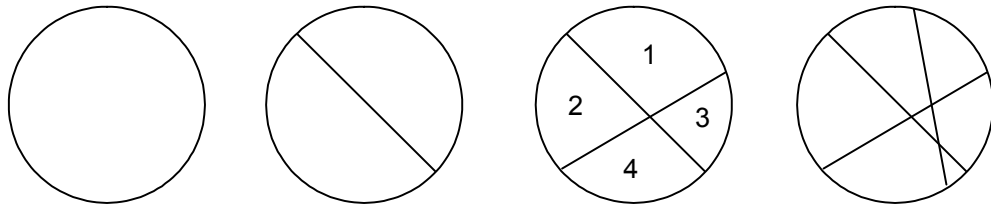


CIRCLES, REGIONS AND CORDS

1. Chords and regions



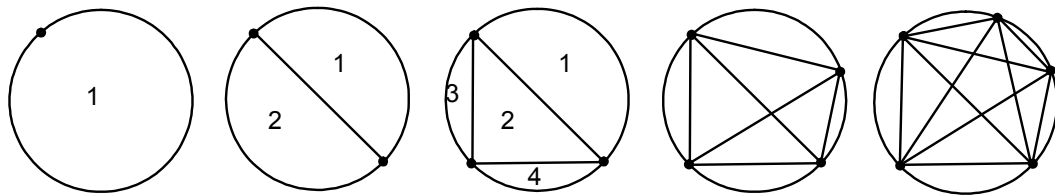
2 chords divide a circle into 4 regions.

What is the *maximum* number of regions into which 6 chords will divide a circle?

And 20 chords?

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4					

2. Points and regions



If 3 points on a circle are joined 4 regions are formed.

What is the *maximum* number of regions into which 6 points on a circle will divide the circle if the points are joined?

And 20 points?

# points (p)	1	2	3	4	5	6	20
# regions (R)	1	2	4	8			

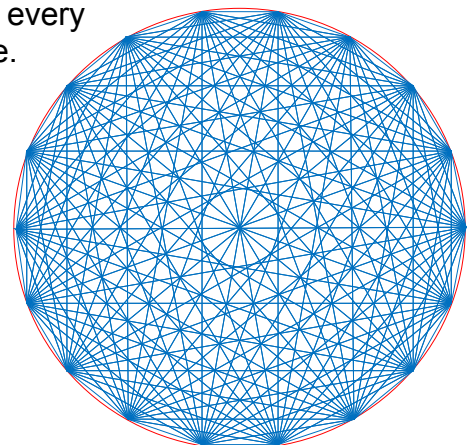
3. Mystic rose

(a) In this figure, there are 18 points on the circle, and every point is connected to every other point on the circle.
How many chords are there all together?

(b) In another circle there are 465 chords.
How many points are there on the circle?

(c) Suppose you have such a completed rose with 25 points and you decide to add one more point.
How many *extra* chords must you draw?

Can you generalise for n points to $n+1$ points?

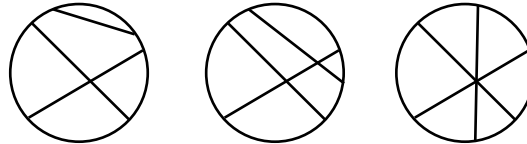


SOME NOTES

INDUCTION

1. Chords and regions

First *understand* the situation! *Maximum* number of regions implies that the chords cannot be drawn like these below; each new chord must cut every other chord and no three chords should be concurrent (cut in one point).

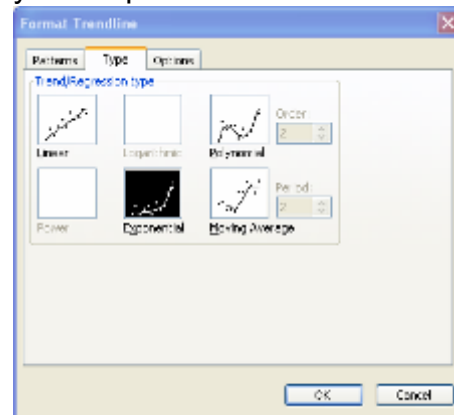
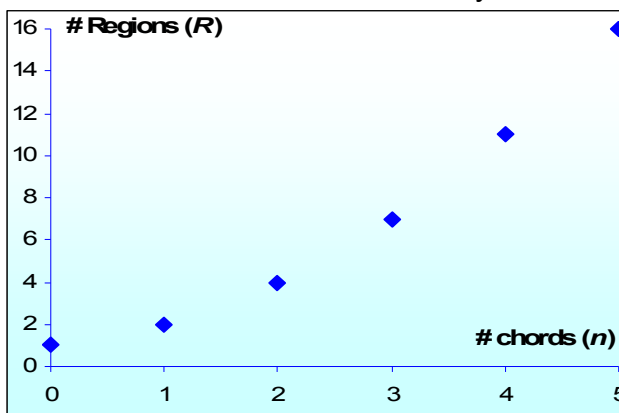


An *inductive* attack tries to find a pattern in the *numbers*.

