BOEING BOEING[1]

1. The activity

Page 1

A Boeing aircraft uses 2 700 ℓ of fuel for a flight of 200 km.

How much fuel will the aircraft use on a flight of
1. 400 km
2. 600 km
3. 700 km?

Page 2

Compare your answers against the following data supplied by Boeing.
If necessary, resolve any conflicts . . .

Flight distance (km)	200	300	400	500	600
Fuel used (ℓ)	2 700	3 800	4 900	6 000	7 100

Describe, in words, how the amount of fuel used varies as the flight distance varies.

Page 3

1. How much fuel will the aircraft use on a non-stop flight of
   (a) 700 km
   (b) 350 km
   (c) 480 km
   (d) 1 800 km?
2. What is the longest non-stop flight that the aircraft can undertake?
   Its fuel tanks can hold a maximum of 23 000 ℓ fuel.

2. Overview

This activity intends to make explicit some aspects of:

- *modelling* as a *mathematical process,* and
- the power of a *mathematical model* as a tool to *describe* and *analyse* the relationship between variables to obtain additional information about the situation.

The first page is not really intended as a “trap” for participants, but as a vehicle for them to reflect more deeply on the nature of the problem-situation and on their own thought processes, and to develop an empathy for children doing mathematics.

We have found that many participants assume that the variables are in direct proportion, i.e. tacitly assume the model $y = mx$ and use its *properties*. So the activity lends itself for reflection on the nature of modelling. After participants have reached consensus on page 1, don’t comment but go on to the second page *at a brisk pace*. The second page provides *additional data* that will confirm the doubts of people who did not want to use direct proportion, or it creates *conflict*, so that participants will have to review their “theory” of how the situation works. This page intentionally requires no calculations – people rush into calculations too soon! Rather this format is to have people *reflect* on the nature of the underlying model.

Given the additional information on page 2, participants will have no problem in now understanding the situation as “it uses 1100 litres for every 100 km plus 500 litres for the take off”. This serves as a new model that they can use to solve the questions on the last page.

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1 Brief notes written for presenters of a teacher development workshop at the level *teacher educator* → *teachers.* It also applies to the level *teacher* → *pupils.*

Note:
In previous workshops, we have found that teachers now criticise the problem statement - that is OK, there is place for that. But actually they are side-stepping the issue and looking for excuses why they made a mistake!! *Help them to accept that they have made a mistake!* They chose the wrong model! It does not mean they are stupid! Let's now learn from it – that is part of the culture of learning we want to create in our classrooms! If they now reflect about why they made the mistake, they can learn much about the effect of classroom culture, about the nature of mathematical modelling, and about children's dilemmas when solving problems – pupils are exactly in this same position when we pose "rule of three" problems (given three values, find the fourth unknown). For example, take this problem:

> *A tree casts a shadow 9 m long. At the same time a pole 3 m high casts a shadow 2 m long. How high is the tree?*

How should Grade 7 and 8 pupils know that it is a proportional situation?? Many, many pupils will "add" (i.e. their answer is 10 m), using the *wrong model*. Why?? One possible explanation is that they simply use an *inductive* strategy, i.e. they try to find a relationship between the *numbers*:

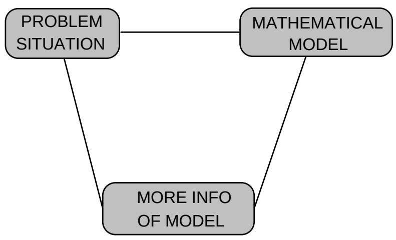
	Pole	*Tree*
Shadow	*2*	*9*
Height	*3*	*?*

They do not really consider the *structure* of the situation, i.e. the way the variables behave in the *physical* situation. *How should they know?* If you do not already know the mathematical *structure* of the situation, you actually have *too little information*, so you must be aware of the *assumptions* that you make! Similarly, most participants did not think about the structure, otherwise they would have realised that the aeroplane uses more fuel at take-off than at cruising speed. If we did realise that, our response should be that we need additional information.

And by the way: this problem was designed especially for teachers on this course. It can be used with pupils, but it is intended for teachers to reflect about their own though processes and to try to better understand pupils struggling with mathematics. Therefore, it is also not very useful to here discuss the difficulties that pupils will have with the activity. Especially the objection that many pupils do not know about aeroplanes is not valid here!

