers. What can we say about three consecutive numbers? Would you agree with the statement: "given any three consecutive natural numbers, one of them is divisible by 3"?

PROBLEM 72:

Convince yourself that given *any* three consecutive natural numbers, one of them is divisible by 3. See this Excel worksheet.

The mathematical mind will wonder if the structure "**three** … divisible by **3**" is general! Is here a more general pattern?

Given **two** consecutive natural numbers, is one always divisible by **2**?

Given **four, five, six** … consecutive whole numbers, is one always divisible by **4, 5, 6, …**?

Can you convince yourself that in any ***n*** consecutive whole numbers one of the numbers must be divisible by ***n***?

If we accept the statement that in any n consecutive natural numbers one of the numbers must be divisible by n, we have solved Problem 70 (a): $m^3 - m$ can always be written as the product of three consecutive numbers (see (1) above), so one of them is divisible by 3, so $m^3 - m$ is divisible by 3. Put differently: 3 is a factor of the right-hand side of (1), therefore it is a factor of the left-hand side of (1) and that proves the conjecture.

We can also prove that $m^3 - m$ is divisible by 6 by looking at the structure of (1) with different eyes, i.e. through the lens of even and odd numbers: m is either even or odd. If m is even, then $m^3 - m$ is divisible by 2. Do you agree? So $m^3 - m$ is divisible by 2 and by 3, therefore it is divisible by 6. Are you convinced of this argument ("*If* a number is divisible by 2 and by 3, *then* it is divisible by $2 \times 3 = 6$")? If m is odd, then both $m - 1$ (the previous number) and $m + 1$ (the next number) are even and divisible by 2. Therefore $m^3 - m$ is divisible by 2 and by 3, therefore it is divisible by 6.

PROBLEM 73:

Prove that if m is odd, then $m^3 - m$ is divisible by 24.

There are other possibilities to solve Problem 70. For example, just as we know that any whole number is either even or odd, we also know that if we divide any whole number by 3, the remainder is either 0 or 1 or 2. This is merely a different, and stronger formulation of the statement that in any three consecutive whole numbers one is divisible by 3.

PROBLEM 74:
Convince yourself of the following. See this Excel worksheet.
Any natural number is either even or odd. So any natural number can be written as $2n$ or $2n + 1$, $n \in N_0$, i.e. if you divide a natural number by 2, the remainder is 0 or 1.
If you divide any natural number by 3, the remainder is 0, 1 or 2, so any natural number can be expressed as $3n$ or $3n + 1$, $3n + 2$, $n \in N_0$ (for a given value of n, $3n$, $3n + 1$ and $3n + 2$ are three *consecutive* whole numbers).
Generalise …

If we accept the statement above as true, we can solve Problem 70 (a) with the following reasoning: m can be expressed as either $3n$ or $3n + 1$ or $3n + 2$, $n \in N_0$. In the first case we have $m^3 - m = (3n)^3 - 3n = 27n^3 - 3n = 3n(9n^2 - 1)$, which is clearly divisible by 3. Note that we are now using a different general theorem as the basis for our reasoning, namely "if a number can be expressed as $3k$, $k \in N_0$, then the number is divisible by 3." Our proof is therefore to show that the special case $m^3 - m$ can be written as $3k$, $k \in N_0$ (P), which implies that $m^3 - m$ is divisible by 3 (Q).

In the second case, if m can be expressed as $3n + 1$, we have

$m^3 - m$
$= (3n + 1)^3 - (3n + 1)$
$= 27n^3 + 27n^2 + 9n + 1 - 3n - 1$ (refer to the use of Pascal's triangle!)
$= 3n(9n^2 + 9n + 2)$
$= 3k$ where $k = n(9n^2 + 9n + 2)$ is a whole number

So $m^3 - m$ can be expressed as $3k$, $k \in N_0$, 3k is divisible by 3, so $m^3 - m$ is divisible by 3.

PROBLEM 75:
Complete the above proof by analysing the case if m can be expressed as $3n + 2$, $n \in N_0$.

The purpose of the above was to demonstrate that in order to prove a statement Q through Modus Ponens, we need to find a relevant premise P and an argument to show that P implies Q. The statement P and the implication P $\Rightarrow$ Q usually are facts or theorems we already know and accept as true. If not, we first have to prove them as *sub-problems*. In our case we have considered three general possibilities:

- If the sum of the digits of a number is divisible by 3 (P), then the number is divisible by 3 (Q).
- If we have three consecutive numbers(P), one of them is divisible by 3 (Q).
- If we can express a number as $3k$, $k \in N_0$ (P), then the number is divisible by 3 (Q).

