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THE GAINS AND THE PITFALLS OF REIFICATION – THE CASE OF ALGEBRA

ABSTRACT. Algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the problem to which they are applied. Not less important, it depends on what one is able to perceive and prepared to notice. It is this last statement which becomes the leading theme of this article. The main focus is on the versatility and adaptability of student's algebraic knowledge.

The analysis is carried out within the framework of the theory of reification according to which there is an inherent process-object duality in the majority of mathematical concepts. It is the basic tenet of our theory that the operational (process-oriented) conception emerges first and that the mathematical objects (structural conceptions) develop afterward through reification of the processes. There is much evidence showing that reification is difficult to achieve.

The nature and the growth of algebraic thinking is first analyzed from an epistemological perspective supported by historical observations. Eventually, its development is presented as a sequence of ever more advanced transitions from operational to structural outlook. This model is subsequently applied to the individual learning. The focus is on two crucial transitions: from the purely operational algebra to the structural algebra 'of a fixed value' (of an unknown) and then from here to the functional algebra (of a variable). The special difficulties experienced by the learner at both these junctions are illustrated with much empirical data coming from a broad range of sources.

When you look at an algebraic expression such as, say, $3(x+5)+1$, what do you see? It depends.

In certain situations you will probably say that this is a concise description of a *computational process*. $3(x+5)+1$ will be seen as a sequence of instructions: Add 5 to the number at hand, multiply the result by three and add 1. In another setting you may feel differently: $3(x+5)+1$ represents a certain *number*. It is the product of a computation rather than the computation itself. Even if this product cannot be specified at the moment due to the fact that the component number x is unknown, it is still a number and the whole expression should be expected to behave like one. If the context changes, $3(x+5)+1$ may become yet another thing: a *function* – a mapping which translates every nubmer x into another. This time, the formula does not represent any fixed (even if unknown) value. Rather, it reflects a change. The things look still more complicated when a letter appears instead of one of the numerical coefficients, like in a $(x+5)+1$. The resulting expression may now be treated as an entire *family of functions* from R to R. Alternatively, one may claim that what hides behind the symbols is a function of two variables, from R^2 to R.

There is, of course, a much simpler way of looking at $3(x+5)+1$: it may be taken at its face value, as a mere *string of symbols* which represents nothing. It is an algebraic object in itself. Although semantically empty, the expression

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may still be manipulated and combined with other expressions of the same type, according to certain well-defined rules.

At this point somebody may wonder whether the above observations on the semantics of algebra are of any practical significance. Indeed it is. For instance, when faced with such an equation as $(p+2q)x^2+x = 5x^2+(3p-q)x$ one will not be able to make a move without knowing whether the equality is supposed to be numerical or functional – whether the question is that of the value of x for which the equality holds (this value should be expressed by means of p and q), or that of the values of the parameters p ad q for which the two functions, $(p+2q)x^2 + x$ and $5x^2 + (3p+q)x$, are equal. The different interpretations will lead to different ways of tackling the problem and to different solutions: in the former case one finds the roots of the equation using the formula $x_{1,2} = (-b \pm \sqrt{\Delta})/(2a)$; in the latter, the values of parameters p and q are sought for which the coefficients of the same powers of x in both expressions are equal ($p + 2q = 5$ and $3p - q = 1$). It is worth noticing that another possible outlook – the one which leaves symbols without much meaning – may initially render the problem-solver helpless. An automatic reaction is likely to follow. A person may be tempted to do what he or she was conditioned to do when faced with a quadratic equation: regardless of the question that has been asked, he or she may have recourse to the formula for the roots.

The plurality of perspectives which one may assume while looking at such a seemingly simple thing as $3(x+5)+1$ is certainly confusing. In the next sections it will be argued that it is also a source of algebra's strength.

1. PRELIMINARIES: ALGEBRA THROUGH THE LENSES OF THE THEORY OF REIFICATION

Algebraic symbols do not speak for themselves. What one actually sees in them depends on the requirements of the specific problem to which they are applied. Not less important, it depends on what one is *prepared* to notice and *able* to perceive. It is this last observation which will be the leading theme of the present article. The main focus will be on the versatility and adaptability of the algebraic knowledge of the student. The question that will be addressed is to what extent the learner is capable of seeing and using the variety of possible interpretations of algebraic constructs. Before tackling this problem, however, let us pause for a moment to reflect on the kind of analysis which is to be carried out here.

The distinctions which we made in the opening examples are quite subtle and refer to what is going on in people's heads rather than to what they actually communicate to the world by means of written records. Indeed, the difference between the alternative interpretations of the equation $(p+q)x^2 + 2x = 3x^2 + (p-q)x$ will not always show on a standard test, since whether the student thinks about the expression as a numerical equality or as a mere string of symbols, he or she may eventually apply the same formulae and perform the same manipulations. A painstakingly detailed scrutiny of student's behaviors and utterances (see, e.g., Schoenfeld et al., 1993; the authors call this kind of analysis 'microgenetic') is necessary to have some insight into his or her thinking.

Our 'fine-grained' analysis will be carried out within a certain theoretical framework by help of which the profusion of loose facts will hopefully turn into a meaningful, manageable whole. It will be called here *the theory of reification* (it must be pointed out that other authors may use somewhat different terminology for basically the same or closely related ideas; see the list of possible names in Harel and Kaput, 1991). Like any theoretical model, it emphasises certain aspects of the explored domain while ignoring many others. Even so, it has already proved itself as a tool for analyzing development of several mathematical concepts, notably the concept of function (Sfard, 1992: Breidenbach et al., 1992); also, it was used to introduce some order into the quickly growing bulk of findings about algebraic thinking (Kieran, 1992). Another glance at algebra through it's lenses will be offered in the next chapters.

In the remainder of this section we will confine ourselves to the presentation of the basic tenet of the theory of reification. Different elements of the resulting system of claims will be discussed throughout this paper. For a more exhaustive presentation of the basic ideas the reader is advised to turn to Sfard (1991) and Kieren (1991). Further, Dubinsky (1991) and Harel and Kaput (1991) describe closely related models. Dubinsky's ideas are an elaboration of Piagetean theory of *reflective abstraction* (Beth and Piaget, 1966); Harel and Kaput's observations bear on Greeno's notion of *conceptual entity* (Greeno, 1983). It should also be mentioned that the idea of the process-object duality of mathematical concepts which is central to this paper reminds Douady's (1985) tool-object dichotomy.

In our opening example, several mathematical *objects* have been identified as possible referents of algebraic expression. We have mentioned a number, a function, a family (set) of functions. One interpretation, however, was of a different nature: when $3(x + 5) + 1$ was read as a series of operations, it was the computational *process* rather than any abstract object (except the processed numbers) which gave meaning to the symbols. What was observed in the example apparently pervades the whole of mathematics: the same representation, the same mathematical concepts, may sometimes be interpreted as processes and at other times as objects; or, to use the language introduced elsewhere (Sfard, 1991), they may be conceived both *operationally* and *structurally*. The fact that these two ostensibly incompatible ways of seeing mathematical constructs seem to be present in any kind of mathematical activity, and thus are complementary, is the basic observation which constitutes the point of departure for our model of mathematical learning and problem-solving. The notion of complementarity is used here in the same way in which it appears (due to Niels Bohr) in physics, where to fully account for observed phenomena one must treat subatomic entities both as material particles and as waves.

The distinction between the two models of thinking, operational and structural, is delicate and not always easy to make. The ability to perceive mathematics in this dual way makes the universe of abstract ideas into the image of the material world: like in real life, the actions performed here have their 'raw materials' and their products in the form of entities that are treated as genuine, permanent objects. Unlike in real life, however, a closer look at these entities will reveal that they cannot be separated from the processes themselves as self-sustained beings. Such abstract objects like $\sqrt{-1}$, -2 or the function $3(x+5)+1$ are the result of a different way of looking on the procedures of extracting the square root from -1, of subtracting 2, and of mapping the real numbers onto themselves through a linear transformation, respectively. Thus, mathematical objects are an outcome of *reification* – of our mind's eye's ability to envision the result of processes as permanent entities in their own right.

This basic ontological observation has numerous theoretical implications. It generates a whole system of claims about mathematical problem-solving; it gives rise to a model of concept formation which applies to historical development as well as to individual learning; it provides its own explanation of the difficulties experienced by a student exposed to a new mathematical idea. All these topics will be discussed in what follows here, in the context of algebra. In section 2 we begin our discussion with an epistemological analysis sypported by historical observations. In section 3, we shall assume a psychological perspective and will try to show that to great extent, the model of algebra's formation constructed through the historical and epistemological analysis fits processes accompanying individual learning.

2. WHAT ALGEBRA IS AND HOW IT DEVELOPED

Many theoretical and empirical arguments may be employed to show that in mathematics, operational conception precedes the structural. What is conceived as a process at one level becomes an object at a higher level (see, e.g., Sfard, 1991, 1992). Kaput (1989) seems to make a similar observation when talking about "mental entity building through reification of actions, procedures, and concepts into phenomenological objects which can then serve as the basis for new actions, procedures, and concepts at a higher level of organization" (p. 168). Even though the point of departure for this statement may be quite different from ours (Kaput's ideas seem to bear on Piaget's theory of reflective abstraction), it points to the basic agreement about the roles of mathematical processes and objects, and about their mutual dependence. Freudenthal was one of the most outspoken proponents of the vision of mathematics as a hierarchy of alternating perspectives: "My analysis of mathematical learning process has unveiled levels in the learning process where mathematics acted out on one level becomes mathematics observed on the next" (Freudenthal, 1978, p. 33). Once again, although originating in a different kind of analysis, this assertion points to the same fundamental characteristics of mathematical construction as those implied by our theory: it underlines the fact that mathematics is a multi-level structure where basically the same ideas are viewed differently when observed from different positions.