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4	7	11	16		

If you want to find a *functional* (here *vertical*) relationship by *inspection* (i.e. by just “looking”) you will have to systematically *ask* – is it maybe $+1$?, is it $\times 2 - 2$?, is it ...??. i.e. the only way of “seeing” a pattern is to first *beforehand* formulate a *conjecture* (an unproven theorem) and use it as a *lense* to look at the data and to *test/check* each conjecture and do not stop trying different alternatives until you find one that fits!!

It may help if you represent the data *graphically* and then use your knowledge of the relationship between the shape of a graph and its formula! Here is an Excel *scatterplot* of the data. What kind of a formula do you think may *fit* the points?



If you are using a technology tool like Excel, you may as well go all the way and let it find the *regression formula* (the “curve of best fit”) for you! But it is not so simple – you have to *tell* Excel which model to use! If you choose a quadratic model (why?), Excel gives the formula $y = 0.5x^2 + 0.5x + 1$. *Is it correct?*

If you want to find a formula *analytically* (i.e. algebraically) you have to bring certain *resources* (knowledge) to the situation. For example, knowledge of *recursive* (horizontal) differences may be helpful here:

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4	7	11	16		

Differences: +1 +2 +3 +4 +5

Second differences: +1 +1 +1 +1

Maybe you know: *if the second differences are constant, then the formula is quadratic.* Do you know this theorem? If not, [click here for a brief discussion in the appendix.](#)

Now that we know the formula is quadratic, we merely have to solve for the parameters a , b and c in $R(n) = an^2 + bn + c$. The values in the *database* satisfy the equation, so substituting them makes the equation true, and this leads to three equations in three unknowns:

$$R(0) = 0a + 0b + c = 1 \quad \dots (1)$$

$$R(1) = 1a + 1b + c = 2 \quad \dots (2)$$

$$R(2) = 2a + 4b + c = 4 \quad \dots (3)$$

Here is yet another *inductive, recursive* method: Look at R and the differences *with different eyes*:

$$R(0) = 1$$

$$R(1) = 1 + 1$$

$$R(2) = 1 + (1 + 2)$$

$$R(3) = 1 + (1 + 2 + 3)$$

$$R(n) = 1 + (1 + 2 + 3 + \dots + n)$$

If you *know* that $1 + 2 + 3 + \dots + n = \sum_{i=1}^n i = \frac{n(n+1)}{2} \quad \dots$

or if you do not, maybe you can *deduce* it from your knowledge of the sum of an *arithmetic series*: $S_n = \frac{n}{2} \{2a + (n-1)d\} \quad \dots$

$$\text{Anyway, } R(n) = 1 + \sum_{i=1}^n i = 1 + \frac{n(n+1)}{2} = \frac{n(n+1)+2}{2}, \quad n \in N_0 = 0, 1, 2, 3, \dots$$

You should *check it against the known database* so that at least you are sure the formula is valid for $n = 1$ to 5!!

But, of course, although we here used *algebraic reasoning* to deduce the formula, it is nevertheless based on an *inductive analysis* of the *numbers* in the table, *not on the structure of the situation!*

So you can only pray that the generalisation is *valid* after $n = 5$!!

But how can you be *sure*??

PTO

2. Points and regions

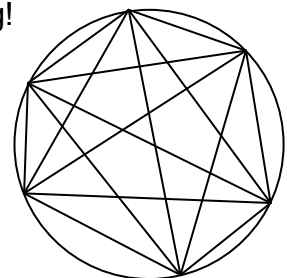
You probably noticed the *recursive pattern* 1, 2, 4, 8, 16, ... in the numbers in the table and then continued the *recursive doubling pattern* to *predict* $R(6) = 32$. Or you may have inductively deduced the *functional formula* $R(p) = 2^{p-1}$, which also yields $R(6) = 32$.

A reminder: *Induction* consists of two processes:

1. *Abstraction* (finding the pattern in the *known set of numbers or database*) – here the pattern in the 6 *known pairs* (1, 1), (2, 2), (3, 4), (4, 8), (5, 16) is $R(p) = 2^{p-1}$, $1 \leq p \leq 5$, $p \in \mathbb{N}$.
2. *Generalisation* (extending the pattern *beyond* the known database) – here assuming that the next pair is (6, 32) and $R(p) = 2^{p-1}$, $p \in \mathbb{N}$.