To prove our specific conjecture that $m^3 - m$ is divisible by 3 (Q), we had to prove

- that the sum of the digits of $m^3 - m$ is a multiple of 3 (P), or
- that $m^3 - m$ can be written as the product of three consecutive natural numbers (P), or
- that $m^3 - m$ can be written as $3k$, $k \in N_0$ (P).

It is very important that we understand the general *structure* of proof through Modus Ponens. To apply the structure to prove a specific conjecture Q, it is very important that we identify appropriate relevant premises (P) based on some known theorem P $\Rightarrow$ Q. Let us further illustrate this important know-how with reference to some school problems.

A few school problems

In this course, in order to develop your perspectives, our problems are not always at the school level, otherwise there is the danger that we will do it automatically or mechanically without reflection and then not learn from it. But it is just as important to use our new perspectives to *look at school-level problems with new eyes!*

You may find school-level problems easy and our course problems difficult and frustrating. Apart from developing our problem solving know-how, it has the added advantage that we can appreciate the fact that our learners experience their school problems just as difficult and frustrating as we experience these difficult unknown problems! Like us, their problem usually is that they do not know what is “the next step” or "what to do now" or “where to start”. We believe that we will be able to help our learners better with their difficulties if we understand and teach the structure of proof through Modus Ponens. So let’s look very briefly at a few school problems.

PROBLEM 76:

Prove that $px + qx = m - mx^2$ has real roots for all p, q, $m \in$ R.

We may find problems such as these easy and solve it routinely, mainly because we have previously done many similar problems. However, for our learners these problems are a challenge, and they often have no direction, e.g. many learners in this problem typically try to solve for x. We must help them to understand the proof structure: We need an implication P (?) $\Rightarrow$ Q (the roots are equal). We get the required premise from a well-known theorem: “If $\Delta \geq 0$ (P), then the quadratic equation has real roots (Q)”. This is the same as the point made earlier that *analysis* is thinking backward, and we can even think of it as writing down the last step first:

$$\begin{aligned} &\Delta \geq 0 \\ &\Rightarrow \text{the roots are real} \end{aligned} \quad \uparrow$$

So we cannot directly prove that the roots are real. We must prove something else, namely our premise P that $\Delta \geq 0$, then we can say that P $\Rightarrow$ Q. So our problem boils down to proving that $\Delta \geq 0$ *for this equation*. Knowing where we want to end, tells us where to start: First we need Δ; we know $\Delta = b^2 - 4ac$; so we need a, b and c; a, b and c are the coefficients of the quadratic equation $ax^2 + bx + c = 0$; so first we need to write our equation in the standard form, then find the value of Δ:

$$\begin{aligned} &px + qx = m - mx^2 \\ \Leftrightarrow\ &mx^2 + (p+q)x - m = 0 \\ &\text{So } \Delta = (p+q)^2 + 4m^2 \end{aligned} \quad \downarrow$$

$$\begin{aligned} &\Delta \geq 0 \\ &\Rightarrow \text{the roots are real} \end{aligned} \quad \uparrow$$

This looks promising, but what to do next? We can manipulate and see what happens, but if we do not know what to look for, we will probably not realise it when we get there. Let's remember what we are trying to do: we want to prove that $\Delta \geq 0$. How will we manage that? Again we need some premise. Where previously we had "If $\Delta \geq 0$ (P), then the roots are real (Q)", what we need now is "If ?? (P), *then* $\Delta \geq 0$ (Q)". The fact we need is that $k^2 \geq 0$ if k is a real number. Now we can continue:

$$px + qx = m - mx^2$$
$$\Leftrightarrow mx^2 + (p+q)x - m = 0$$
$$\text{So } \Delta = (p+q)^2 + 4m^2$$
$$(p+q)^2 \geq 0 \text{ and } 4m^2 \geq 0 \text{ if } p, q, m \in R$$
$$\Rightarrow \Delta \geq 0$$
$$\Rightarrow \text{the roots are real}$$

PROBLEM 77:
Solve for x: $\log(x-3) + \log(x-2) = 2\log x$

We should think what implication P $\Rightarrow$ Q will enable us to find the solution. The required implication is log A = log B $\Rightarrow$ A = B. In order to be able to apply this general structure we must therefore write this specific equation in the form log A = log B, which is usually taught as "write separate logs as one log". In order to do that we must call on two properties of logs, namely log A + log B = log AB and nlog A = log A^n. We leave the rest of the solution.