As we have already pointed out, the process-object duality should be conveyed by algebraic constructs. For several reasons, with a common root in the rigidity of people's ontological attitudes, the ability to grasp the structural aspect is not easy to achieve. Therefore, those crucial junctions in the development of mathematics where a transition from one level to another takes place are the most problematic, and clearly the most interesting. To use Freudenthal's words once again, "If learning process is to be observed, the moments that count are its *discontinuities*, the jumps in the learning process" (p. 78). Thus, in the analysis that follows, the focus is on the singular points in the development of algebraic concepts – at those points where the ontological perspective must undergo an accommodation to make further progress possible.

Let us precede our account of the development of algebra with a remark on the nature of our investigation. The construction of algebra, like that of any other domain of human knowledge, may be scrutinized from several perspectives. One may focus on the logical structure of the discipline and ask about the way the different items of knowledge combine into one coherent system. Let us call this kind of analysis *logical*. Then, there is a *historical* approach, which concentrates on the collective efforts invested through ages into the construction of the given system of concepts. Finally, a researcher may choose to make an inquiry into the *cognitive processes* which constitute individual learning. One can hardly expect that these three kinds of analysis would yield the same result. As some writers put it, it is but a myth that "[t]he [logical] structure of mathematics accurately reflects its history" (Crowe, 1988). One should also be careful not to make automatic projections from history to psychology. After all, the deliberately guided process of reconstruction may not follow the meandering path of those who were the first travellers through an untrodden area. Even so, one may also expect some striking similarities between the pictures obtained through the different kinds of analysis. Garcia and Piaget (1989) made a particularly strong case for the analogy between the historical and psychological developments. Although much caution is advisable, logical analysis should not be dismissed altogether as a potential source of insights about the process of learning. After all, mathematics is a hierarchical structure in which some strata cannot be built before another has been completed. Thus, after making certain fundamental claims regarding the development of algebra on the basis of logical, ontological, and historical analysis, we will show in the next section that they also hold for individual learning.

2.1. *Algebra as Generalized Arithmetic: the Operational Phase*

Throughout our analysis we will call some kinds of algebra 'operational' and other kinds of algebra 'structural'. This is not to say that at any of the different stages in algebra's development only one type of ingredient – either operational or structural – was present. The complementary nature of the distinction between processes and objects makes it clear that this would be impossible. Also, it is obvious that if some processes (operational ingredient) are considered, there must be certain objects (structural ingredient) to which these processes are applied. Our claims that certain kinds of algebra were operational in their character while others were structural should be understood as referring to what constituted the *primary*

TABLE I
Rhetoric algebra – Examples

1. *Babylonia, second (?) millenium B.C.* (after Boyer, 1985, p. 34)	
The problem:	Find the side of the square if the area less the side is 14,30 (the numbers are presented on basis 60).
Solution:	Take half of one, which is 0;30, and multiply 0;30 by 0;30 which is 0;15; add this to 14,30 to get 14,30;15. This is the square of 29;30. Now add 0;30 to 29;30, and the result is 30, the side of the square.
2. *Al-Khwarizmi, A.D. 825,* (after Struik, 1986, p. 58)	
The problem:	What is the square which combined with ten of its roots will give a sum total of 39?
Solution:	... take one-half of roots just mentioned. Now the roots in the problem before us are 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course, is 9.

focus of a given type of algebra. Or, to put it differently, a statement that, say, algebra was initially operational in its character means that the most advanced, central, ideas investigated at this stage were still conceived operationally rather than structurally.

The history of algebra brings much support to the thesis about the precedence of the operational over the structural approach. For thousands of years, algebra was nothing more than a science of computational procedures.

From the developmental point of view, algebra is a continuation of arithmetic. Like arithmetic, it deals (at least at its early stages) with numbers and with numerical computations, but it asks questions of a different type and treats the algorithmic manipulations in a more general way. Those who do not view symbolic representation as a necessary characteristic of algebra agree that both in history and in the process of learning mathematics, algebraic thinking appears long before any special notation is introduced. Typically, it begins with the first attempt to find the unknown number on which a given operation was performed and a given result obtained. In this kind of activity, the usual arithmetic practice of applying a computational algorithm to a concrete number must be reversed: instead of computing, say, the sum paid for a given number of pencils and a notebook, one must 'undo' what was done to the number of pencils when the total has been calculated. Although initially quite simple and intuitively immediate, such reversal stops being a trivial matter when word problems of some complexity start to appear.

Let us dwell for a moment on the earliest stages in the development of the domain. First, let us have a look at two typical pieces of ancient and medieval algebra (see Table I).

It is worth noticing that in spite of the three millennia that separate them, the two samples are not much different in their basic characteristics: they involve quadratic equations (or at least they would lead today's problem-solver to such), they use concrete numbers instead of general coefficients, and they present the solution in the form of verbal prescription for finding the unknown. The approach is thus purely operational: the focus is on numerical processes and there is no hint of abstract objects other than numbers.

The *rhetoric algebra* (it is how the verbal algebra is referred to by historians) was practiced from the earliest times till the sixteenth century. This is also the kind of algebra encountered by today's school children well before any formal notation is introduced. Naturally, what children are expected to solve 'rhetorically' is much simpler and the words in which they put their solutions sound differently; still, it is basically the same type of mathematics: verbal and operational.

At this point it must be stressed that the operational character of algebra is not inseparable from its being verbal. It is true that as long as algebraic ideas are dressed in words and in words only, it is difficult to imagine the more advanced structural approach, where the computational processes are considered in their totality from a higher point of view, and where operational and structural slants meet in the same representations. To put it differently, words are not manipulable in the way symbols are. It is this manipulability which makes it possible for algebraic concepts to have the object-like quality. It is the possibility of performing higher-level processes on the processes represented by compact expressions that spurs structural thinking. Thus, introduction of a symbolic notation seems *necessary* for reification. On the other hand, it is *not sufficient* for the transition to the structural mode. As already stated in our first example, operational conceptions may also be conveyed through symbolic representations. In any case, the verbal means perpetuate operational thinking. This may be one of the reasons why the absolute reign of the operational algebra lasted thousands of year.

2.2. *Algebra as Generalised Arithmetic: the Structural Phase*

Before we make a further step into the history of algebra, we should make some preliminary theoretical clarifications.

First, it is important to explicitly stress an important implication of the closing remark in the last section: the history of algebra is not a history of symbols. True, from a certain stage on, algebraic concepts become practically inseparable from symbols just like the artist's conception of a picture or of a sculpture is inseparable from its physical embodiment. Indeed, the basic concepts of modern algebra can hardly be conveyed by any means other than algebraic symbols. Moreover, new algebraic knowledge is constructed through manipulations and investigations of formal expressions, and therefore the changes in symbolism parallel conceptual metamorphoses. Thus, from the moment modern algebraic notation was introduced, the history of algebra and the history of symbols, although certainly different from each other, became so intimately intermingled that it is practically impossible to tell the story of one of them without telling the story of the other.

Another important issue which should be explicated at this point regards the correlation between the advancement from operational to structural algebra and the difficulty of demands which the algebraic ideas place on their user. The point we wish to make is that climbing the hierarchy of algebraic ideas is not necessarily tantamount to increase in the sophistication of a person's thinking. One may even feel tempted to say that the opposite is the case: the transition from operational to structural algebra, although a significant step forward in the degree of abstraction and generality, results in facilitating the performance rather than in adding complexity. Although reification itself may be difficult to achieve, once it happens, its benefits become immediately obvious. The decrease in difficulty and the increase in manipulability is immense. What happens in such a transition may be compared to what takes place when a person who is carrying many different objects loose in her hands decides to put all the load in a bag. To fully appreciate the facilitating impact of reification (attained through an appropriate symbolization), it is enough to have a glimpse at the samples of rhetoric algebra presented in Table I. The algorithms which seem so easy and obvious when performed through formal manipulations on concise formulae become very intricate when dressed in verbal, purely operational, representation. Thus, the medieval mathematicians deserve our highest esteem for systematically dealing with problems as advanced and complex as cubic and quartic equations without the ingenious apparatus of structural symbolic algebra (the complex algorithms were presented in a rhetoric manner by Cardan in his famous *Ars Magna* of 1545). As far as their capabilities and the sophistication of thinking are concerned, they can only be compared to the best of today's mathematicians dealing with the most advanced problems of modern mathematics. We should keep all these facts in mind later, when focusing on issues of individual learning and problem solving.

2.2.1. *Algebra of a fixed value (of an unknown).* As previously stated, algebraic symbolism is unrivalled in its power to squeeze the operationally conceived ideas into compact chunks and thus in its potential to make the information easier to comprehend and manipulate. If introduced earlier, the parsimonious notation could have changed the rate of algebra's development, something that the science of computations seems to have needed badly since the day it was born. In comparison to geometry, where the means for structural thinking in the form of graphic representations were readily available, the progress of algebra was slow and hesitant. By the end of the sixteenth century algebra approached such a degree of complexity that without a transition to a structural mode its further development would have been stymied. Historians of mathematics have often wondered why the thinkers of the past, having such strong incentives for a radical change of the method, did not come across the idea of non-verbal representations much earlier.