In this case the abstraction (1) is correct but the generalisation (2) is not! Unfortunately our expectation that the pattern is continued beyond the fifth case is wrong!

In fact $R(6) = 31$! Check for yourself by physically counting the number of regions in this sketch. *Understand the problem!* The prerequisite of *maximum* number of regions implies that no three chords should be *concurrent* (cut in one point).



This example serves to remind us that inductive reasoning, powerful as it may be in discovering new patterns, is prone to error. *That is why we need a structural proof!!*

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks. Kasner & Newman, 1940

But can we find the *correct* formula? Let's try differences again:

# points (p)	0	1	2	3	4	5	6		20
# regions (R)	1	1	2	4	8	16	31		

First differences: 0 1 2 4 8 15

Second differences: 1 1 2 4 7

Third differences: 0 1 2 3

Fourth differences: 1 1 1

The fourth differences are equal, so the generating formula is a fourth degree polynomial. So the formula is of the form $R(p) = ap^4 + bp^3 + cp^2 + dp + e$. So we must find a, b, c, d and e. But $R(0) = 1$, so $e = 1$. So we must find a, b, c, d, i.e. solve four unknowns and so we need four equations:

$$R(1) = 1a + 1b + 1c + 1d = 0 \dots\dots\dots (1)$$

$$R(2) = 16a + 8b + 4c + 2d = 1 \dots\dots\dots (2)$$

$$R(3) = 81a + 27b + 9c + 3d = 3 \dots\dots\dots (3)$$

$$R(4) = 256a + 64b + 16c + 4d = 7 \dots\dots\dots (4)$$

Solving these four simultaneous equations, we get:

$$R(p) = \frac{1}{24}p^4 - \frac{1}{8}p^3 + \frac{23}{24}p^2 - \frac{2}{3}p + 1$$

$$= \frac{p^4 - 6p^3 + 23p^2 - 18p + 24}{24}$$

Or you could use the known database in a technology software package to easily find the regression formula for you. My Excel Trendline gives

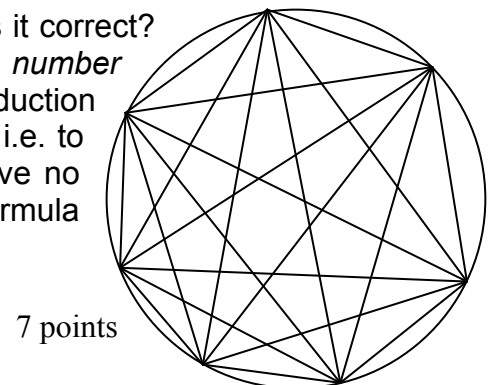
$$R(n) = 0.0417n^4 - 0.0833n^3 + 0.4583n^2 + 0.5833n + 1$$

Do you agree that this technology formula is *equivalent* to the above?

You may want to check that this formula generates the correct values, i.e. 1, 2, 4, 8, 16, 31:

# points (p)	1	2	3	4	5	6	7
$R(p) = 2^{p-1}, p \in N$	1	2	4	8	16	32	64
$R(p) = \frac{p^4 - 6p^3 + 23p^2 - 18p + 24}{24}$	1	2	4	8	16	31	57

This formula produces the next value $R(7) = 57$. But is it correct? Remember, this formula is generalised from the *number pattern*, i.e. through *induction*, so we must wonder if induction will get us into trouble yet again! To check $R(7) = 57$, i.e. to check if 7 points on a circle yield 57 regions, you have no choice but to draw and count! You cannot trust this formula as a model of the situation, you cannot be sure!



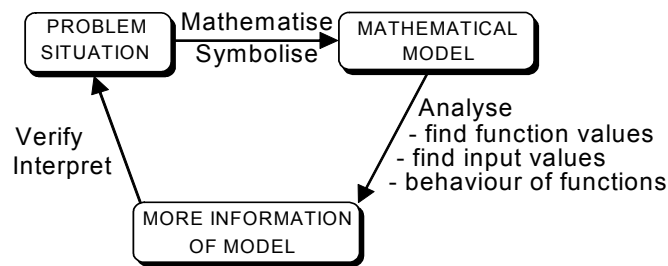
You can only be *sure* if you reasoned with the *structure of the situation*!