3. Mathematical model

A *mathematical* model serves as a *description* of the situation, but also gives us a powerful tool to *analyse* the situation and find *additional information* about the model and about the situation. The power of such a model is that it *simulates* the physical situation, and we can therefore find more information about the physical situation without having to manipulate the actual physical situation, but by manipulating the mathematical model in stead.



Such a model can be represented

- in *words* (here: 1100 litres per 100 kilometres plus 500 litres for the take off), or
- an equivalent symbolic representation as a *formula*

$$y = {}^{1100l}\!/_{100\,km}\, x + 500 = 11x + 500 \text{ , or}$$

- as a *table*, or
- as a *graph*.

The representation used to a large extent also determines the solution strategies used.

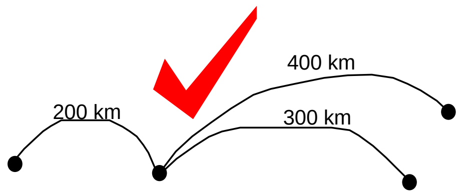
4. The fit of the model, modelling

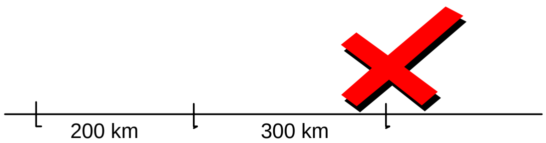
The model will of course only generate sensible/acceptable/valid information about the situation if it really mimics/simulates the situation. Few models are really isomorphic to the situation, because we make many *assumptions*, and in many cases actually *choose* to ignore important characteristics of the physical situation in order to keep our model simple. For example, in using matric and most of university science to calculate the distance a projectile will travel, we ignore the fact that the earth rotates and we ignore air resistance. When we say 1 apple costs 25c therefore 20 apples cost 20 $\times$ 25c (an arithmetical model), we are ignoring the possibility of a discount on bulk purchases. Provided there is a "good fit" between model and situation, the model should generate acceptable information about the physical situation.

In many situations we have (too) little information and we necessarily make many assumptions in formulating a mathematical model for a situation. Therefore the model we *choose* must be *tested* against our understanding of the concrete situation through logical analysis. We have to conduct a *thought experiment*, which means that we must compare the *known properties of our model* against the *known characteristics of the situation*. For example, what model may describe the relationship between the number of car accidents and the age of drivers? Young people make many accidents (the insurance companies surcharge people under 25!), then they make less as they become more mature/responsible/better drivers, but I think old people may again be involved in relatively more accidents. So we have a situation where the values first decrease, reach a minimum and then increase, so the model could be quadratic, because that is a *property* of quadratic functions.

5. Understanding the situation

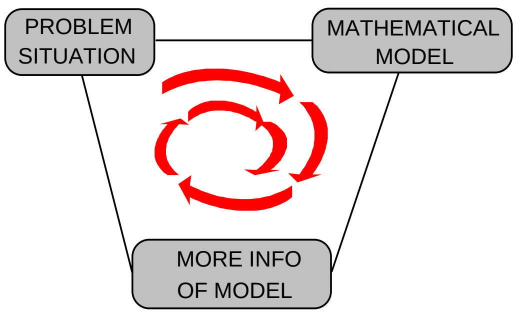
One reason why many people use an inappropriate model in *Boeing Boeing* is that we interpret the problem statement incorrectly. Here "flight" means a journey by air. It therefore refers to "full flights" between different destinations of varying distance, including the take off and landing, e.g. the fuel used between Cape Town and PE, or Cape Town and Bloemfontein ... It does not refer to the fuel used at different (cumulative) distances at cruising altitude on the same flight, e.g. from Cape Town to Johannesburg. Maybe that is why people did not consider the higher fuel usage at take off?





6. Validating (the results of) our model

The *results* generated by a model must always be checked against the physical situation to make sure that our model is an adequate "fit". In *Boeing Boeing* we have the data supplied by Boeing. Our chosen model on page 1 does not generate acceptable data! So we *revise* our model. This is a typical mathematical way of working!