PROBLEM 78:
Solve for x: $9^{x-1} \times 3^{x+1} = \frac{1}{27}$

We should think what implication P $\Rightarrow$ Q will enable us to find the solution. The required implication is $a^x = a^y \Rightarrow x = y$ ($a \neq 0$, $a \neq \pm 1$). In order to be able to apply this general structure we must therefore write this specific equation in the form $a^x = a^y$, which is usually taught as "make the bases the same". In order to do that we must call on the properties of exponents. We leave the rest of the solution.

The point made here is that "write this equation in the form log A = log B" and "write separate logs as one log" may seem the same, but they are not. What is missing in our teaching and in children's understanding is the underlying logic of Modus Ponens, of understanding that each step in our mathematical work is based on using some implication P $\Rightarrow$ Q, that we should understand the logic and that we should understand that a large part of doing mathematics is identifying the appropriate P in order to can deduce P. The logic above was *not* "write in the form log A = log B" ("write as one log"), but "log A = log B $\Rightarrow$ A = B"!

PROBLEM 70 continued:
Prove that $m^5 - m$ is divisible by 5 for all $m \in N$.

It was Rene Descartes who said: “Each problem that I solved became a rule which served afterwards to solve other problems”. This is the same perspective as Polya’s heuristic: “Have you seen it before? Have you seen the same problem in a slightly different form?” It is natural to here try the same approach we used for our $m^3 - m$ problem above.

Like in the $m^3 - m$ problem, we have three possibilities. We make a choice based on our intuition of which possibility holds most promise. So let us try the implication “If $m^5 - m$ can be expressed as the product of 5 consecutive natural numbers, then $m^5 - m$ is divisible by 5.” So our *plan* or *sub-problem* is to prove the premise that $m^5 - m$ can be expressed as the product of 5 consecutive natural numbers. Now to carry out the plan (Polya's phase 3). We will again put our hope in factorisation:

$$m^5 - m = m(m^4 - 1)$$
$$= m(m+1)(m-1)(m^2+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

This is very disappointing – we clearly have three consecutive numbers ($m - 1$, m and $m + 1$), but not five! What does this mean? It could mean that the statement is not true, so let’s check by specialising for a few numbers. Let us at the same time try to understand the underlying structure:

$m - 1$	m	$m + 1$	$m^2 + 1$	$(m - 1)m(m + 1)(m^2 + 1)$	Divisible by 5?
0	1	2	2	0	Yes
1	2	3	5	30	Yes
2	3	4	10	240	Yes
3	4	5	17	1020	Yes
4	5	6	25	3000	Yes

It certainly looks like the answer is divisible by 5. Of course, these few cases do not guarantee that it is always true – we need a deductive argument! But our numerical examples give us little indication of *why* the answer should be a multiple of 5. It seems that for every value of m, at least one of the four factors is divisible by 5. We may try to prove this conjecture, but how? I leave it for the moment …

PROBLEM 79:
Prove that if $m \in N$, then at least one of $m - 1$, m, $m + 1$ or $m^2 + 1$ is divisible by 5.
Why is at least one of $m - 1$, m, $m + 1$ or $m^2 + 1$ is divisible by 2?
Why is at least one of $m - 1$, m, $m + 1$ or $m^2 + 1$ is divisible by 3?
Deduce that $m^5 - m$ is divisible by 30 for $m \in N$.

I am rather going to persevere with trying to prove the premise that $m^5 - m$ can be written as the product of five consecutive whole numbers. We already have

$$m^5 - m = m(m^4 - 1)$$
$$= m(m+1)(m-1)(m^2+1) \text{..............................} (1)$$

Surely, if $m^5 - m$ is divisible by 5, we will expect that it can be expressed as the product of five consecutive whole numbers? I conjecture that it should be possible to end with

$$m^5 - m = (m-2)(m-1)m(m+1)(m+2) \text{ ....} (2)$$

So the question is, can we manipulate equation 1 to transform it into equation 2? Then the problem is solved. There does not seem much to do with equation 1, so let's work from equation 2 to equation 1, i.e. from the "bottom" to the "top":

$$m^5 - m = m(m+1)(m-1)(m^2+1) \text{ ....} (1)$$

$$= m(m-1)(m+1)(m^2-4) \text{..............................} (3)$$
$$= (m-2)(m-1)m(m+1)(m+2) \text{ ....} (2)$$

Now I grab at equation 1 and try to change 1 into 3, working from top to bottom, making an ingenious substitution:

$$m^5 - m = m(m^4 - 1)$$
$$= m(m+1)(m-1)(m^2+1) \text{ ....} (1)$$
$$= m(m+1)(m-1)[(m^2-4)+5] \text{ ....} (4)$$
$$= m(m+1)(m-1)(m^2-4) + 5m(m+1)(m-1) \text{ ...} (5)$$

$$= m(m-1)(m+1)(m^2-4) \text{ ....} (3)$$
$$= (m-2)(m-1)m(m+1)(m+2) \text{ ....} (2)$$