PTO

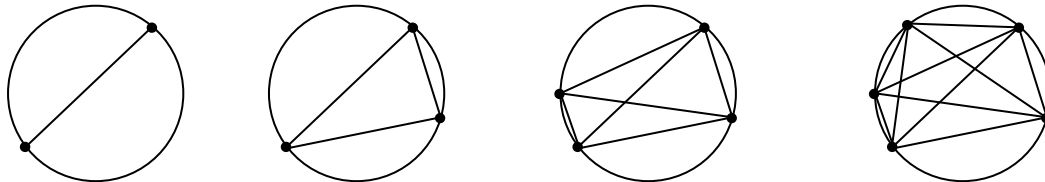
3. Mystic rose

Students often simply formulate a guess like "It's 18×18 ", or "it's 18×17 " and then cannot *justify* it, expecting the teacher to tell them if they are right or wrong. This is *not* what mathematical thinking is about! We must learn to see our efforts not as *answers*, but as *conjectures*, as *public statements* that should be discussed, explained, verified and justified through *logical argument*. And we should learn to value logical arguments, and not accept mere *authority* of someone like the teacher as support or justification for our solutions!

Now let's try again! Of course you can count, but that will be a rather daunting task and it will be prone to error! The essence of mathematics is to construct *mathematical models* that mimic the real situation, and then we manipulate mathematical objects in stead of real-life objects to *predict* unknown information.



Let's begin with an *inductive approach* and let's use some *heuristics*: Let's investigate some special cases, let's do it systematically, let's organise our resulting data in a table, try to find a pattern in the data and then use the pattern as a model to solve the original problems. Here are some special cases, where it is very easy to count the cords:



# points (n)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10				

What we want is a *functional (vertical) formula* expressing C in terms of n . But it is not always so easy to find a formula through *inspection (just by "looking")*. In the table below I identify some easily observed patterns, suggesting a relationship:

# points (n)	2	3	4	5	6	7	18	n
		↓ ×1		↓ ×2		↓ ×3		
# chords (C)	1	3	6	10	15	21		

I now "fill in the gaps" using the pattern:

# points (n)	2	3	4	5	6	7	18	n
	↓ × $\frac{1}{2}$	↓ ×1	↓ × $1\frac{1}{2}$	↓ ×2	↓ × $2\frac{1}{2}$	↓ ×3		
# chords (C)	1	3	6	10	15	21		

I now write the numbers in an *equivalent*, but *more useful* form:

# points (n)	2	3	4	5	6	7	18	n
	$\downarrow \times \frac{1}{2}$	$\downarrow \times \frac{2}{2}$	$\downarrow \times \frac{3}{2}$	$\downarrow \times \frac{4}{2}$	$\downarrow \times \frac{5}{2}$	$\downarrow \times \frac{6}{2}$		
# chords (C)	1	3	6	10	15	21		

Please notice that for the purpose of “seeing” the structure in this context, $\frac{3}{2}$ is “simpler” than $1\frac{1}{2}$ and $\frac{6}{2}$ is simpler than 3!!! The conventions you learned in primary school about “always writing in simplest form” are totally irrelevant in context!

To generalise our pattern, we must observe what is unchanging (*invariant*) and what changes. The invariant part is clear: every value is multiplied by something and divided by 2. It is this *invariant structure* that must be continued and *generalised* to C(18) and C(n):

# points (n)	2	3	4	5	6	7	18	n
	$\downarrow \times \frac{1}{2}$	$\downarrow \times \frac{2}{2}$	$\downarrow \times \frac{3}{2}$	$\downarrow \times \frac{4}{2}$	$\downarrow \times \frac{5}{2}$	$\downarrow \times \frac{6}{2}$	$\downarrow \times \frac{?}{2}$	$\downarrow \times \frac{?}{2}$
# chords (C)	1	3	6	10	15	21		

We must now remove the *noise* and concentrate only on the variable part, so that we can more easily *identify the structure*:

# points (n)	2	3	4	5	6	7	18	n
	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow	\downarrow
numerator	1	2	3	4	5	6		

The functional (vertical) relationship is easily seen as -1 and extended to 18 and n :

# points (n)	2	3	4	5	6	7	18	n
	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$	$\downarrow -1$
numerator	1	2	3	4	5	6	17	$n - 1$

We can now answer our original question: $C(18) = \frac{18 \times 17}{2}$ and $C(n) = \frac{n \times (n - 1)}{2}$.