7. Properties of models

Many people will on the first page assume a proportional model, i.e.

$$y = {}^{2700\ l}/_{200\ \mathrm{km}}\, x = 13{,}5\,x$$

People may not be explicitly aware that this is the model they are using, because very few will actually write down a formula here – they simply base their answers on the intuitive assumptions that

$200\text{ km} \xrightarrow{\times 2} 400\text{ km}$

$2\,700\ \ell \xrightarrow{\times 2}$

OR

$200\text{ km} \xrightarrow{+\,200} 400\text{ km} \xrightarrow{+\,200} 600\text{ km}$

$2\,700\ \ell \xrightarrow{+\,2\,700} \xrightarrow{+\,2\,700}$

But this is exactly the point! The "methods" or techniques that we use actually reflect the *properties* of the underlying (intuitive, not-formulated) model. The inappropriate model $y = \mathrm{m}x$ is the *only* model with this multiplicative property, in which the variables increase in the same proportion! (With model we here mean the collection of $y = \mathrm{m}x$, $y = \mathrm{m}x + \mathrm{c}$, $y = \mathrm{a}x^2 + \mathrm{b}x + \mathrm{c}$, $y = \frac{k}{x}$, $y = \log x$, $y = \sin x$, etc.). Here are examples:

×2 ×3

x	1	2	3	4	9
$3x$	3	6	9	12	27

×2 ×3

×2 ×3

x	1	2	3	4	9
$3x+1$	4	7	10	13	28

×? ×?

×2 ×3

x	1	2	3	4	9
x^2	1	4	9	16	81

×4 ×9

×2 ×3

x	1	2	3	4	9
$1/x$	1	1/2	1/3	1/4	1/9

÷2 ÷3

However, both the model $y = \mathrm{m}x$ (inappropriate in *Boeing Boeing*) and the model $y = \mathrm{m}x + \mathrm{c}$ (appropriate in *Boeing Boeing*) have the same "constant differences" properties, *because* they both have constant *gradients*, and this adds to the confusion of choosing an appropriate model:

+1 +1 +6

x	1	2	3	4	9
$3x$	3	6	9	12	27

+3 +3 +18

+1 +1 +6

x	1	2	3	4	9
$3x+1$	4	7	10	13	28

+3 +3 +18

Even after people have realised that direct proportionality is an inappropriate model, many people again use the inappropriate property to calculate f(1800) as, for example, 3×f(600) = 21 300, emphasising how problematic the notions of property of models is!

Similarly many people solve the maximum flight distance problem by saying they know from question 1(d) that f(1800) = 20 300 litres. This is only 2 700 litres short of the full tank of 23 000 litres. Look in the table: f(200) = 2 700. Therefore the answer is 1800 km + 200 km = 2000 km. This looks innocent, but it is based on the assumption that the property f(1800 + 200) = f(1800) + f(200) is valid, which it is not! In general, the classical property that f(a + b) = f(a) + f(b) is true *only* for the model y = mx:

1+2=3 2+3+4=9

x	1	2	3	4	9
$3x$	3	6	9	12	27

3+6=9 6+9+12=27

1+2=3

x	1	2	3	4	9
$3x+1$	4	7	10	13	28

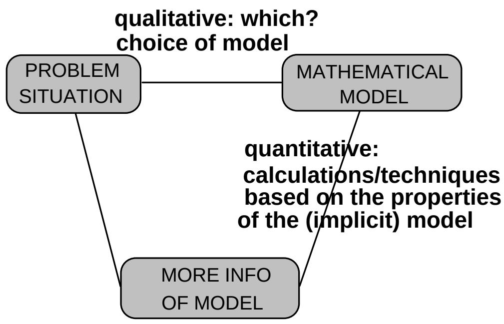
$4+7 \neq 10$

We glibly use this property when we say that the cost of 2 apples plus the cost of 3 apples is just as much as the cost of 5 apples. But it applies *only* to a proportional model! Do we know it?

8. Qualis vs quantum

It is exactly the failure to distinguish between these different properties and shared properties of functions that underlie much of pupils' errors in ratio and proportion, e.g. adding in stead of multiplying. See Olivier, 1992 [2].

In *Boeing Boeing* it is clear that most people do not adequately first reflect on their chosen model, i.e. on the fit of properties of the model and the characteristics of the situation. People jump in and start to calculate without first adequately reflecting on the appropriateness of their chosen model. Someone, I think Freudenthal, said: the moment we start calculating, we stop/kill the thinking! Should we not have realised that the aircraft uses more fuel at take-off than when cruising at constant speed at its desired altitude? Problem is, do people actually know/understand the properties of the models to enable them to reflect on them??



To my mind we are not attempting to achieve it in traditional approaches (except that a parabola has an axis of symmetry, which practically has very little use). I am in effect saying we are teaching the wrong things! We put all our energy into developing *methods* and *techniques* like “adding”, “multiplying” and factorising polynomials, and not enough into understanding the *meanings* and properties of polynomials as function rules. This is equivalent to the problem in primary school where we teach pupils *how* to multiply, but they do not know *when* or *where* to multiply!