Although so natural to us, to them the concept of symbolic notation was evidently not at all obvious. In fact, the difficulty lies probably not so much in the idea of using letters instead of numbers and operations (these appeared from time to time even in ancient writings), as in the necessity to imbue the symbolic formulae with the double meaning: that of computational procedures and that of the objects produced. In arithmetic it is easy to keep these two meanings separate by putting them in different expressions: $2 + 3$ denotes the operation, 5 is the outcome. No such separation is possible in algebra, in an expression like $a + b$ or $3(x + 5) + 1$. Here, the process cannot be actually formed; no added value results from the operations. The formula, with its operational aspect salient (it contains 'prompts' for action in the form of operators), must be also interpreted as the product of the process it represents. Even our most abstract thinking, however, is shaped by metaphors provided by sensory experience (Lakoff and Johnson, 1980), and this experience speaks with force against the idea of a process which produces no added value and ends up being treated as its own product. Indeed, nothing like that is possible in real life: we just cannot eat a *recipe* for a cake pretending it is the cake itself (even though we can *imagine* the cake or ourselves eating the cake)! Thus, our intuition rebels against the operational-structural duality of algebraic symbols, at least initially. (The disbelief with which new kinds of numbers were invariably greeted throughout history is another example of a phenomenon which may probably be ascribed to the same ontological dissonance: such objects like 3/4, -2 or $\sqrt{-1}$ were born out of operations of division, subtraction, and square-root extraction which did not seem to produce anything at all.)

True, once we manage to overcome this difficulty, it is quickly forgotten. To those who are well versed in algebraic manipulation (teachers among them), it may soon become totally imperceptible. Our eyes are easily blinded by habit and by our own ontological beliefs. Nevertheless, much evidence for the difficulty of reification may also be found in today's classroom, provided those who listen to the students are open-minded enough to grasp the ontological gap between themselves and the less experienced learners. In sections 3.2 and 3.3 we shall substantiate this claim with many examples.

Historical facts indicate that the idea of operational-structural duality was also difficult for generations of mathematicians. It was probably Diophantus (c. 250 A.D.) who made the first significant step in the direction of a structural approach to computational procedures. By systematically intermingling letters with words he created for himself a special brand of algebra, known as *syncopated*. While solving word problems, he constructed such expressions as $10-x$ and $10+x$ (in fact, he wrote equivalent strings of Greek letters) and manipulated them as if they were genuine numbers (e.g. he multiplied them obtaining $100 - x^2$; see Fauvel and Grey, 1987, p. 218). The fact that thirteen centuries after Diophantus mathematicians still preferred the awkward verbosity of rhetoric algebra bespeaks the inherent difficulty of his way of thinking.

Diophantus did not go beyond the use of algebraic expressions in which a letter denoted an unknown but fixed value, and where the resulting expressions represented the numbers obtained by combining the unknown with other numbers. We shall say that what he developed was algebra *of a fixed value*, as opposed to *functional algebra*, where letters represent changing rather than constant magnitudes. The idea of a letter as variable – as a symbol instead of which any number may be substituted – so obvious to us nowadays, never occurred to Diophantus. To demonstrate his solution to a problem as, say, "Find two numbers given their sum and product", he used to choose concrete numbers as givens. It seems that the idea of an algebraic expression as a representation of the final result is something quite different – and more difficult to accept – than using formulae as temporary representations of manipulations on the unknown. It certainly requires a full-blown structural view of algebraic expressions – the ability to relate to the above parametric formula as if it was a number and not just an operation which could not be implemented.

2.2.2. *Functional algebra (of a variable).* It was not until the sixteenth century that algebraic expressions came to denote functions rather than fixed values. The breakthrough occurred in several steps, the first of which was introduction of special symbols for operations and relations, followed by the idea of a letter as a parameter (a given).

The French mathematician François Viéte (1540–1603) was the first to replace numerical givens with symbols. This invention led to a far-reaching conceptual change in algebra. First, the process-product duality of an algebraic expression almost imposed itself on the mathematicians since it was a part and parcel of the idea of using letters as unspecified numbers (operations on letters, say $3(x+5)+1$, could not be actually implemented, so in order to proceed and do something to the resulting number one had no choice but to refer to the formula as if it stood also for the product of the computation). Second, once the letteral formulae were accepted as representing certain objects, a formal algebraic calculus was created which, among other things, specified the ways equations should be manipulated to be solved. This was a drastic change in comparison to the operational algebra, where problems were solved mainly by reversing computational processes. Third, after the new invention was transferred (mainly by Descartes and Fermat) to geometry to serve as an alternative to the standard graphic representations, and then applied in science (by Galileo, Newton, and Leibniz, among others) to represent natural phenomena, algebra was ultimately transformed from a science of constant quantities into a science of changing magnitudes. By that time, the quest for logical foundations of algebra began. The meaning of algebraic expressions and of their symbolic ingredients was found to be elusive and hard to capture into a mathematical definition. Such names as 'generalized number' or 'variable number' were soon disqualified as lacking precision (see, e.g., Frege, 1970). The problem was eventually solved by abandoning the idea of defining the variable as such and by offering instead an interpretation for an algebraic formula in its totality. *Function*, a new kind of abstract mathematical object, was created to serve as a referent for such expressions as $3(x+5)+1$ or x^2+2y+5.

The problematic nature of the new concept, noted and analyzed in detail by historians and by psychologists (see, e.g., Kleiner, 1989; Dubinsky and Harel, 1992), is a separate theme on which we will not elaborate in the present paper (another article in this volume is devoted exclusively to this issue). As it has much bearing on our subject, however, an understanding of the inherent difficulty of the notion of function is necessary for those who wish to have a deep insight into the process of learning algebra.

2.3. *Abstract Algebra: Algebra of Formal Operations and Algebra of Abstract Structures*

Abstract objects, such as different kinds of numbers or functions, emerge at these junctions in the development of mathematical knowledge where some new processes are introduced, which are to be applied to certain other, already well-known, processes. An abstract object mediates between the two: it may be viewed as a product of the lower-level process and it lends itself to the higher-level manipulations. Thus, with respect to a given object, these two, lower- and higher-level, processes may be called, respectively, *primary* and *secondary*. For example, the idea of rational number originates in dividing integers by integers (primary process), but an entity like 3/4 fully crystallizes as a number in its own right only when the rules of manipulating it and combining it with other numbers (secondary processes) are established. In algebra, primary processes are the arithmetical operations on numbers, the secondary processes are those for which these numerical operations serve as inputs. The latter kind of processes expresses itself in manipulations on algebraic formulae. Thus, the idea of function constitutes a conceptual bond between numerical calculations and formal algebraic manipulations. It acts as a link through which new algebraic knowledge is tied to the system of arithmetical concepts.

After the Vietean type of algebra established itself as the leading tool for doing mathematics, the next step was to climb to a higher point of view from which the present secondary operations – those performed on functions and expressed in manipulations on formulae – could be watched in their totality and become an object of a systematic study (this is a typical development: a process which is regarded as secondary at one level will become primary at a higher level).

This stage in the development of algebra began in Britain in the third decade of the nineteenth century. Although from now on the story goes well beyond school algebra, it is worth telling for reasons which will become clear when today's students' views on symbolic formulae and equations will be analyzed (section 3.3).

Until the nineteenth century, algebra was regarded as a "universal arithmetic" – a discipline which specialized in expressing in a general way the rules governing arithmetical procedures. However, this interpretation greatly limited the scope and the force of the operations on algebraic formulae (e.g., a restriction $a > b$ was a necessary supplement to the expression $\sqrt{a-b}$). Now, when the focus of attention was shifted to the formal manipulations themselves, mathematicians felt an urge to set algebra free from any confinements. A group of British formalists (A. de Morgan, G. Peacock and D. F. Gregory) suggested that algebra should be relieved from the ballast of its original interpretation. From now on, an algebraic formula should be treated as a thing in itself, interpretable in many different ways but devoid of a meaning of its own. The algebraic expression became an empty vehicle waiting to carry an arbitrary semantic load. The formalist school was interested not so much in the potential 'cargo' as in the rules that governed the movements of the vehicles. As Gregory put it, algebra was to become a science "which treats the combinations of operations defined not by their nature, that is by what they are or what they do, but by the laws of combinations to which they are subject" (Gregory, 1840; as quoted by Novy, 1973, p. 194). Here, the word *operations* is used to denote primary processes, while *combinations* are clearly the secondary processes. Thus, the British formalists initiated a new, higher-level operational stage in algebra. It was the first step in the development of *abstract algebra.*

Although the story of algebra does not end here, it is where our historical account stops. The science of abstract structures such as groups, rings, fields or ideals, initially developed in the nineteenth century, does not belong to our subject as it is not taught at secondary level. Just to complete the picture let us remark that with the advent of group theory, a new structural phase began – a natural successor to the operational higher-level algebra of the British formalists.

The numerous stages in the development of algebra are sumarized in Table II. The scheme reinforces the claim that was made at the very outset: algebra is a hierarchical structure in which what is conceived operationally at one level must be perceived structurally at a higher level. Understanding the nuances of the different interpretations of algebraic expressions and their mutual relations is very important for our further discussion, in which the learning of algebra by today's school children will be analyzed.

3. THE DEVELOPMENT OF ALGEBRAIC THINKING – PSYCHOLOGICAL PERSPECTIVE

3.1. *Preliminary Remarks: Competence in Algebra as a Function of Versatility and Adaptability in the Interpretation of Symbols*

As we have shown in the above historical account, the development of the long sequence of possible approaches to algebra and to its symbolic constructs took thousands of years. Today, to solve one little problem from a standard textbook, the learner must often resort to all the different perspectives together. The example in Table III shows the meandering route through diverse outlooks that one has to take to solve the parametric equation discussed in the introduction to this paper. The problem-solver oscillates between the operational and structural approach, and between one structural interpretation to another.