We have our solution, but it was not so easy to find the functional formula! To emphasize that different people see the same situation differently *because they bring different background knowledge (resources) as lenses to the situation*, let’s investigate the *recursive* (horizontal) pattern of differences $+2, +3, +4, \dots$

# points (n)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10	15	21		
Differences:		2	3	4	5	6		
Second differences:			1	1	1	1		

So the second difference is constant, so the formula is quadratic of the form $C(n) = an^2 + bn + c$ and we merely have to solve for the parameters a , b and c :

$$C(1) = 1a + 1b + c = 0 \dots (1)$$

$$C(2) = 4a + 2b + c = 1 \dots (2)$$

$$C(3) = 9a + 3b + c = 3 \dots (3)$$

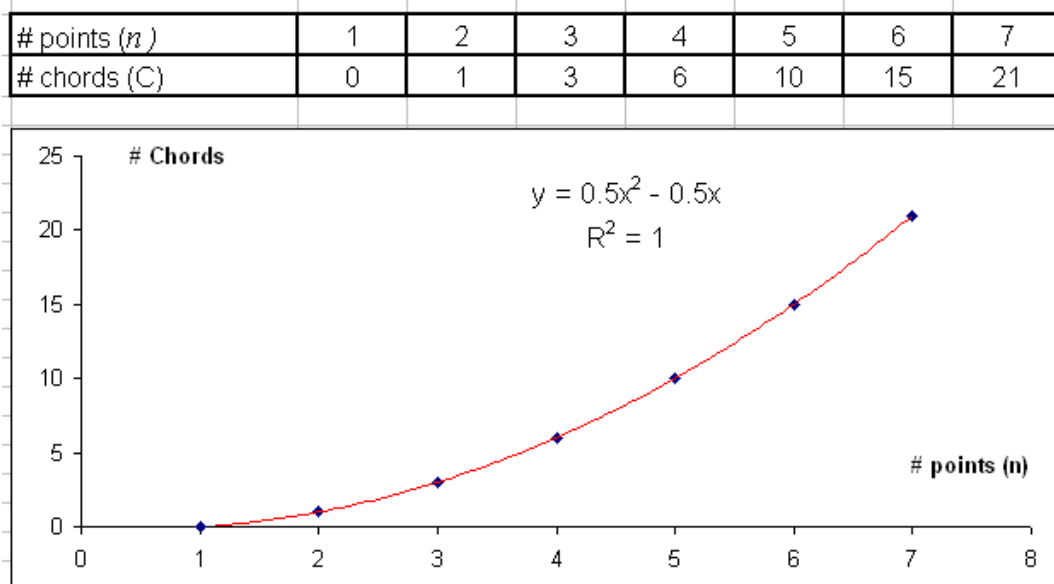
$$(2) - (1): 3a + b = 1 \dots (4)$$

$$(3) - (2): 5a + b = 2 \dots (5)$$

$$(5) - (4): 2a = 1 \Rightarrow a = \frac{1}{2}, b = -\frac{1}{2}, c = 0$$

$$\text{So } C(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2}, \quad n \in \mathbb{N}$$

Another approach is to use a computer package to find the regression formula, what Excel calls the Trendline:



This formula generates $C(18) = 153$. But is it correct? You cannot trust this inductively generated formula as a model of the situation, you cannot be sure! Do you really want to check by actually physically counting $C(18)$??

Solving the question $C(n) = 465$ is very laborious without the formula, but with the formula it becomes very easy, illustrating the power of Algebra:

$$\begin{aligned}
 3 \text{ (b)} \quad C(n) &= 465 \\
 \Rightarrow \frac{n(n-1)}{2} &= 465 \\
 \Rightarrow n^2 - n - 930 &= 0 \\
 \Rightarrow (n-31)(n+30) &= 0 \\
 \Rightarrow n &= 31
 \end{aligned}$$

$$\begin{aligned}
 3 \text{ (c)} \quad C(n+1) - C(n) & \\
 &= \frac{(n+1) \times (n+1-1)}{2} - \frac{n \times (n-1)}{2} \\
 &= \frac{n^2 + n - n^2 + n}{2} \\
 &= n
 \end{aligned}$$

PTO

DEDUCTION

All the previous work was *induction*, i.e. deducing patterns from *numbers*, which are *special cases*. But inductive conclusions may be wrong! And induction does not *explain the form* of the result.

To *prove* a statement (to show *that* it is true) and to explain *why* it is true, we need to reason *deductively*, i.e. *generally*, using the *structure of the situation*.

We can prove *that* a statement is true using **complete mathematical induction**¹. For example, for **1. Chords and regions**:

To prove that $R(n) = 1 + \frac{n(n+1)}{2}$, $n \in N_0$ by mathematical induction, we have to prove:

$$1. R(k) = 1 + \frac{k(k+1)}{2} \Rightarrow R(k+1) = 1 + \frac{(k+1)(k+2)}{2}$$

$$2. R(0) \text{ is true, i.e. } R(0) = 1 + \frac{0(0+1)}{2} = 1$$

Suppose there are already k chords drawn in the circle. Then the next, i.e. the $(k+1)^{\text{st}}$ chord will cut each of the previous k chords and therefore pass through $(k+1)$ regions, therefore adding an additional $(k+1)$ regions.