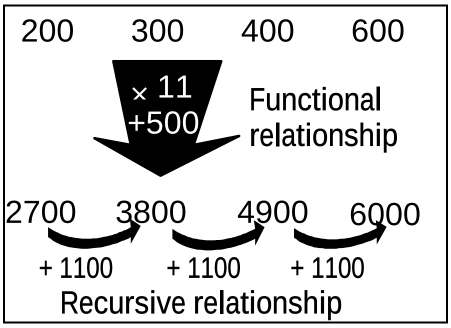
[2] Olivier, A.I. (1992). *Developing Proportional Reasoning.* In M. Moodley, R.A. Njisane & N.C. Presmeg (Eds), **Mathematics Education for Pre-Service and In-Service**: 297-313. Shuter & Shooter, Pietermaritzburg.

9. Solution strategies

Even though the verbal description “1100 litres per 100 km plus 500 litres for the take off” can easily be written down as the *functional* formula in symbols

$$y = {}^{1100\ l}\!/_{100\ \mathrm{km}}\, x + 500 = 11x + 500$$

I suspect that very few will do it. Why not??



On the last page, all the questions can be easily solved by plugging the given values into the functional formula. Many will not do it, but rather intuitively use the *recursive* relationship

$$f(100n + 100) = f(100n) + 1100$$

and then use *extrapolation* and *interpolation*, which are merely forming *equivalent* rates, e.g. (notation just for “us”):

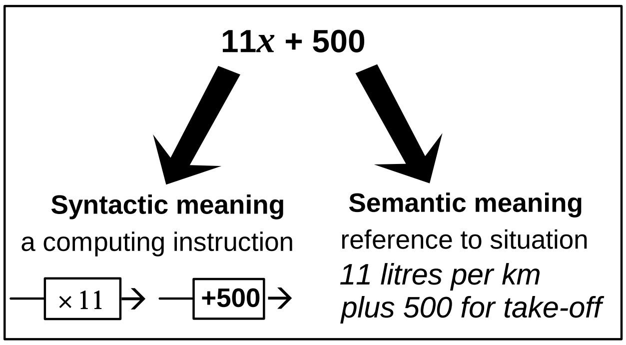
$$f(700) = f(600) + 1100 = 8200$$
$$f(350) = f(300) + \tfrac{50}{100} \times 1100 = 4450$$
$$f(480) = f(400) + \tfrac{80}{100} \times 1100 = 5780$$
$$f(1800) = f(200) + \tfrac{1600}{100} \times 1100 = 20300$$

Of course, most people will not really use this general structure, but rather handle each case intuitively. For example, for f(480): 100 km uses 1 100 litres, so 10 km uses 110 litres, so 80 km uses 880 litres. So f(480) = f(400) + 880. It is clear that these people are actually using sound knowledge of the rate of change (gradient, derivative) of the function! (Only problem is, if the model was $y = x^2$, I think many people would use the same method!)

We have these two approaches, each with its strengths and weaknesses: a “formal” approach would plugs in each value into the *functional* relationship $y = 11x + 500$. It is powerful because you handle all values exactly the same; with the calculator it does not matter how nasty the values are. But, people using the *recursive approach* are also using *valuable important* mathematics: they understand the idea of *gradient*:

$$\frac{1100\ l}{100\ km} = \frac{2200\ l}{200\ km} = \frac{3300\ l}{300\ km} = \frac{550\ l}{50\ km} = \frac{110\ l}{10\ km} = \frac{880\ l}{80\ km} = 11\,{}^{l}\!/_{km}$$

This method, however, becomes very difficult for nasty values, e.g. 384. On the other hand, in many cases “formula people” do not really understand the situation as well as the “gradient people”! You may test them (at this point it is too late!), but I have found that most people who deduced the formula $y = 11x + 500$ directly from the “numbers” in the table do not understand the meaning of the 11 as coming from the rate 1100 ℓ/100 km. In other words, people do not necessarily understand the *semantic* meaning of the formula in terms of the practical situation. And there you have it: On the second page people nicely formulated “1100 ℓ for every 100 km plus 500 ℓ for the take off” and now they have $y = 11x + 500$ and they do not connect the two, i.e. they do not understand the semantic meaning of the formula.