Now, suddenly, if I have not been expecting it, the cold realisation dawns that equation 5 definitely is not equivalent to equation 3, so I cannot complete the proof as I anticipated! But at the same time I have a deep insight, an *Aha Erlebnis!* Equation 5 has two terms, and each term is divisible by 5 (are you sure? Why?), therefore the whole expression is divisible by 5. Are you sure? Why? The reason is that "If in $\frac{a+b}{c}$, a is divisible by c and b is divisible by c, then a + b is divisible by c." Do you agree? Therefore $m^5 - m$ is divisible by 5. QED – we are finished.

But not quite – Polya says we must "look back" (phase 4)! One aspect of looking back is to try to generalise our result. It is clear that equation 5 is also divisible by 2 and by 3, and therefore $m^5 - m$ is divisible by $5 \times 2 \times 3 = 30$!

Another aspect of "looking back" is to find possible errors – we should always re-check our assumptions and each logical inference in our work. So we must wonder why I could not complete my original plan, namely to write $m^5 - m$ as the product of five consecutive numbers. The answer is that I made an error in reasoning! In fact, I made two errors in reasoning! My plan was based on the implication "If a number is divisible by 5, then it can be written as the product of five consecutive whole numbers". This is clearly false, as a quick specialisation shows: 5, 10, 15, 20, … are all multiples of 5, but cannot be written as the product of five consecutive whole numbers! The smallest multiple of 5 that can be written in this form is $1 \times 2 \times 3 \times 4 \times 5 = 60$. The statement is generally untrue, e.g. 4 is divisible by 2 but cannot be written as the product of 2 consecutive whole numbers (although 2 can), 9 is divisible by 3 but cannot be written as the product of 3 consecutive whole numbers (although 6 can). The source of my error is probably twofold: First, I expected my successful method of writing $m^3 - m$ as the product of three consecutive whole numbers to generalise to writing $m^5 - m$ as the product of five consecutive whole numbers, but it turns out that $m^3 - m$ is a *special case!* Second, I now realise that I have wrongly assumed that the converse of a theorem is also true! While it is true that "If a number can be written as the product of n consecutive whole numbers, then it is divisible by n", its converse "If a number is divisible by n, then it can be written as the product of n consecutive whole numbers", is false! Let's just say that *doing* mathematics makes you *humble!* Making mistakes is probably part and parcel of creatively work in mathematics – there are many examples in the history of mathematics. It was the philosopher Karl Popper who said: "We are fallible, and prone to error; but we can learn from our mistakes."

Another aspect of "looking back" is to investigate alternative solution methods. So let's revisit the following problem:

PROBLEM 80:
$m^5 - m = (m - 1)m(m + 1)(m^2 + 1)$
Prove that if $m \in \mathrm{N}$, then at least one of $m - 1$, m, $m + 1$ or $m^2 + 1$ is divisible by 5, and therefore that $m^5 - m$ is divisible by 5.

We start using the basic structure of consecutive numbers: $m - 1$, m, $m + 1$, $m + 2$ and $m + 3$ are five consecutive whole numbers, therefore one of them is divisible by 5. Now we must use this fact to solve Problem 70:
If one of $m - 1$, m or $m + 1$ is divisible by 5, then $m^5 - m$ is divisible by 5.
If none of $m - 1$, m or $m + 1$ is divisible by 5, then $m + 2$ or $m + 3$ is divisible by 5.
First take the case if $m + 2$ is divisible by 5. If $m + 2$ is divisible by 5, we can express it as $5k$, $k \in \mathrm{N}$. We will show that it implies that $m^2 + 1$ is divisible by 5, by showing that $m^2 + 1$ can be expressed as $5n$, $n \in \mathrm{N}$:

$m + 2 = 5k$, $k \in \mathrm{N}$
$\Leftrightarrow m = 5k - 2$
$\Rightarrow m^2 = (5k - 2)^2 = 25k^2 - 20k + 4$
$\Rightarrow m^2 + 1 = 25k^2 - 20k + 5 = 5(5k^2 - 4k + 1)$, which is divisible by 5.
$\Rightarrow m^5 - m$ is divisible by 5.

PROBLEM 80 (a):
Complete the proof for the case if none of $m - 1$, m, $m + 1$ or $m + 2$ is divisible by 5, but $m + 3$ is divisible by 5.

PROBLEM 70 (b) continued:
For which $n \in$ N is $m^n - m$ divisible by n?