Some 'real life' examples of such process-object swinging in algebraic problem solving may be found in Moschkovich et al. (1992). Gray and Tall (1991) point to a similar phenomenon in arithmetic. Mason (1989) notes the frequent occurrence and the importance of the "delicate shift of attention from seeing an [algebraic] expression *as* an expression of generality, to seeing the expression *as* an object or property". All the researchers agree than the "flexibility [of the perspective] is a hallmark of competence" (Moschkovich et al., 1992). The reason for this, as suggested by the theory of reification, may be summarized as follows (for a

TABLE II
Stages in the development of algebra

Type	Stage	New focus on	Representation	Historical highlights
1. Generalized Arithmetic	1.1. Operational	1.1.1. Numeric **computations**	Verbal *(rhetoric)*	Rhind papyrus, c. 1650 B.C.
			Mixed: verbal+symbolic *(syncopated)*	Diophantus, c. 250 A.D.
	1.2. Structural	1.2.1. (Numeric) **product** of computations (*'algebra of a fixed value'*)	Symbolic (letter as an **unknown**)	16th century, mainly Viete (1540–1603)
		1.2.2. (Numeric) **function** (*'functional) algebra'*)	Symbolic (letter as a **variable**)	Viete, Leibniz (1646–1716), Newton (1642–1727)
2. Abstract Algebra	2.1. Operational	**Processes on symbols** (combinations of operations)	Symbolic (no meaning to a letter)	British formalist school (de Morgan, Peacock, Gregory), since 1830
	2.2. Structural	**Abstract structures**	Symbolic	19th and 20th century: theories of groups, rings fields, etc., linear algebra

much more comprehensible treatment of this issue, see Sfard, 1987, 1991, 1992). The operational way of thinking dictates the actual actions to be taken to solve the problem at hand, while the structural approach condenses the information and broadens the view. The abstract objects serve as landmarks with the help of which the problem-solving process may be navigated. Since a jump from operational to structural mode of thinking means a transition from detailed and diffuse to general and concise – from the foot of a mountain to its top – it is only natural that it is accompanied by an increase in student's ability to cope with the task at hand.

Now, let us take a closer look at the notion of flexibility purported to be the source of competence. This particular trait of algebraic thinking seems to be a function of two parameters: the *versatility* of the available interpretations,

TABLE III

Oscillating between approaches when solving an algebraic problem

The problem: For what values of the parameters p and q the equation $(p+2q)x^2 + x = 5x^2 + (3p-q)x$ holds for *every* value of x?

A possible solution

A step in the solution (decision, operation)		The approach applied	
(1)	Each of the component formulae represents a family of quadratic functions. The task is to find these members of the two families that are equal to each other. Two polynomial functions are equal if the coefficients of the same powers of x are equal. Thus, to answer the question we have to solve the system of equations: $p+2q=5$ $3p-q=1$	(1)	Here, the formula is interpreted as representing a **family of functions** (one may think about two **parabolas** and ask for what p and q these two curves overlap)
(2)	Let's start with solving the first equation with respect to p: $p=5-2q$ $3p-q=1$	(2)	*First interpretation* $p+2q=5$ is seen as **a string of symbols** to be manipulated according to rules *Second interpretation* $p+2q$ is **a number**; subtraction of $2q$ (also a number) from $p+2q$ and 5 preserves the equality.
(3)	Let's substitute $5-2q$ instead of p in the second equation $p=5-2q$ $3(5-2q)-q=1$	(3)	$5-2q$ is treated as **a number** (a product of the process which it represents)
(4)	Let's solve the second equation with respect to q: $3(5-2q)-q = 1$ $15-6q-q = 1$ $15-7q = 1$ $-7q = -14$ $q = 2$	(4)	the formulae are seen as **strings of symbols** to be subjected to formal operations, according to the rules
(5)	Let's substitute 2 instead of q in the first equation and compute the value of p $p=5-2*2=5-4=1$	(5)	the expression $5-2q$ which turned into $5-2*2$ is interpreted as **computational process**
(6)	Let's formulate the answer: (the functions) $(p+2q)x^2+x$ and $5x^2+(3p-q)x$ are equal iff $p=1$ and $q=2$	(6)	Back to the **functional** outlook

and the *adaptability* of the perspective. These two parameters appear to be quite independent. In certain circumstances a person may display his or her ability to see an expression as a process, in another context he or she may view it as the product of this process, and in still another situation as a function. One would say, therefore, that the versatility of his or her outlook is quite impressive. This, however, does not necessarily mean that the person will always be able to adapt the perspective to the task at hand. Although such adaptation would sometimes occur as smoothly and imperceptibly as the accommodation of an eye to a changing perspective, in certain circumstances it may be equally difficult as alternating between different perceptions of a cube represented in a two-dimensional picture. For instance, one may be well aware, in principle, that such expressions as $(p + 2q)x^2 + x$ and $5x^2 + (3p - q)x$ can represent functions, but this particular interpretation would not occur to him or her spontaneously at a time of solving the problem presented in Table III. Thus, pointing to the potential versatility of student's conceptions is not enough to arrive at a good assessment of their algebraic competence. Equally important, the adaptability of their outlook should be tested.

It is one of our basic theoretical assumptions that the assortment of perspectives available to the student grows gradually, roughly following the logical-historical path presented in Table II. The hierarchical structure of algebra and of its different interpretations makes this conjecture fairly plausible. A direct jump, say, over the wide gap separating functional algebra from operationally interpreted algebra may end in broken bones. In the discussion that follows, we shall focus our attention on two critical junctions in school algebra: first, we shall consider the transition from purely operational conception of a symbolic formula to the dual process-product interpretation (from cell 1.1.1. to 1.2.1. in Table II); second, we will investigate the passage from here to the functional approach (to cell 1.2.2.). We shall make an effort to find out how difficult these two steps are for the pupil and what phenomena may be regarded as symptoms of such difficulties. In the last section, the looking glass will be turned at the final outcome of schooling: the question of versatility and adaptability of the conceptions with which the students typically leave the school will be addressed.

3.2. *Toward the Structural Outlook*

Although past investigations did not originate in one consistent conceptual framework, their findings, when combined together and re-examined through the lenses of the theory of reification, will often lead to new insights and to a more comprehensive picture of learning. To make these insights even more powerful, we shall reinforce them with samples of interviews and observations carried out by ourselves with children of various ages and competences. The interviews focused on the notion of propositional formula (equation, inequality). Our interviewees included

Group 1: Six seventh-graders (age 12–13) of an average and slightly over-average ability who, by the time we met them for the first time, were already acquainted with the notion of an algebraic expression but not with the concept of equation;

Group 2: Four ninth-graders (age 14–15) of above-average ability who were supposed to be well versed in basic algebra, including linear and quadratic equations and linear inequalities, and were familiar with the notion of function in general and with linear functions in particular;

Group 3: Four tenth-graders (age 15–16) of above-average ability who had had a long experience with algebra in many different contexts, including analytic geometry and calculus (thus, could be expected to be well acquainted with the functional approach).

There was also another kind of inquiry. Each of the children from Group 1, after he or she was taught to solve linear equations of the type $ax + b = cx + d$, was asked to help in explaining the subject to his or her peer for whom the whole issue of equations was still completely new. In this way, we hoped to get another perspective on the children's thinking. We believed that since the necessity to convince an uninitiated is a strong motivational force, listening to children giving explanations to other children may be a more powerful method of inquiry than asking direct questions. All the interviews and the teaching meetings were recorded either on video- or on audio-tape.

It must be emphasized that the present paper is by no means a systematic report of the above research, nor an attempt at a comprehensible presentation of all the results. The large-scale study from which our interviews were but a small part will be summarized elsewhere. Here we shall only avail ourselves of those selected samples which bring our message with particular clarity.

3.2.1. *First step: toward algebra of a fixed value (recognition of a process-product duality).* Since the transition from purely operational outlook to the dual, process-product interpretation of algebraic formulae occurs in close vicinity to the point at which arithmetic meets with algebra, much data relevant to our topic may be found in the research devoted to this crucially important junction. In the following brief summary, enriched with our own observations, we shall try to find out how the transition from operational algebra to structural algebra of a fixed value expresses itself in student's behavior. With this goal in mind, we shall analyze pupil's initial understanding of formulae, of the equality sign, and of equations.

It is one of our basic theoretical theses that an operational conception naturally precedes a structural. Findings collected by ourselves and by other researchers gave this supposition strong support: it seems that even without direct intervention from a teacher, the learners initially interpreted algebraic expressions as computational processes. The following are only a few out of many examples that may be brought to illustrate this point.

Those who listen carefully to the language used by the pupils will usually notice that the way many children initially refer to algebraic expressions indicates their operational outlook. It may be easily seen in the interviews quoted by Booth (1988, p. 21). For instance, a child asked by an interviewer to write down the length of a space-ship's path composed of y 11-light-years long segments said: "What, shall I write what I would do?"; and after she eventually contrived the formula $11 * y$, she exclaimed to the interviewer: "What, is that all it was? Why didn't you say so? I thought you wanted an answer." Thus, for this child the expression was a mere prescription for the sought-for quantity, not the quantity itself. In our own teaching experiment with children of Group 1, such references to algebraic expressions as 'this is an exercise that must be done' were noted time and again even when a pupil was explaining what this 'thing' (say $8x$) was which he or she decided to subtract from both sides of an equation. The following fragment of a dialogue between the interviewer (I) and the Group-1 pupil Ayala (A) is particularly enlightening, as it indicates that the operational approach to algebraic formulae is inherited from arithmetic. Ayala was trying to explain how her friend Irit solved an equation.

I: How did Irit go from here [$15x = 8x + 35$] to here [$7x = 35$]?