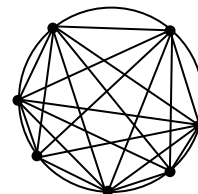
$$\text{So, if } R(k) = 1 + \frac{k(k+1)}{2}$$

$$\text{then } R(k+1) = 1 + \frac{k(k+1)}{2} + (k+1) = 1 + \frac{(k+1)[(k+1)+1]}{2}$$

which proves the implication in (1) and the rest is obvious ...

Mystic rose

A deductive approach will use the structure of the situation, not the number answers for specific cases!. Analyse the *special case* when we have 7 points on the circle, and try to develop some clever way of counting the chords that will bring out the *structure* of the situation.



It should be clear that there should be 6 chords at every point on the circle, *because we are connecting every point with every other point, except itself*. So for 7 points there are 7×6 chords altogether. However, this is still not correct – *we have counted every chord twice!* Therefore, the number of chords in a 7-point Mystic Rose is $\frac{7 \times 6}{2}$, which of course confirms the answer we previously obtained for $C(7)$ through recursion.

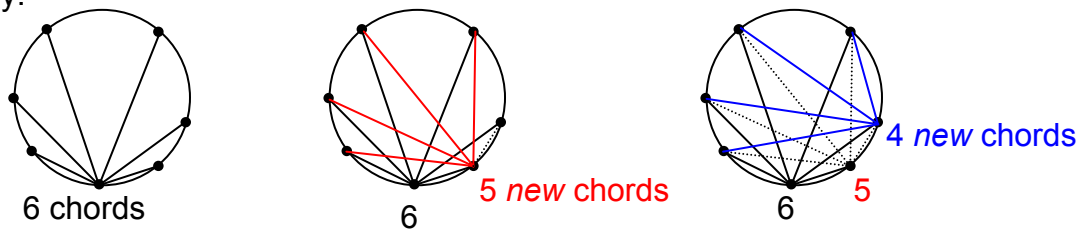
In deduction, and generally in mathematical thinking, we are not concerned with the numerical answer, but with the *method* or *structure*. So if we understand the structure in the one example $C(7) = \frac{7 \times 6}{2}$, we can *generalise* the structure – we must be able to see

the general in the particular. So it should be clear that for 18 points, there will be 17 chords at each point, so 18×17 in total, except that we counted each chord twice, so $C(18) = \frac{18 \times 17}{2}$. In *general*, if we have n points, each point is connected to $n - 1$ points,

so $C(n) = \frac{n \times (n-1)}{2}$. *So deduction confirms and proves our previous result!*

¹ What we are calling “induction”, i.e. generalising from *a few special cases*, is really better named “incomplete induction”, in contrast to the method of *Mathematical induction*, which is “complete induction”, because it considers *all cases!*

Looking back, or discussing the problem with others, we may realise that there are other ways of looking at the structure. Our approach was to count the chords at each point. This counted each chord twice, that is why we divided by 2. Now look at it differently:



It is clear that at the first point there are 6 chords. At the second point there are 5 new chords, because the chord to the first point has already been counted at the first point. Similarly, at the third point there are 4 new chords, etc.

So we have $C(7) = 6 + 5 + 4 + 3 + 2 + 1$

Again we emphasise that we do not want the numerical *answer*, but want to understand the *structure* of the situation. The result is not simply the sum of arbitrary numbers – the structure is clear: it is a decreasing sequence because we are not double-counting chords; 6 is not just any number, but is one less than 7 because we are drawing chords to every other point except itself. *This means that we can see the structure of the situation in this one example. We say we see the general in the particular.*

We can now without further ado say that $C(18) = 17 + 16 + 15 + 14 + \dots + 3 + 2 + 1$. Please notice that you can find $C(18)$ by adding this sequence manually, but it is very longwinded to solve (b), i.e. solve *the equation* $C(n) = 465$ without a formula!!

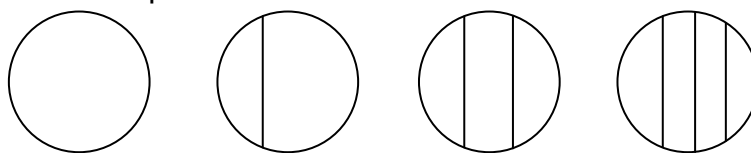
When you *do* Mathematics, problems always lead to new problems!

New problem:

Develop a short method (formula) to calculate $17 + 16 + 15 + 14 + \dots + 3 + 2 + 1$. Generalise!