We have proven the special cases that $m^n - m$ divisible by n for $n = 3$ and 5. For what other n will it be true? Most of us will probably conjecture that it is true for n an odd number. Certainly it is easy to check another special case: if $n = 1$, $m^1 - m = 0$ is divisible by 1. So we know that it is true for $n = 1$, 3 and 5 and this strengthens our conjecture about odd numbers. But we cannot be sure! Trying to prove it for $n = 7$ and 9 can be very difficult. Let's try for $n = 9$:

$$\begin{aligned} m^9 - m &= m(m^8 - 1) \\ &= m(m^4 - 1)(m^4 + 1) \\ &= m(m^2 - 1)(m^2 + 1)(m^4 + 1) \\ &= m(m - 1)(m + 1)(m^2 + 1)(m^4 + 1) \end{aligned}$$

This is good practice in factorisation but it is going nowhere! Another important aspect of mathematical know-how is that we should exercise *control* over our activity. We should know when not to go on a wild goose chase and when to abort an effort and rather try something else. Mathematicians never jump into doing hard work such as above before they do not at least *convince themselves psychologically that the conjecture is true*.

If you have to prove a theorem, do not rush. First of all, understand fully what the theorem says, try to see clearly what it means. Then check the theorem, it could be false. Examine the consequences, verify as many particular instances as are needed to convince yourself of the truth. When you have satisfied yourself that theorem is true, you can start proving it.

Polya, 1945

Psychological conviction comes mainly through specialisation and analogy. So before you jump in and get strangled in heavy manipulation that does not seem to lead anywhere, first check some special cases, using a calculator for the calculations, as shown here.

m	$\frac{m^7 - m}{7}$	$\frac{m^9 - m}{9}$
1	$\frac{0}{7} = 0$	$\frac{0}{9} = 0$
2	$\frac{126}{7} = 18$	$\frac{510}{9} = 56,66\ldots$
3	$\frac{2184}{7} = 312$	$\frac{19680}{9} = 2186,66\ldots$

These special cases provide *counter-examples* that the conjecture is *not true* for $n = 9$! So our conjecture that $m^n - m$ divisible by n for n odd is not true! The special cases do suggest that $m^n - m$ divisible by n for $n = 7$, so I believe it and I will be willing to invest time and energy to try to prove it! How will you adapt your conjecture?

We have not tested if $m^n - m$ divisible by n for n and even number. Special cases for $n = 2$ show that $m^2 - m$ is divisible by 2 and it is easily proven deductively: $m^2 - m = (m - 1)m$, where $(m - 1)$ and m are two consecutive whole numbers, so one of them is divisible by 2, because one is even and one is odd! What about other even numbers?

PROBLEM 70 (c):
Show that $m^n - m$ is not divisible by any even n except $n = 2$.

We now know that $m^n - m$ divisible by 2, 3, 5, 7, … Now finish Problem 70(b):

PROBLEM 70 (b):
For which $n \in$ N is $m^n - m$ divisible by n?

Equivalence

Let's now return to our original problems in A. We have shown that the use of one-way implications P $\Rightarrow$ Q was logically invalid as proof, and that the proof itself and the thinking behind the proof really depended on analytical reasoning, i.e. P $\Leftarrow$ Q. We therefore suggested re-writing synthetic proofs as the reverse of the analytical thinking and even suggested that we write from the bottom up!

But we can do better than that. We can combine the two one-way implications P $\Rightarrow$ Q and P $\Leftarrow$ Q in a two-way implication P $\Leftrightarrow$ Q where it is valid, and then our analysis (from top to bottom) will simultaneously be our synthesis (from bottom to top)!

If P is the statement $x = 2$ and Q the statement $2x = 4$, it is clear that P $\Rightarrow$ Q (i.e. we can deduce Q from P) and also P $\Leftarrow$ Q (i.e. we can deduce P from Q). In such a case we can write P $\Leftrightarrow$ Q. We say that P and Q are *logically equivalent* and it means that *we can replace the one with the other whenever we want*. As another example: if P is $(x + y)^2$ and Q is $x^2 + 2xy + y^2$ then P $\Rightarrow$ Q and P $\Leftarrow$ Q, so we can write P $\Leftrightarrow$ Q and we can interchange the one with the other as we wish. It does not matter which one of the two one uses, because you can always deduce the one from the other when needed.

However, we must be careful. For example, if P is the statement $x > 4$ and Q the statement $x > 2$, it is clear that P $\Rightarrow$ Q is valid (if a number is greater than 4, it is also greater than 2), but P $\Leftarrow$ Q is invalid (if a number is greater than 2, we cannot conclude that it is greater than 4!).

