BUILDING BRIDGES BESPREKING

Vooraf

The purpose of this lesson is, for us to

1. analise the activity *mathematically* (what are the mathematical ideas in the activity) and *didactically* (what are the cognitive requirements to solve the activity), to develop *appreciation* that such activities are worthwhile to teach and learn
2. observe and reflect on Schoenfeld's categories: the resources, heuristics, beliefs, control and disposition children bring to the activity – what they can do, what they cannot and why
3. observe and reflect on how learning (= "doing mathematics") is an individual and social process
4. learn to manage learning opportunities and monitor outcomes.

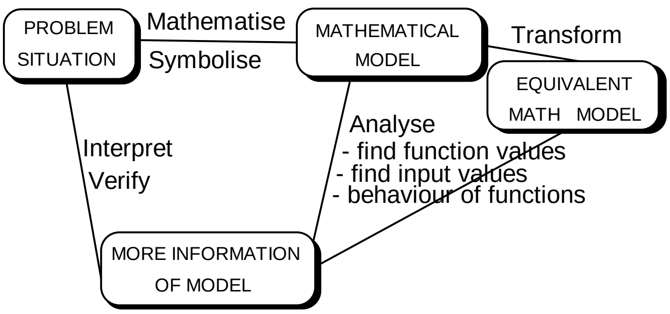
The mathematics

This is an interesting situation, illustrating the power of mathematics to *model* a "real-life" situation. See also Reconceptualising School Algebra (p. 15).

It is a typical *functional* situation: We have two *variables* where the one variable (the size of the gap, d cm) is *dependent* on the other variable (the temperature, t ^{o}C). To understand the nature of the relationship between the variables, we must understand the scientific principle that an increase in temperature leads to an expansion of the metal, and therefore to a *decrease* in the size of the gap. It is therefore a *decreasing function*. (Note that in the *real world*, the friction of e.g. a heavy train at high speed can increase temperature of the rail tremendously, so in a real-real problem the independent variable should be the temperature of the rail, not the air temperature!)

Different representations of functions

The function is here given in *words* and can be transformed to a *table*, a *formula* or a *graph*, depending on which representation is more convenient for a particular purpose. These transformations are important skills in learning algebra, as shown below:



From...to	**Words**	**Table**	**Graph**	**Formula**
Words				
Table				
Graph				
Formula				

One of the aims of the lesson is too observe what representation children will spontaneously use. They should have all the necessary prerequisite knowledge to implement any approach – do they *use* this knowledge in new situations, is their knowledge transferable? (This is an indication of the *quality* of their knowledge!)

The words and the table show that there is a *constant rate of change* of 0,05 cm/^{o}C. It is this *constant rate of change* (gradient) that defines it as a *linear function*, with a straight line *graph* and *formula* of the form $y = mx + c$.

The questions can all be solved *numerically*, directly from the word formulation or the table representation. But the advantages of a formula should be clear!

The activity involves the following function *problem types*:

- Finding function values, i.e. given t, find $d(t)$.
- Finding input values, i.e. solving equations, e.g. given $d(t)$, find t.
- Working with the behaviour of function values (a decreasing function, rate of change, domain and range).
- Finding a formula.

Semantics and syntactics

One aspect to understand is the *syntactical meaning* of the algebraic expression: the expression is a shorthand notation telling us exactly which operations to carry out on which numbers in which order. So $d = 2 - 0{,}05t$ tells us that to calculate the value of d for any value of t, we should multiply the value of t by 0,05 and then subtract the result from 2.

The other aspect to understand is the *semantic meaning* of the formula, the *meanings* it derives from its *physical referents*, e.g. t is not just an abstract number, it is the temperature; d is not just a number, it us the size of the gap.

> *The power of mathematics is immense, and at all stages symbols make a major contribution to this power. But without the ability of the mathematician to invest them with meaning, they are useless.* Richard Skemp, 1971, 89

Physical referents

This activity creates an opportunity to reflect on the *connections* between the physical situation and the models in the form of a formula, table and graph. For example, we "see" the physical situation in the formula $d = 2 - 0{,}05t$: At $t = 0^{o}$C the size of the gap is 2 cm, and we know that as t increases the size of the gap becomes *physically* smaller; *mathematically*, we *subtract* a positive amount from 2, making the gap numerically smaller. However, if the temperature *decreases* below 0^{o}C, we know, physically, that the gap becomes larger, and the mathematics must behave accordingly, e.g. for $2 - 0{,}05(^{-}1)$ to become larger, it will have to be $2 \mathbf{+} 0{,}05$ ("a minus times a minus is a plus"), illustrating again that *we construct mathematics to fit reality*!

A further essential aspect of the sematic meaning of the formula is to understand the *physical referents* for the different parts in the formula! So it is important that we know that in $d = 2 - 0{,}05t$ the "2" is the gap size at 0^{o}C and the "0,05" is the gradient or rate of change – how *fast* the size of the gap changes as the temperature changes. Furthermore, this rate of change is specific to the specific bridge material – it is the so called *linear expansion coefficient* of the material, which is unique to each substance, like density.

Parameters

Therefore, to solve the desert problem, we should be able to change these "constants" to *parameters,* e.g. $d(t) = d(0) - kt$ where $d(t)$ is the gap size at t ^{o}C, $d(0)$ is the gap size at 0 ^{o}C and k is the expansion coefficient. These two parameters can be varied to suit the conditions: If we make $d(0)$ larger, then $d(0) = 2{,}8$ cm will guarantee that the gap does not close up before 56 ^{o}C. However, note that if $d(0) = 2{,}8$ cm, then $d(-10) = 3{,}3$ cm, which is probably too large! Why is that? The other option is to specify a different material for the rail with a *larger* expansion coefficient, at least k = 0,36 cm/^{o}C. Why? See this interactive Excel worksheet dynamically illustrating the effects of changing the parameters on the graph of the model, and therefore illustrating both the physical problem and its solution. In essence we are here dealing with the *transformation of functions*: $f(x) + k$ as a vertical shift, and $kf(x)$ as stretching and shrinking which is for a straight line interpreted as a change in the gradient.