10. Solving the equation

So at a formal level, it is relatively easy to solve question 2. Replacing y with 23 000 leads to the *equation*

$$23000 = 11x + 500$$

which can also easily be solved at the formal level.

At an informal level, many people interpret the situation as the *forward action*, where the problem is therefore interpreted as finding the *input number* that will yield 23 000 as the *output number:*

$$? \longrightarrow \boxed{\times 11} \longrightarrow \quad \longrightarrow \boxed{+500} \longrightarrow \mathbf{23\,000}$$

This forms a sound understanding of the meaning of an *equation.* Such equations can, generally, be solved through a trial-and-improvement strategy (although this specific case is difficult), that actually develops a sound understanding that the *solution of an equation* means finding the value that will make the statement *true*. At the same time such an approach handles the letter symbols as *variables* and not merely as an unique unknown:

x	$11x + 500$	= 23 000?
1 000	11 500	*Too small*
2 000	22 500	*Too small, but close*
2 100	23 600	*Too big*
2 050	23 050	*Too big, but close*

At another level the solution procedure can be interpreted as reversing the sequence to work *backwards:*

$$? \longleftarrow \boxed{\div 11} \longleftarrow \quad \longleftarrow \boxed{-500} \longleftarrow \mathbf{23\,000}$$

Again, such an approach forms a powerful basis for understanding the formal procedure:

$$\begin{aligned} 11x + 500 &= 23000 \\ -500{:}\Leftrightarrow \quad 11x &= 22500 \\ \div 11{:} \quad \Leftrightarrow \quad x &= \tfrac{22500}{11} \end{aligned}$$

11. Interpreting data; domain and range

Values generated by the model, while perfectly acceptable in abstract mathematics, are not necessarily acceptable in the physical situation. That is why interpretation and validation are such important processes in modelling. While the abstract mathematical model $y = 11x + 500$ has no restrictions (i.e. x, $y \in$ R), the practical situation places restrictions on acceptable results. The linear model $y = 11x + 500$ has no maximum, yet in the practical situation there is a maximum capacity of fuel tanks and therefore a maximum flight distance.

Similarly, the abstract solution of the equation $23000 = 11x + 500$ is $2045\tfrac{5}{11}$. Yet in the practical situation this is not a sensible answer given the fact that the distances and fuel consumption are probably only approximate and given the necessity of safety factors. 2 000 km is probably an acceptable answer, although the *passengers* will all feel safer with a larger safety factor!

12. Some implications for teaching

Just like many participants here used an inappropriate model because, to a certain degree, they had insufficient information (the classic rule of three: given three of the four values, find the fourth), so our children struggle to choose appropriate models in rule-of-three situations. Indeed, how must children choose the appropriate model in such situations??

"The" solution clearly is that we must not *superficially* rush through many different contexts (speed-distance, enlarging photographs, gears, orange juice-mixtures, etc. etc.) and especially not formulated as rule-of-three problems, but we should stay with situations longer in-depth and really study the *behaviour* of the variables and reflect on the *properties* of the models, so that they know when to use which models and which properties. *This is best done through many tables of values in which both recursive relationships (gradient) and functional relationships (formula) are emphasised.*

If we want to interpret algebra as the study of the relationships between variables, then analysing such relationships means that we should be able to clearly describe how a change in the *independent variable* influences the values of the *dependent variable.* If "x" increases by 1, what happens to "y"? If x doubles in value, what happens to the value of y?

Just because we traditionally jot down formulae and use them to plug in *specific* values, does not by far mean that we handle such situations as a study of variables or as a study of the relationships between variables. Let's give an example: I argue that we all know the formula $P = 4s$ for the perimeter of a square. And we have all found the perimeters of hundreds of specific squares. Yet, plugging in single values for s at a time essentially handles s and P as specific constants and not as *variables.* Which in many cases means that we have not even begun understanding the *dynamics* of the relationship. Can we answer the question: What happens to P if s increases by 1? And what happens to P if s doubles? We only really develop such understandings through the substitution of many values in table form *and reflecting on the patterns and relationships:*

Sidelength (s)	1	2	3	4	5	6	7	8
Perimeter (P)	4	8	12	16	20	24	28	32

Similarly, we calculate isolated areas of squares without ever really analysing how
A changes as s changes.
How does A change if s changes by 1 unit?
How does A change if s doubles?

Sidelength (s)	1	2	3	4	5	6	7	8
Area (A)	1	4	9	16	25	36	49	64

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Alwyn Olivier
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