So let's now re-write our original proofs in A using *equivalence*:

$$\sin^2\theta + \cos^2\theta = 1 \text{ ---------------- P}$$

$$\Leftrightarrow \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = 1$$

$$\Leftrightarrow y^2 + x^2 = r^2 \text{ provided } r \neq 0 \text{ -------- Q}$$

$$\text{So Q} \Rightarrow \text{P is valid. Q is true. Therefore P is true.}$$

$$\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta} \text{ ------------ P}$$

$$\Leftrightarrow \sin^2\theta = (1-\cos\theta)(1+\cos\theta) \text{ and } \sin\theta \neq 0, \cos\theta \neq 1$$

$$\Leftrightarrow \sin^2\theta = 1 - \cos^2\theta$$

$$\Leftrightarrow \sin^2\theta + \cos^2\theta = 1 \text{ ------------- Q}$$

$$\text{So Q} \Rightarrow \text{P is valid. Q is true. Therefore P is true.}$$

(D) **EQUIVALENCE**

Analysis ↓ ↑ **Synthesis**

In each case, the *plan for the proof* was found through the *analysis* P $\Rightarrow$ Q, which of course is not valid as *proof*. However, the proof as we have now written it in D is completely valid from two perspectives:

- *Using the Modus Ponens proof structure:* P $\Leftarrow$ Q is valid, Q is true, therefore P is true. The validity of the proof therefore lies in the converse implications. Reading the proof from the bottom to the top represents our *synthesis*.
- *Using equivalence:* each statement is equivalent to the other (we can deduce each statement from the other, working top-down or bottom-up. Therefore P and Q are equivalent, P $\Leftrightarrow$ Q, which also means that the truth of the one implies the truth of the other. So again: Q is true, therefore P is true!

I trust that you thoroughly understand the difference between the invalid statement "Q is true, so P is true" in A and the valid statement "Q is true, so P is true" in D!

To end

I do not claim that all learners will necessarily excel in mathematics if we teach *analysis*. Mathematical problem solving is complex. For example, Schoenfeld[16] distinguishes the following factors influencing learners' mathematical problem solving:

- "*Resources*": mathematical knowledge that the individual knows, can recall and can use in the situation.
- *Heuristics*: problem solving strategies (strategic knowledge) like make a sketch, introduce notations, investigate special cases, make a table, find a pattern, ...
- *Control*: the selection and monitoring of knowledge and heuristics – to know how long to persist with a line of reasoning and when to abort and try something else.
- *"Belief systems*": perspectives on the nature of mathematics, the learning of mathematics and the own ability …

What I do claim is that:

- We have to teach the logic of proof thoroughly, otherwise learners continually make mistakes like the Converse trap.
- Explicit (conscious) knowledge and understanding of *analysis* is an essential heuristic for successful mathematical problem solving.
- The logic of proof and analysis are not explicitly mentioned in the curriculum, so it is not taught, thus *depriving* learners of the *opportunity* to be creative and to learn mathematics as a constructive process. We do not provide them with the necessary tools of the trade!
- The perception that mathematics is only, or really, "tidy mathematics", undermines and sabotages any effort at independent problem solving.
- The previous two points are main contributory factors to learners' lack of confidence and expertise in mathematical problem solving!
- The culture in the community of mathematicians[17] to publicly show only the formal, "tidy" aspect of mathematics, contributes to and entrenches the misconception about the nature of mathematical activity among learners, teachers and the public.

PROBLEM 81:

Prove that $\frac{a+b}{2} \geq \sqrt{ab}$, a, b are positive real numbers.

PROBLEM 82:

If we add any fraction and its reciprocal (e.g. $\frac{5}{6} + \frac{6}{5}$), what is the smallest possible value of the sum? Supply a general proof or explanation of your conclusion!

PROBLEM 83:

Prove that the logarithms of the terms of a geometric sequence form an arithmetic sequence.

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[16] Schoenfeld, A. (1985): *Mathematical problem solving* is essential reading! See also online: http://gse.berkeley.edu/faculty/ahschoenfeld/Schoenfeld_MathThinking.pdf

[17] There are exceptions – mathematicians that can write describe their thinking from a meta-perspective, e.g. Poincaré, Hadamard, Polya, Schoenfeld, Davis & Hersch, …

PROBLEM 84:

Use the method of equivalence to prove that $\frac{1+\tan\theta}{1-\tan\theta}=\frac{\cot\theta+1}{\cot\theta-1}$.

Discuss the *logic* of the proof.
(You are not asked to use the LHS-RHS proof structure!)