A: She subtracted an exercise, 8 times x, and she subtracted it also from the other side of the equation.

I: What do you mean when you say 'exercise'?

A: 8 times x is an exercise, it is something you must do. She takes off this exercise. It's like when you have $1 + 2 + 4 = 3 + 4$. Then you can take off 3, and then at the other side you take off 'one plus two'.

I: $1 + 2$ is an exercise and 3 is not an exercise?

A: 3 is a number, it's a result of an exercise. This [points to $1 + 2$] is the exercise: and $8x$ is an exercise and $15x$ is an exercise. We subtract the same exercise from both sides so that what is left is the same.

(Ayala's statements indicate that she is somehow half-way between operational and structural outlook: she already manipulates algebraic expressions as if they were objects, but the language she uses is still operational.)

The conviction that a formula is nothing else than a process waiting to be performed may be responsible for what Collis (1974) called the inability to accept the lack of closure – a student's difficulty with complex expressions not followed by an equality sign and not complemented by the 'result' of the computation written on the other side of this sign (see also Chalouh and Herscovics, 1988). This inability may be responsible for the striking result obtained at the British national survey of 15 year olds (as quoted by Bell, 1992): a wide gap in students' performance was noted in the following, seemingly not too different, problems: 'What is x if $2x+7 = 45$?' and 'If $A = L * B$ tells us how to work out A, what formula tells us how to work out L?' The success rates on these two questions were 73% and 39%, respectively. The dramatic difference may probably be ascribed to the fact that in the second problem, in which some of the letters played the role of parameters ('givens'), the final result (the value of L) would have to be presented by means of a formula and not of a number. This must have seem unacceptable to those for whom an algebraic expression was still only a process (as it probably was even for Diophantus, who did not refrain from using formulae to present intermediary calculations, but would not use them for the final outcome). Similar difficulty was experienced by Gay, a 15 year old pupil (group 2), when he was trying to simplify the expression $kx - x = -2$. Although Gay had already two years of algebra behind him, and he was regarded by all his past and present teachers as talented in mathematics, he could not cope with the problem. No idea seemed to come to his mind.

I: $kx - x$ – can't you present it in a different way?

G: No. There is a multiplication here, kx, so what can I do?

I: And if I wrote $3x - x$, would you be able to do anything?

G: $3x - x$? It's $2x$.

I: So? Isn't $kx - x$ similar?

G: But this ... but this does not work ... I don't know the value of k.

I: So what?

G: So what can I write? $k - x$?

I: What have you done here [$3x - x$] to get $2x$? What did you do to 3?

G: I subtracted 1.

I: So?

G: So what? Shall I take one off? I don't know ... If I subtract 1 from k I'll be left with the same mess. As if it was ... See, I don't know how to write it.

I: Here [$3x - x$] you subtracted one from 3 and multiplied x by the result, right? Here [$kx - x$] you subtract one from k and...

G: Multiply by x ... But how do I subtract one from k? How do I write it? $k - 1$?

Not surprisingly, when algebraic expressions are seen as processes rather than objects, the equality sign is interpreted as a 'do something signal' (Behr et al., 1976; Kieran, 1981) and not as a symbol of a static relation. The expression on the left-hand side is a process, whereas the expression on the right-hand side must be a result. Once again, the idea seems to come from arithmetic where the sign '=' is used as a prompt for the implementation of a 'program' appearing to the left of this sign. Nowadays, this outlook may get additional reinforcement from the fact that this is exactly the way we use the '=' key in hand-held calculators. It serves here as a 'run' command. When treated in this way, the equality symbol looses the basic characteristics of an equivalence predicate: it stops being symmetrical or transitive. Indeed, young children seem to have no qualms about solving word problems with the help of a chain of non-transitive equalities. For instance, when asked 'How many marbles do you have after you win 4 marbles 3 times and

2 marbles 5 times?', the child would often write:

$$3 * 4 = 12 + 5 * 2 = 12 + 10 = 22$$

(see also Vergnaud et al., 1979). The 'one-way', non-symmetric approach to the equality sign, so symptomatic of the operational perspective on algebra manifested itself with force in the following little incident between a Group-1 pupil Danna and her friend and 'student' Zohar. After Zohar had successfully solved the equation $7x + 157 = 248$, she looked baffled and stymied when presented with the next example, $112 = 12x + 47$. Whereas the observer was puzzled, Danna immediately suggested the reason for Zohar's helplessness. "She doesn't know what to do because the order of the equation confuses her, it's not like it should be", she said.

The spontaneity of the operational outlook is exhibited in the easiness with which many young children handle simple linear equations of the form $ax+b=c$. As was noted more than once in different studies (e.g., Kieran, 1988, 1992; Filloy and Rojano, 1989), for young pupils it is often intuitively obvious that in order to solve this kind of problem, one must just 'undo' what had been done to the unknown. We have witnessed many examples of such spontaneous reversal of the computation in our first interviews with Group-1 pupils. The following little story is one of them. When Snir came to the interview, he knew nothing about equations. The interviewer mentioned to him that in an equation like $7x + 157 = 248$, $7x$ means '7 times x' and without any further explanations asked for a solution. Snir immediately concluded: "Here, I have to find a number so that 7 times this number plus 157 is 248. First, 248 minus 157 is 91. Now, the number times seven ... 91 divided by 7 is 13. The number is 13." The operational character of Snir's algebra is underlined by its rhetorical presentation.

All the above observations point in the same direction: the operational outlook in algebra is fundamental and the structural approach does not develop immediately. Moreover, as we have shown in our historical outline and reinforced with theoretical argumentation, there is an inherent difficulty in the idea of process-object duality – of a recipe which must also be regarded as representing its own product. This difficulty cannot be expected to disappear without some struggle.

Davis (1975) was probably one of the first writers who realized the significance of what came to be known in literature as the 'name-process dilemma' (in fact, the term 'process-product dilemma' seems more adequate, as there is no indication that the student distinguishes between the name of a thing and the thing itself; incidentally, the inability to sever a sign from the signified may be one of the reasons why the duality of algebraic expressions is sometimes so difficult to grasp). Davis pointed out to what until now might have well gone unnoticed by the majority of teachers: the idea of duality is not self-evident and may be perplexing for a student.

A particularly convincing symptom of this difficulty may be the phenomenon of 'didactic cut' in learning to solve equations, noted by Filloy and Rojano (1985, 1989) and confirmed by others (e.g., Herscovics and Linchevski, 1991, 1993).

These researchers discovered that, whereas the solution of an equation of the form $ax + b = c$ is intuitively accessible to most pupils, the equation with an unknown appearing on both sides, such as $ax + b = cx$ or $ax + b = cx + d$, evidently poses a problem. Since in the former equation the equality sign still functions like in arithmetic – operations on one side and the result on the other – they called it 'arithmetical'.

On the grounds of our general claims and assumptions, the apparent difficulty at this particular point in learning is not surprising. The cut runs along the demarcation line between operational and structural algebra. As long as only arithmetical equations were concerned, there was no need to hold the dual process-product outlook. The computational operations and their results remained separated by an equality sign and each side of an equation preserved its particular ontological identity – that of a process and that of an object, respectively. This division of roles is no longer in force in the non-arithmetical equation. The expression on the right-hand side, expected to be a *product* of the left-hand expression, is in fact a process. Without the dual outlook, which would turn this last expression into an object, the equation does not make much sense. This may be clearly seen in the following typical dialogue, taken from the interview with Snir, a thirteen year old pupil of more than average ability (Group 1). Snir, for the first time, is faced with a non-arithmetical equation $15x + 12 = 8x + 47$.

S: One must find something ... that when I multiply 15 by a number and I add 12 to it, it will be equal to eight times a number, and I add 47 to it. Let's start with something simple. Here, times 3 and times 1 [writes 3 over x on the left-hand side and 1 over x on the right-hand side]. No, it's not that ... I don't know.

I: What are you looking for when you are solving the equation?

S: We must find two things. Something that when I multiply it or divide it ... doesn't matter ... will be equal to the other thing. One must find the way of making them equal, there are two equations here.

Much can be learned from this little exchange. It was difficult for Snir to make sense of the equation which looked to him like 'two equations', two processes waiting to be performed while the relation between them was not clear at all (at this point, Ayala said that there are 'two exercises' here). The meaning of the equality sign here, so evident to the experienced person, turned out to be far from obvious for the beginner: what aspects of the two processes should be equal? The object operated upon? The object obtained? Or maybe both? Interestingly enough, for Snir it was obviously the equality of results that was required, whereas the numbers for which this equality would hold could be different. Indeed, he looked for two different values of x on the two sides of equation. We were initially perplexed by this interpretation, and even more so when we noticed the same unexpected approach in almost all interviews with Group-1 children and with their peer-students. The phenomenon seemed surprising because at this stage of learning the pupils already knew the convention that different occurrences of the same letter in a given expression signify the same number. It seems that this principle collapsed in the face of an equality which could not be interpreted on the basis of the previous knowledge (as was stated by Herscovics and Linchevski, 1991, the problem of the double interpretation may be easily solved by an explicit statement of the convention regarding different appearances of the same letter in a given equation; even so, we find it significant that this restriction does not occur to the student spontaneously in spite of former instruction).

The difficulty with non-arithmetical equations becomes even more visible when such an equation must be solved. Here, the technique of 'undoing' no longer works. The structural conception of an algebraic formula is a prerequisite for the comprehension of the strategy that must be used – that of adding, subtracting, multiplying and dividing both sides by the same expression. Indeed, the idea can only be accepted by those for whom the sides of an equation and the expressions with which these sides are operated upon are objects while the equality sign is a symbol of equivalence. That this is not always the case may be seen from the following exchange between Danna and Zohar, the two Group-1 students quoted above:

D: [To solve $15x = 8x + 35$] you subtract $8x$ from both sides now.