Chords and regions²

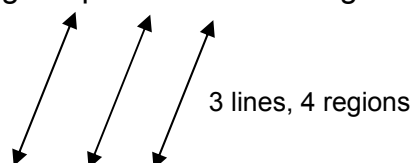
A typical mathematical approach is to *first look at simpler cases*. So let's first understand the simpler case if no chords intersect:



We can say:

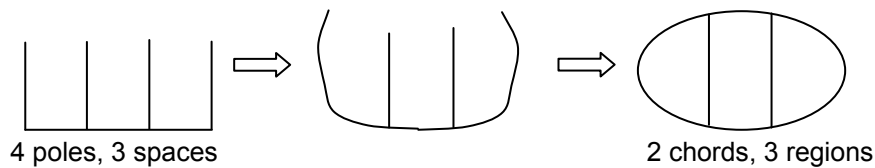
$$R(n) = 1 + n, \text{ i.e. } \# \text{ regions} = 1 + \# \text{ chords}$$

Note that this is not an inductive generalisation from the numbers in the above sketches, but is a *structural statement!* As an analogy, the structure is not different from n parallel lines dissecting the plane into $1 + n$ regions:

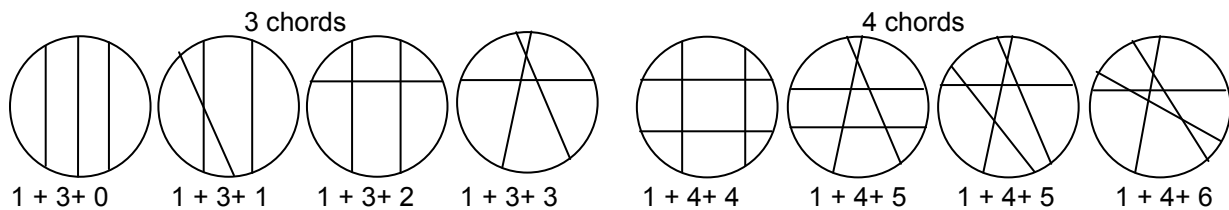


² Compare Ticky de Jager at <http://www.tickey.co.za/>

Or if we want to think in terms of a boundary: it can be seen as a generalisation of the well-known structure where a fence with n poles has $n - 1$ spaces (regions) between the poles. Now imagine that the two end-poles become the circle ...



Now we have to consider the more general case when the chords intersect. Another typical mathematical attitude is not to merely construct a new relationship, but to construct it in such a way that the relationship still applies to the previous special case. So we do not want to change our present formula too much! Now look at these examples:



Convince yourself, structurally, that in all cases (except when 3 chords are concurrent), we have:

$$\# \text{ regions} = 1 + \# \text{ chords} + \# \text{ intersections}$$

This is a generalization of our earlier formula, i.e. this relationship also applies to the case where the chords do not intersect, because then the number of intersections is 0!

How many intersections are there? In the case of the maximum number of regions, if we have n chords each intersecting each of the other $n - 1$ chords, the total number of intersections is $\frac{n \times (n - 1)}{2}$. So we now have the formula:

$$R(n) = 1 + n + \frac{n \times (n - 1)}{2}$$

If you simplify this formula, you get the *equivalent form* we had before:

$$R(n) = 1 + \frac{n \times (n + 1)}{2}$$

This proves that our inductive result is correct, and it also gives us insight into *why* the result takes this form.

PTO

Points and regions

Now we investigate the case when we have p points on the circle. Convince yourself that the reasoning is the same as in the previous case:

$$\# \text{ regions} = 1 + \# \text{ chords} + \# \text{ intersections}$$

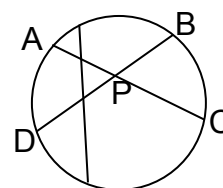
We already know from the Mystic Rose problem that the number of chords is $\frac{p \times (p-1)}{2}$.

To find the number of intersections, we need to activate knowledge about permutations and combinations:

Every intersection is fixed by four points on the circle. For example, P is fixed by A, C, B, D.

So the number of intersections is the number of ways in which we can choose four points out of p :

We have p choices for the first point, $p-1$ for the second, $p-2$ for the third and $p-3$ for the fourth. So the total number of intersections is $p(p-1)(p-2)(p-3)$.