PROBLEM 85: SCHOOL PROBLEMS

Take several trigonometry identities from your school mathematics textbook and prove the identities using the method of equivalence. Compare it with the usual, LHS-RHS-separately method. Which is easier? Why?

PROBLEM 86:

Prove that for all a, b, c, $d \in$ R:

$$a^2 + b^2 + c^2 + d^2 = ab + bc + cd + da \quad \Rightarrow \quad a = b = c = d$$

PROBLEM 87:

Prove that $f(n) = n^5 - 5n^3 + 4n$ is divisible by 120 for all $n \in$ N.

PROBLEM 88:

Prove that $2n^3 + 3n^2 + n$ is a multiple of 6 for all $n \in$ N.

PROBLEM 89:

Prove that $n(n^2 - 1)(3n + 2)$ is a multiple of 24 for all $n \in$ N.

PROBLEM 90:

If we divide any whole number by 3, the possible remainders are 0, 1 or 2.
If we square a whole number and then divide by 3, what are the possible remainders? Check your answer and give a general proof.
If we square a whole number and then divide by 5, what are the possible remainders? Check your answer and give a general proof.

PROBLEM 91: AVERAGES

Check if you agree: The average of the four consecutive numbers 1, 2, 3, 4 is $2\frac{1}{2}$ and the average of the first and last number is also $2\frac{1}{2}$.

Is this a coincidence or is it true for *any* four consecutive numbers? Why?

What if not? What if it was not *four* numbers, but three, or five, or six, … or *n*?
What if not? What if it was not consecutive *natural* numbers, but consecutive *odd* numbers, or consecutive *even* numbers, or …?

PROBLEM 92:

P is a point inside equilateral ∆ABC. P is 3 cm from AB, 4 cm from AC and 5 cm from BC. Find the length of a side of ∆ABC.

PROBLEM 93: SUM = PRODUCT?

Check, continue and explain:

1. $4 + \frac{4}{3} = 4 \times \frac{4}{3}$

   $5 + \frac{5}{4} = 5 \times \frac{5}{4}$

   $6 + \frac{6}{5} = 6 \times \frac{6}{5}$

2. $4 + \frac{4}{3} + \frac{16}{13} = 4 \times \frac{4}{3} \times \frac{16}{13}$

   $5 + \frac{5}{4} + \frac{25}{21} = 5 \times \frac{5}{4} \times \frac{25}{21}$

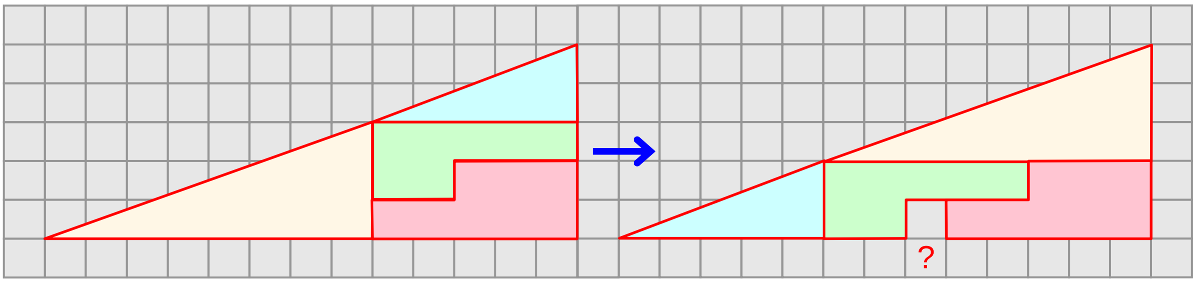
   $6 + \frac{6}{5} + \frac{36}{31} = 6 \times \frac{6}{5} \times \frac{36}{31}$

Note: Only a foolhardy person would want to generalise and say that multiplying a fraction is the same as adding the fraction! One counter-example will disprove the conjecture. But situations like this nevertheless are *sometimes* true. The mathematical mind will always want to know

- *Why* are these special cases true? (*Explain* the structure.)
- Exactly *when* is it true? (Find *all* the cases for which it is true, i.e. *generalise.*)

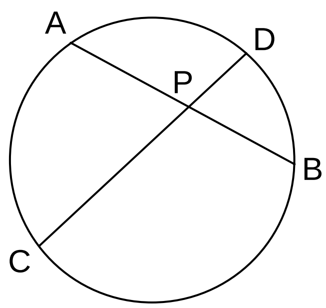
PROBLEM 94: THE MISSING CM^2

The four figures are arranged in a triangle below left, and then re-arranged as shown to the right. Where does this gap come from?