Domain and range

The *domain* and *range* of the function also are very important here: Although there are no restrictions on the values of the variables in the *abstract* model $d = 2 - 0{,}05t$, the domain and range are determined by the *physical* situation: d cannot be less than zero. Therefore $d \geq 0$ and therefore $t \leq 40$ (*this is the desert problem!*) and this should also be reflected and interpreted in the graph and in our interpretations of the mathematical solutions to the practical problem.

We can therefore use the function (graph or formula) to *extrapolate* (extend beyond the known values) to find unknown values, but should do so with care. This is like *inductive generalisation*, the pitfall of induction, that the pattern is not necessarily valid beyond the known *database*, because the structure does not continue. Here, the structure holds only for the *physical domain and range*.

Because temperature and length are *continuous* quantities (unlike a *discrete* quantity like "number of triangles"), we can also *interpolate* (extend *inside)*, i.e. find values *between* known values, e.g. *calculate* d between 15^{o}C and 16^{o}C, or between $15{,}3^{o}$C and $15{,}4^{o}$C. That means that we can connect points on the graph with a smooth curve and calculate values between known values. (This is illegal for discrete situations like the number of triangles, or even for continuous situations for which we have only discrete data, e.g. the height of a plant measured every week.)

Constant gradient

We can apply the particular useful and *unique* property of the linear function, that the differences in the variables are *proportional*, i.e. $\Delta d = k\Delta t$, i.e. there is a constant gradient, and in particular that a value exactly between two d-values is the *average* of the corresponding t-values, i.e. $t = \frac{t_1 + t_2}{2} \Rightarrow d(t) = \frac{d_1 + d_2}{2}$. This true for no other type of function, e.g. it is not true for $y = x^2$ (check!).

Children's approaches
What do you anticipate?

Fasilitering
Onthou, ons probeer 'n probleemgesentreerde benadering (Onderrig via probleemoplossing), dus wys ons nie vir kinders hoe om die probleem op te los nie, maar wil hê dat hulle hul oplossings op vorige kennis bou, en by mekaar leer. Hoe reageer ons op verskillende metodes?

Ons het Graad 9 leerlinge ... As hulle redelik die formule gebruik, moet ons *probeer* om by die woestynprobleem en bespreking uit te kom. Doen vraag 7 (grafieke) slegs as hulle gou maklik klaarmaak ...

Agterna refleksie (Lana Schreuder - 2006)

Wiskundige inhoud:
Die inhoud van hierdie les is miskien te moeilik vir graad 8's. *[Kyk gerus wat is in die leerplan. AIO]* Dit was hoegenaamd nie moontlik om ver te vorder met die vrae nie, maar dit is nie altyd die belangrikste om al die vrae klaar te kry nie. Die struktuur van 'n formule is nog nie vasgelê by die leerders nie. Ons het nie uitgekom by vraag 4 waar dit met negatiewe getalle te doen het en wat dan met die gaping gebeur nie.

Die leerders:
Hulle was 6 leerders in twee groepies van 3 by tafels. Meeste van hulle het begin om 0,05 van 2 af te trek, maar meeste van hulle het ook so aangegaan met aftrek al die pad tot by 10 en 20. Ek het vir hulle gevra hoekom hulle dit op daardie manier doen en 'n paar riglyne probeer gee om nie net die hele tyd af te trek nie. Ek moes eers vir hulle sê dat hulle dit anders moet probeer op 'n ander manier en nie net bloot met aftreksomme nie, want wat gebeur as die temperatuur baie hoog word?

Die leerders het nie genoeg geweet om veranderlikes te gebruik nie. Nêrens maak hulle gebruik van x of y om uit te kom by 'n formule nie. Sommige leerders het 'n metode ontwikkel sonder om 'n formele formule te skryf. Byvoorbeeld een meisie:

Vir 10°C: 10 × 0,05 = 0,5
2cm – 0,5 = 1,5

Dit is korrek in stappe gedoen, alhoewel sy nie in een stap die *struktuur*, naamlik 'n *formule* kon skryf nie. Sy kon wel mooi verduidelik hoe sy tot hierdie stappe gekom het.

'n Ander leerder in die ander groep het wel in een stap gedink. Sy skryf:

Gaping = 2 cm – (*Temp* × 0,05 cm)

Daarna kon sy ook maklik vraag 2 beantwoord.

Die leerders dink in terme van 'n *metode* (nadat ek gesê het hulle moenie net aftrek nie). Hulle kom uit op stappe wat gevolg kan word om die antwoord te kry. Ek het hulle gelei deur te vra: "wat maak jy by 1°C om die gapinggrootte te kry?", en as hulle dat vir my kan verduidelik, "wat maak jy nou vir 10°C?".Die stappe kan dan vir enige temperatuur gebruik word, maar hulle kon nie dit in 'n algemene formule omskryf nie. Ek dink daardie kennis is nog nie opgedoen nie.

Die fasiliteerder:
Ek wonder soms hoe ver ek as onderwyser die leerders moet help. Moet ek vir hulle sê gebruik 'n x vir die gaping en t vir temperatuur en probeer jou metode in een stap skryf, of is wat hulle gedoen het genoeg? Wat hulle gedoen het, is tog al kennis wat hulle het om toe te pas. En die metode is die belangrikste, so dis seker genoeg.