But the order in which we choose the points does not matter, so we must divide by the number of ways in which we can arrange 4 things in a row: We have 4 choices for the first, 3 for the second, 2 for the third and 1 for the fourth, i.e. $4 \times 3 \times 2 \times 1$ choices. So the number of intersections is $\frac{p \times (p-1)(p-2)(p-3)}{4 \times 3 \times 2 \times 1}$

So $R(p) = 1 + \# \text{ chords} + \# \text{ intersections}$

$$= 1 + \frac{p \times (p-1)}{2} + \frac{p \times (p-1)(p-2)(p-3)}{4 \times 3 \times 2 \times 1}$$

If you “simplify” this, you get the *equivalent form* we had before:

$$R(p) = \frac{p^4 - 6p^3 + 23p^2 - 18p + 24}{24}$$

But, because we here reasoned *deductively*, we have now *proved* that the formula is generally valid, so we can *trust* and *know* that the value $R(7) = 57$ is correct!!

PTO

REFLECTION

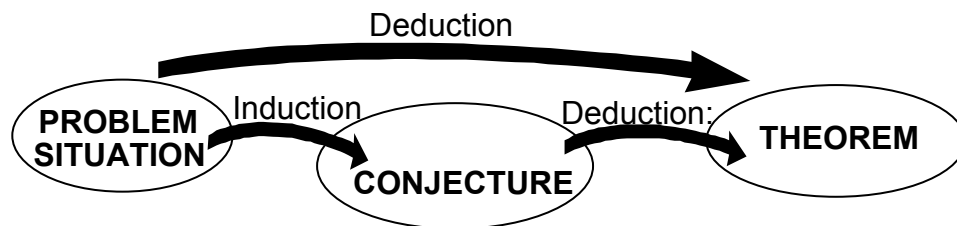
Let us repeat: inductive reasoning is a powerful method to discover new relationships. But we can never be sure that an inductive pattern will not somewhere break down, even after millions of cases. To be *sure that* it is always true (validity), and to *explain why* it is true – why the pattern has this form and not another, we must use *deductive reasoning*, i.e. reason using the structure of the situation.

This means that in mathematical activity there are two approaches:

- One can work deductively, or
- One can work inductively, but to make sure, it should be *followed by deduction*.

But just induction on its own is not adequate! When we use induction we cannot be sure our conclusion is correct, no matter how many cases we check.

The following diagram depicts the relationship between induction and deduction and the status of knowledge (a conjecture is not yet proved; a theorem is a proved conjecture):



APPENDIX: THE POLYNOMIAL DIFFERENCE THEOREM

Complete the tables³ of the following *special cases* to convince you *intuitively* that:

If the n^{th} difference is constant, then the model is a polynomial of the n^{th} degree ... (1)

n	1	2	3	4	5	6		
$y = 2n + 3$	5	7	9	11	13	15		

1st difference: 2 2 2 2 2

n	1	2	3	4	5	6		
$y = n^2$	1	4	9	16	25	36		

1st difference: 3 5 7 9 11

2nd difference:

n	1	2	3	4	5	6		
$y = n^3$	1	8	27	64	125	216		

1st difference: 7 19 37 61 91

2nd difference:

3rd difference:

n	1	2	3	4	5	6		
$y = 2^n$	2	4	8	16	32	64		

1st differences: 2 4 8 16 32

2nd difference:

3rd difference:

Can you *prove* the theorem? Take as special case the general quadratic function $f(n) = an^2 + bn + c$:

$$f(n+1) - f(n) = a(n+1)^2 + b(n+1) + c - (an^2 + bn + c)$$

$$= 2an + (a^2 + b)$$

That means that the differences between consecutive terms is a first-degree function of n .

Similarly, the differences between consecutive terms of a 3rd degree function is a 2nd degree function, the 2nd differences is a 1st degree function and the 3rd differences is a 0th degree function (constant!).

Our tables above in fact show that:

If the model is an n^{th} degree polynomial, then the n^{th} difference is constant ... (2)

This is not what we had to prove – (1) is in fact the *converse* of (2)! But because the above reasoning is reversible, we have therefore proven both conjectures!

Note:

- This is a property *only* of polynomial functions. A function like 2^n , i.e. 1, 2, 4, 8, ... can *never* generate a constant difference. So we know that if a difference is *not* constant, the model is *not* a polynomial!
- If the first difference is constant, the formula is of the 1st degree i.e. $y = mx + c$. Algebraically this constant difference is the *gradient* m of the function and this is the reason why the graph is a straight line!
- Of course, a constant difference for discrete values is the *definition* of an Arithmetic Sequence! You should make the connections between sequences and functions, i.e. between $T_n = a + (n - 1)d$ and $f(x) = mx + c$!
- The differences of course represents the *derivative*; We know that $f'(x^n) = nx^{n-1}$, so it is not surprising that the successive difference polynomials are of decreasing degree!
- This theorem is important in the curriculum, e.g. in modelling and to find the equation of the straight line through two points and the equation of the parabola through three points.

³ Also see the Excel development at <http://www.sun.ac.za/mathed/DifferencesA.xls>