Click here for an interactive version of the problem.

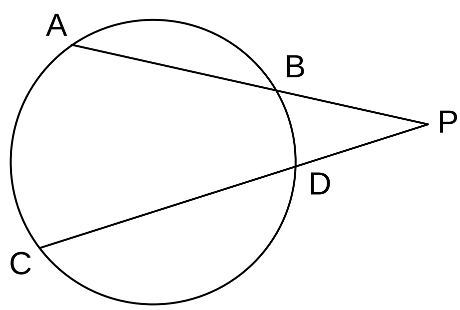
PROBLEM 95: CHORDS, SECANTS AND TANGENTS



1. ABCD is a cyclic quadrilateral with AB and CD intersecting at P.
   Prove that PA.PB = PC.PD

Click here for an interactive applet.

2. Specialise by checking the nature of the result when:
   - P is the centre of the circle.
   - AP = CP and PB = PD (when AC // DB).



3. Generalise by proving that the relationship in 1 applies also when AB and CD intersect *outside* the circle. Why is it a generalisation?

4. Now specialise by taking the following limiting cases (i.e. deduce the result directly from the relationship in 3):
   - Secant PBA becomes a tangent to the circle at B
   - Secant PDC *also* becomes a tangent at D.

   Show that you can deduce that PA = PB.

PROBLEM 96: EVERY TRIANGLE IS ISOSCELES

Analyse the logic of this proof ....

Theorem: **Any triangle is isosceles**

Construction: Let ABC be *any* triangle. Bisect angle A to meet the perpendicular bisector of BC in O. Draw perpendiculars from O to meet the (extentions of the) sides of $\triangle$ABC in P, Q and R.

Proof:

If O is *inside* the triangle, as shown in Figure 1:

$\Delta OBR \equiv \Delta OCR$ ...... $s, \angle, s$

$\Rightarrow OB = OC$ .... (1)

$\Delta APO \equiv \Delta AQO$ ...... $\angle, \angle, s$

$\Rightarrow OP = OQ$ .... (2)

$\Rightarrow AP = AQ$ .... (3)

$\Delta OPB \equiv \Delta OQC$ ...... $90°$, hyp, s: from (1) and (2)

$\Rightarrow PB = QC$ .... (4)

$(3)+(4): AP + PB = AQ + QC$

$\Rightarrow AB = AC$, i.e. ΔABC is isoceles.

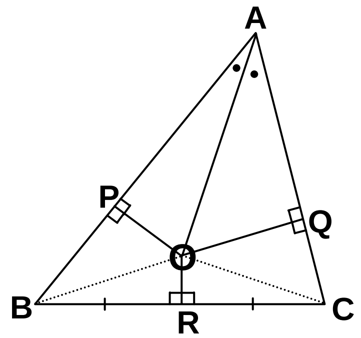


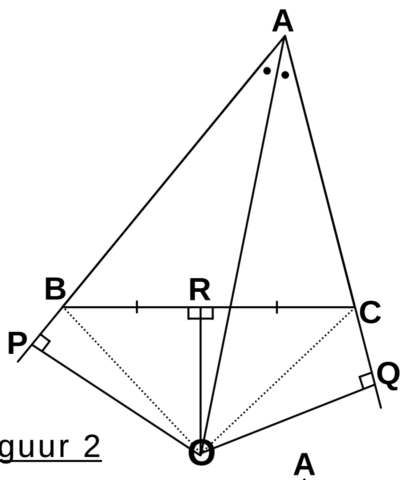
Figure 1

As O *buite* die driehoek lê, soos in Figuur 2:

Dan geld (1), (2), (3) en (4) presies soos hierbo.
Neem nou (3) – (4):

$AP - PB = AQ - QC$

$\Rightarrow AB = AC$, d.i. ΔABC is gelykbenig.



Figuur 2

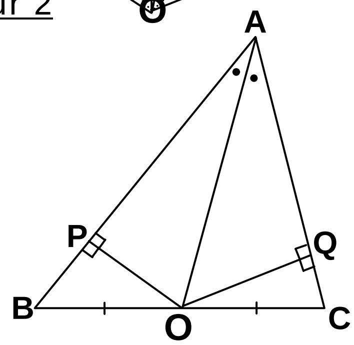
As O *op* BC lê (Figuur 3):

$OB = OC$ .... konstr. (1)

Die res van die bewys volg presies soos in Figuur 1:

$AP + PB = AQ + QC$

$\Rightarrow AB = AC$, d.i. ΔABC is gelykbenig.



Figuur 3

Dus, *enige* driehoek is gelykbenig.

JUMP TO 2.6 EQUATIONS