Z: But I don't know how much is $8x$, so how can I subtract it? ... I don't even know whether the x's on both sides are equal.

The utterances by Zohar and similar statements by other students leave little doubt as to the source of the difficulty they experienced at this point. The children were not able to relate to a formula as a representation of a ready-made object. For them it was still a process, and how can a process be subtracted from another process? (In fact, Ayala's utterances quoted above hint at a possibility that some children may be able to put up with the idea of 'arithmetic operation' on processes; the question is whether this kind of understanding is really consistent and effective).

The last point to be made in this section regards solving word problems using equations. Equation requires suspension of actual calculations for the sake of static description of relationships between quantities. This approach does not comply with the pruely operational view of algebra.

Moreover, the declarative, structural mode of the algebraic presentation reverses the order in which the operations must be performed if the 'answer' is to be found. These may be the reasons why in many studies, even quite advanced students have been found to prefer verbal-operational mode of solving word problems rather than symbolic-structural way of putting things (Clement et al., 1979; Soloway et al., 1982; Sfard, 1987; Harper, 1987).

Let us end this section with a quotation from Davis (1975):

> Many major cognitive adjustments – 'accommodations' rather than 'assimilations' – are required if one is to do the necessary mental flip-flop and start seeing the equal sign in new ways, and even seeing $3/x$ as an 'answer' instead of a problem. It is not entirely clear that this whole new point of view can be acquired by gentle accumulation of small increments. It may resemble the

geological phenomenon of earthquake more than the phenomenon of erosion or dust deposit. [p. 29]

The conjecture ventured here by the author is perfectly in tune with the theory of reification: the transition from purely operational to a dual process-object outlook is probably not a gradual smooth movement toward a higher level. Like any reification it is likely to be a quantum leap toward a higher vantage-point.

3.2.2. *Second step: toward the functional algebra.* The passage from the algebra of a fixed-value (of an unknown) to the functional algebra (of a variable) is not as well documented in the literature as the previous transition – that from purely operational to the dual, process-product, approach. True, much has been written about the notion of function – about the way it develops and about the difficulty with which it is acquired and applied by the majority of people. Although all this is of some relevance to our present subject, it does not provide us with the kind of direct information that is needed to understand the specific problems of the functional approach to algebra. Among the issues that should be addressed are such questions as student's ability to think about algebraic formulae in terms of functions and his or her readiness to apply this outlook whenever appropriate (thus, once again, the problem we are facing is that of versatility and adaptability of pupil's algebraic thinking).

There are several reasons why our present issue is somehow more difficult to cope with than the previous one. For one thing, the significance, or even the very existence, of the particular turning point we are now talking about may be less clear than the former. Even if recognized, it is quite elusive. Whereas it is quite obvious where to look for the transition from the operational to process-product outlook (after all, it is a part and parcel of the passage from arithmetical to symbolic algebra!), it is not that easy to pinpoint the moment in learning when the functional approach becomes truly necessary. Also, distinguishing one functional approach from any other in student's solutions to standard school problems is certainly not a trivial matter.

On top of it, the modern school curricula often introduce the functional approach almost from the beginning, so that, at the face of it, there is not much sense in talking about a *transition* to this approach. The Israeli method of teaching algebra may serve as an example. An advanced structural outlook is assumed here almost simultaneously with the introduction of algebraic symbols. Let us draw a sketchy picture of the method.

Algebraic expressions are introduced before they become a part of an equation or inequality. The 12–13 year old child begins her or his algebraic education with modelling different 'real-life' situations and numeric relationships using symbolic formulae. At this stage he or she is not requested yet to compute any values, just to describe the state of affairs assuming that all the numbers are given. Thus, from the very start, the letters are applied as *variables* rather than as unknowns. *Equations* and *inequalities* are introduced slightly later, as two different, but closely related, instances of a single mathematical notion: *propositional formula* (PF, from now on). This universal construct is defined as 'a combination of symbols (names of numbers, letters, operators, predicates, and brackets) that turns into a proposition when names of numbers are substituted instead of the letters'. The idea of PF is introduced as early as seventh grade, and then equations and inequalities are dealt with simultaneously. Every PF has its *truth-set* (TS, for short), namely the set of all the substitutions that turn this PF into a true proposition. Any two PFs with the same truth sets are called *equivalent. Solving an equation or an inequality* means finding its TS. As a consequence of this approach, even the solution procedures are described in set-theoretic terms: to solve, say, an equation E, one must find the simplest possible PF which is equivalent to E. The basic steps which may be taken to transform an equation into an equivalent PF are called *elementary (permissible) operations* (in our language, these are the secondary processes of school algebra).

Although the concept of function is officially introduced some time later (eight grade, sometimes only ninth grade), the above method of teaching provides a good example of a structural, functional approach: letter is presented as a variable, an algebraic expression as a function of this variable, and propositional formulae are interpreted as comparisons between functions. Notably, there is much emphasis on graphical representations of the truth sets, which in the case of an equation with two variables, x and y, often leads to the graph of a function $y = f(x)$. This uncompromisingly structural way of dealing with the subject is certainly very attractive due to its mathematical elegance, consistency, and universality. However, since it is somehow at odds with the epistemological and historical order in which the algebraic concepts seem to be related to each other, one cannot be sure that the functional approach is the best slant to begin with.

In a series of interviews carried out in Group 2 (age 14–15, Grade nine) and Group 3 (age 15–16, Grade ten), it was our goal to assess students' familiarity with the functional approach and their competence in using it in different contexts. In preliminary talks with our interviewees, we tried to ascertain that all of them were beyond the 'didactic cut' and that they had assimilated the fixed-value approach. Indeed, we found that they could cope with many kinds of equations and inequalities and could explain the necessary moves in terms of operations on both sides of a propositional formula (e.g., "Here I added $2x$ to both sides... it is permissible because when I do it on both sides, one thing balances the other and the two sides remain equal").

Our first task was to decide in what kind of problems the functional approach becomes indispensable or at least more helpful than any other outlook. We scrutinized several textbooks and identified three representative examples: a quadratic inequality, a system of equations with an infinite truth set (singular equations) and a system of parametric equations. An explanation for why the functional approach is indeed vital in dealing with each of these problems is given below.

Let us precede the explications and the examples of our findings with a methodological remark. While presenting the interviewee with our three questions, we were well aware that he or she may not yet be acquainted with the techniques for answering some of the problems (e.g., quadratic inequalities). For us, it was an advantage rather than a drawback. Since we were interested in diagnosing the student's ways of interpreting algebraic constructs, we had to prevent his or her conceptions from being buried under mechanical algorithmic behavior, typically displayed in solving standard problems. In theory, even our younger interviewees were endowed with all the information necessary to actually implement even those tasks which were not yet exercised at school. The intriguing question was whether their algebraic knowledge was versatile and adaptable enough to allow them at least to understand the problems. In the following short account of our clinical interviews we will not try to give any statistics or generalizations. We shall confine ourselves to three brief case studies which seem to us most enlightening and quite typical. The fact that all our interviewees were considered by their mathematics teachers as being perceptive and successful makes these three episodes particularly telling.

Problem 1: Quadratic inequality

$$\boxed{\text{Solve:} \quad x^2 + x + 1 > 0}$$

Inequalities were introduced to Alon (age 15) more than a year before we met. By the time of the interview he was already well versed in solving linear inequalities. Thus, for the reasons we explained above, it was clear that presenting him with this kind of problem would not provide us with much information. Like for all Israeli pupils, an inequality was supposed to be for Alon just a special case of a propositional formula, not much different from an equation. To us, however, the former kind of PF seemed more difficult, as it clearly demanded a more advanced structural outlook. Whereas equations may be understood and solved on the grounds of a fixed-value approach, namely by treating a letter as a certain unknown number and each side of an equation as a concrete product of operations on this number, an inequality requires that the values of the component formulae are tested and compared for different values of the letter. Thus, in the inequality, the letter plays the role of a variable and the component expressions – of functions of this variable (it goes without saying that the more basic operational approach is out of question: unlike the equality sign, the symbol '$>$' cannot be interpreted as a 'do something' signal). Perhaps the most efficient way of solving a quadratic inequality such as the one we have chosen is to look at the graph of the appropriate function (in our case: $x^2 + x + 1$) and choose those segments of the x-axis which run below the graph. Although all the pupils we talked to had some previous experience in drawing parabolas, our expectations as to their performance were not very high. A study by Even (1988) with prospective mathematics teachers has already shown that "relating solutions of equations to values of corresponding functions in a graphical representation" is a tough task with which not many learners can cope. Indeed, not even one of our interviewees, whether the younger or the older, resorted to graphing. Not even one of them solved the inequality. The following dialogue between Alon and the interviewer is quite typical.

(1) I: [Pointing to the inequality] What is it? What do we call such a thing?

(2) A: Quadratic equation.

(3) I: Equation?

(4) A: No, inequality.

(5) I: What do we look for when we solve it?

(6) A: We try to find out what the left-hand side is equal to.

(7) I: What do you mean?

(8) A: We check how much greater than zero it is and whether it really is greater than zero.

(9) I: Could you be more precise? What are you looking for? What do you want to get and to write down in the end?

(10) A: That $x^2 + x + 1$... I want to find x^2 and x... then I can substitute and check whether it is true, this equation... this inequality.

(11) I: Say it again. What are you looking for? Chairs, pens, tables?

(12) A: A number.

(13) I: A certain number?

(14) A: Yes, one number.

(15) I: The special number that... what?

(16) A: That I can substitute and find a solution.

(17) I: What do you mean by 'find the solution'?

(18) A: Solution, it means that the inequality is true.

From what Alon is saying it seems that he is 'stuck' in the fixed-value view of algebraic formula. In utterances 6 and 9 he refers to $x^2 + x + 1$ as if it had only one, well defined, value. Statements 10, 12, 14, and 16 also indicate that the letter x is for him just a code name for a certain concrete number. With this approach, Alon could not be expected to be able to cope with the problem. To solve the inequality he turned to some old routines, obviously hoping that they will do the job for him even if he could not tell why.

(19) I: Do you know how to do what you want to do?

(20) A: Maybe... [writes $x_{1,2} = (-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1})/2$]. I have here the square root of -3. There is no solution.

(21) I: So?

(22) A: So this [points to the inequality] is not true.

(23) I: What do you mean?

(24) A: That whatever I substitute, it will be less than zero, or maybe zero, but it won't be more than zero.

Alon was unable to assume the functional approach which would allow him to deal with the inequality in a meaningful way. This handicap led him to a mechanical behavior grounded in habits which he never tried to revise. The mere occurrence of a square root of a negative number was for him a signal that the answer should be 'no solution' (while, in fact, any number fulfills the inequality!). His response was given automatically and there was no attempt on his part to postpone the decision in order to check how it related to the particular contents of the question he was asked.

Problem 2: Singular system of equations

$$\boxed{\text{Solve:}\quad \begin{cases} 2(x-3) = 1-y \\ 2x+y \;\;= 7 \end{cases}}$$

Singularities and the things that happen at the fringes of mathematical definitions are often the most sensitive instruments with which student's understanding of concepts may be probed and measured. This is also the case with regard to singular simultaneous equations.

Without a functional approach to algebraic formulae, one is not likely to realize that a system of linear equations may have infinitely many solutions. If the letters in the equations represent unknown but fixed numbers, how can anybody expect that one or both of these fixed numbers will be 'any number'? Moreover, in the case of two linearly dependent equations, the truth set itself is, in fact, a function: to each value of x there is a corresponding value of y. To be prepared for this possibility, the student must realize that each of the component equations may be considered as representing a function, and the graphs of the two functions can coincide in any number of points. Only if he knows this, will he be capable of interpreting in the correct way the tautological equality, such as $0 = 0$, which is usually obtained as the final output of the routine solving procedure applied to a system of linearly dependent equations.

Unlike in the case of quadratic equations, the issue of singular equations should have been known to all our students, even to the youngest among them. This subject is usually touched upon almost simultaneously with the introduction of systems of equations (Grade nine), and it is given additional treatment at more advanced stages of learning. Even so, some of the interviewees responded to Problem 2 with such answers as: "The truth set is empty" or "x and y may be any numbers" (no functional relationship between x and y specified).

Let us take a closer look at the particular way in which the problem was tackled by 15-year old (ninth grader) Mariella. After several transformations, Mariella arrived at

$$\begin{cases} 2x+y=7 \\ 2x+y=7 \end{cases}$$

(1) M: This is the same equation. Now, shall I solve it? [Starts humming.] Just a moment, $2x$ ... let's take $2x$ off [Writes: $y = 7 - 2x$.] Let's put $7 - 2x$ instead of y ... [Substitutes in the second equation; writes: $2x + (7 - 2x) = 7$; simplifies and arrives at $7 = 7$.]

(2) I: So?

(3) M: So x equals zero [writes $x = 0$]. Do you want y as well?

(4) I: I don't know, you decide. How do you usually complete such a task in the class? What do you write in the end?

(5) M: The solution.

(6) I: Well, so write the solution.

(7) M: If x is 0 then y is 7. Now, I shall put it into the first equation... No, in the second. So, 2 times 0 is 0 ... [arrives at an equality $7 = 7$]. So my solution is true: (0,7).

(8) I: Is it the only solution?

(9) M: Yes.

Instead of constantly monitoring the problem from a more advanced point of view in an attempt to look for meaningful shortcuts, Mariella slips into a less stressful, to her taste obviously more secure, algorithmic mode. At the point where the two equations became identical – the fact that she explicitly noted in (1) – the functional outlook combined with the knowledge of linear functions would have given her an immediate answer. Mariella remains unaware of this simple option and uses the routine substitution method instead.

Similar to Alon, Mariella might be thinking in terms of fixed values rather than of variables and functions. Her moves could be dictated by a tacit assumption that x and y represent specific numbers. Alternatively, she might just be manipulating symbols in a routine way without actually giving any thought to the meaning of the letters. Her expectation to find an expression of the form 'x = number' in the end of the process was strong enough to force her into faulty reasoning (in (3), she concluded that $x = 0$ from $7 = 7$). The conviction as to the general nature of the solution made her insensitive to the mistake. She did not change her mind even when the interviewer interrogated her on the reasons for her decision:

(10) I: How from here [$7 = 7$] did you conclude that $x = 0$?

(11) M: x was cancelled, so I could cancel the 7. And then 0 times x equals 0.

Problem 3: Parametric equations

Is it true that the following system of linear equations:

$$\begin{cases} k - y = 2 \\ x + y = k \end{cases}$$

has a solution for every value of k?

Our own familiarity with variables often makes us insensitive to the subtle difference between equations with numerical coefficients and those with parameters. For people well versed in solving techniques, the question of whether they are operating on numbers or on letters may seem irrelevant. Thus, even for the teachers, the great conceptual difference between the regular kind of equations and parametric equations would not always be clear. (Incidentally, as it often happens with conceptual subtleties, one glance into history could open their eyes.)

In a problem like the present one, the objects that the student is supposed to consider are not just numbers – they are functions. To understand the question, one must realize that each of the equations, $k - y = 2$ and $x + y = k$, represents a whole family of linear functions (or, to put it in different terms, it expresses a family of infinite sets of ordered pairs of numbers), and that for different values of k the system will yield different pairs of such functions. One of the ways to interpret the conjecture presented in Problem 3 is to say that for none of the pairs, the graphs of the two functions are parallel. The conceptual step that must be taken to reach this interpretation may be higher than even the most experienced teachers would guess.

It should be noted that in our particular problem, the functional approach combined with the knowledge of linear functions, if used not only to decipher the question but also to answer it, could lead to an appropriate argument immediately, without any calculations. To show that the truth-set is never empty, it would suffice to realize that the first equation ($k - y = 2$) represents a horizontal straight line, while the graph of the second ($x + y = k$) is oblique. Two such straight lines must meet in a point. An alternative algebraic solution would involve applying the usual solution technique and finding whether some of the operations put any restrictions on the value of k (in this case, there are no such limitations and this explains why the truth set is never empty).

While presenting the problem to our interviewees we had every reason to expect that all of them, and especially the older ones, would at least be able to understand the question. We also had certain hopes that the functional way of proving the claim would occur spontaneously. Indeed, except for the fact that the functional approach to propositional formulae was promoted in the school from the very beginning, all our students were well acquainted with linear functions, with their properties and representations. Moreover, the older pupils had a course on analytic geometry behind them and by the time we talked to them they have been studying calculus for almost a year. In spite of all this, not even one of our interlocutors used their knowledge of linear functions to prove that the existence of a solution is independent of the value of k. Some of them (two from each age group) answered the question using the standard algebraic manipulations. For the others, the problem seemed incomprehensible. We find the following conversation with Dina (age 16, tenth grade) representative and significant.

Dina was helpless when faced with the problem. She asked the interviewer what she was supposed to do. The question was obviously not clear to her at all. After a minute or two of looking at the problem she said, "I am groping in the dark." Here is a fragment of our further exchange with her.

(1) D: [reads the question silently] "... has a solution ..."

(2) I: What does it mean "has a solution"?

(3) D: That we can put a number instead of k and it will come out true.

(4) I: When we say that the system has a solution for every value of k, what is the meaning of the word 'solution'? Is it a number or what?

(5) D: Yes, it's a number.

(6) I: One number?

(7) D: Yes, it's the number that when you put instead of k, then the system is true.

Obviously, Dina had a difficulty with grasping the meaning of the word 'solution' as it appeared here. One possible interpretation of her utterances 3 and 7 is that, for her, the statement 'the equations have a solution' was equivalent to the claim that 'the equations are true' (utterance 3) – as if all the components of the equations had established values. It was probably on the grounds of verbal hints that she decided that in this problem the focus is on k rather than on x and y. In the further exchange, in which the interviewer insisted on getting a clearer explanation, it soon became evident that Dina was far from being firm in her position. Without qualms or explanations, she extended the focus from k to x and y.

(8) I: This word 'solution' here, to what does it refer? Solution of what?

(9) D: Of the equations, $k - y = 2$ and $x + y = k$.

(10) I: What is a solution of these equations?

(11) D: When we substitute numbers...

(12) I: Instead of what?

(13) D: ... instead of x, y, and k, and it comes out true.

(14) I: So, once more, what are the solutions we are talking about in this question [points to the words 'has a solution']?

(15) D: I think ... I think that I need three numbers: x, y, and k.

One possible interpretation is that, similar to Alon, Dina was a captive of the fixed-value approach. She could not raise her sight to a higher vantage point which would help her to realize that the objects to be considered here are functions and not numbers. Seeing these advanced abstract entities through the standard algebraic symbols, let alone operating upon them, was clearly beyond her power. This limitation made her unable to interpret the question in a meaningful, consistent way. It left her confused and helpless. Thus, when asked what she was supposed to look for, she had no choice but to 'shoot at random' with pieces of standard statements which had worked in the past. Needless to say, the way Dina tried to actually solve the problem reflected her confusion: she isolated k from the first equation (wrote $k = 2 + y$) and substituted it in the second ($x + y = 2 + y$). At this point she became stymied and could not make a further move.

Before we end this section it should be mentioned that although none of the pupils quoted above were able to demonstrate the functional approach in the pieces of conversations we chose to present here, two of our three interviewees did display some ability to think in functional terms on other occasions. For example, Dina interpreted the quadratic inequality in a correct way: as a problem of a possibly infinite set of values of the variable which would render the quadratic expression a positive value. Alon solved Problem 3 by manipulating the equations, but seemingly with some understanding.

The message from all this may be presented as follows: the functional perspective was not necessarily beyond the assortment of approaches potentially available to our students. Rather, the problem we have witnessed was that of adaptability: the functional approach was not always accessible; sometimes, even if indispensable, it would not be applied spontaneously.

3.3. *When Something Goes Wrong: Pseudostructural Approach*

Despite the functional approach being promoted in Israeli schools, and in spite of the fact that the students we talked to scored quite high in their achievements in mathematics, what we discovered through the interviews was rather alarming. The usually successful pupils displayed very limited ability when thinking about propositional formulae in the advanced structural terms of functions and truth sets. This fact merits special attention, as a danger may be lurking here of what we called elsewhere *semantically debased* or *pseudostructural* conceptions (see Sfard, 1991, 1992; Linchevski and Sfard, 1991).

Let us clarify these terms. As we explained before, abstract objects, such as functions or sets, play the role of links between the old and the new knowledge (see Fig. 1a). In algebra, function is what ties together the arithmetical processes (primary processes) and the formal algebraic manipulations (secondary processes). Thus, reification of the primary processes, or, in the case of algebra, the acquisition of the structural functional outlook, is a warranty of relational understanding. However, the data we collected up to this point provided sufficient evidence that reification is inherently very difficult. It is so difficult, in fact, that at a certain level and in certain contexts, a structural approach may remain practically out of reach for some students. Once the developmental chain has been broken (Fig. 1b), the process of learning is doomed to collapse: without the abstract objects, the

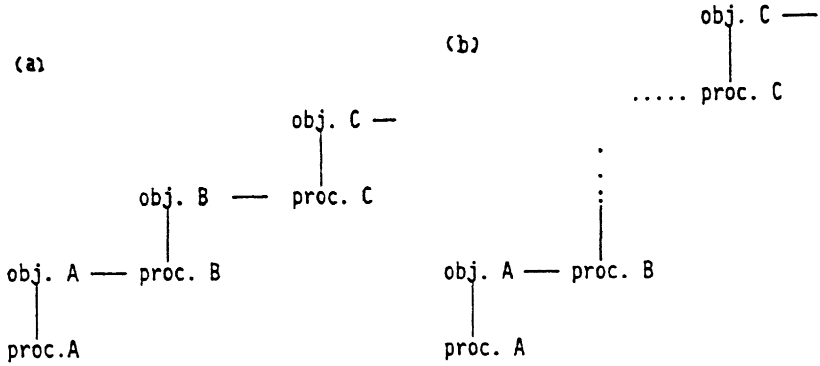

Fig. 1. Development of mathematical concepts as transitions from operational to structural conception. (a) A 'healthy' developmental chain; (b) a broken developmental chain: process B has not been reified, so there is no link between B and C.

secondary processes will remain 'dangling in the air' – they will have to be executed ... on nothing. Unable to imagine the intangible entities (functions, sets) which he or she is expected to manipulate, the student use pictures and symbols as a substitute: a graph of a function or an algebraic formula, a name of a number, the letters 'ϕ' and x' – each of these signs will turn into a thing in itself, not standing for anything else (how literal such identification may be can be learned from the study by Wagner, 1981, which disclosed that for the majority of secondary school students, a change in the name of a variable leads to a completely new equation). In this case, we shall say that the learner developed a *pseudostructural* conception: he or she mistakes a signifier for the signified. In the absence of abstract objects and their unifying effect, the new knowledge remains detached from its operational underpinnings and from the previously developed system of concepts. In these circumstances, the secondary processes must seem totally arbitrary. The students may still be able to perform these processes, but their understanding will remain instrumental.

In the conversations cited in the last section, there were some disquieting incidents in which one may see symptoms of our interviewees' propensity for a pseudostructural approach. One salient example is the way Alon tried to solve Problem 1 (utterances 19–24). The boy was obviously indifferent to the difference between a quadratic inequality and a quadratic equation, and it was just the form of the left-hand side expression $x^2 + x + 1$ that provided him with clues for his decisions. He applied the formula for the roots mechanically and interpreted the outcome like he used to do when solving equations. Another illustration is Dina's inability to pinpoint the difference between the role of the parameter k and that of the variables x and y, and her confusion as to the meaning of the term 'solution' in the particular context of Problem 3. Both students acted as if they were not aware that the strings of symbols might be interpreted in many different ways, depending on the context. They didn't seem to recognize the existence of any objects external to the letters themselves (except, perhaps, some unknown numbers, but this interpretation soon proved unhelpful).

Both these cases seem indeed to be typical examples of pseudostructural thinking: both students acted as if they were handling some kind of object, but their thinking was completely inflexible and the appropriate kind of structural interpretation was unavailable. That this kind of conceptions are quite widespread was clearly shown in our study (Sfard and Linchevski, 1993), in which 280 secondary-school students (age 15–17) were directly interrogated on the meaning of such algebraic notions as *solving equations, permissible operations, equivalence of equations*. The vast majority of the pupils could not provide any sensible justification for the permissible operations and it was obvious that for them these were no more than arbitrary 'rules of the game'. To be more precise, in the eyes of many respondents, solving equations and inequalities was tantamount to performing a certain algorithm. In this game, an expression of the form '$x =$ number' or '$x >$ number' was but a 'halting signal'.

We have already observed that many pupils seem unable to cope with singular propositional formulae – equations and inequalities in which the variable disappears at a certain stage of the solution process. We illustrated this claim with the case of Mariella (Problem 2). Now a possible reason for the difficulty becomes even more clear. If the student, unable to see the abstract objects behind the symbols, is 'programmed' to regard a problem as solved only when an expression of the form '$x =$ number' or '$x >$ number' is obtained, then in a situation in which such an expression does not appear at all he or she must feel lost and helpless. The following is an excerpt from our interview with 15 year old Naomi which renders these speculations even more plausible.

While solving Problem 2, Naomi arrived at a stage where the first of the two equations turned into $1 = 1$.

(1) N: One equals one. It's true, but it gives us nothing. Maybe I shouldn't have opened the brackets [in the first equation] ... I don't know...

(2) I: What can you say now about the solution of the system of equations?

(3) N: That maybe there is one solution.

(4) I: What is it?

(5) N: One. The number 1. Or in fact ... in fact I don't think so.

(6) I: So?

(7) N: There is no solution. An empty set.

(8) I: Where do you infer it from?

(9) N: Because we did all this ... we isolated $2x$, etc. ... to arrive at the value of y. We substituted this [$7 - y$ instead of $2x$ in the first equation] and we were left without x, only with y. And then there was no y as well. Our goal was to find the value of y and we didn't succeed. So I think that we have here an empty [truth] set.

In the comment about her lack of success (line 9), Naomi was obviously referring to the fact that she did not arrive at an expected expression of the form '$y =$ number'. Since her perspective was restricted to the situation in which such an expression does appear in the end, this 'failure' in the *process* of finding the solution was interpreted as an absence of the *result* of solving.

4. AFTERWORD: WHERE WE ARE AND WHERE WE SHOULD GO

The episodes discussed in this article give rise to the suspicion that most of the time algebraic formulae are for some pupils not more than mere strings of symbols to which certain well-defined procedures are routinely applied. In these students' eyes, the formal manipulations are the only source from which the symbolic constructs may draw their meaning.

At face value, this outlook is very close to the view promoted by Peacock and his colleagues. In fact, the students' conception and the formalist position cannot be equated, and the differences are certainly more significant than the similarities. The belief about the nature of the symbolic manipulations is where the formalists and today's students part. While dealing with symbols, the formalists focus on *combinations of operations* (Gregory, 1840). The operations that the high-school student is supposed to master, namely those that can be interpreted as a generalization of arithmetic calculations, are for the formalists but a point of departure, mere inputs to the processes they are really interested to investigate. In other words, the formalist's algebra begins where the school algebra ends. Besides, although both the mathematician and the pupil view the formal operations as arbitrary, for the formalist such an approach is a matter of a deliberate choice, while for the student it is an inevitable outcome of his or her basic inability to link algebraic rules on the laws of arithmetic.

We started this article with a list of possible perspectives one may assume while dealing with algebraic constructs. We stressed all along in our discussion the importance of the versatility and adaptability of student's thinking. The conclusions we draw are not very encouraging. In a quite consistent way, all our findings have shown that more often than not, the pupil cannot cope with problems which do not yield to the standard algorithms. Since it became clear that the functional approach is not easily accessible even for the better students, it is not so difficult to understand why the mechanistic, pseudostructural approach may eventually dominate student's thinking like an overgrown weed, leaving no room for other, more meaningful perspectives.

As we explained more than once, the sense of meaningfulness comes with the ability of 'seeing' abstract ideas hidden behind the symbols. However, "mathematical objects and structures that the teacher can 'see' are unlikely to be apparent to the student" (Cobb, 1988). On the other hand, it would be a mistake to claim that student's activities and decisions are entirely devoid of an inner logic. With Davis (1988) we shall say that "students usually *do* deal with meanings, and when instructional programs fail to develop appropriate meanings, *students create their own meanings* – meanings that are sometimes not appropriate at all" (p. 9). Although somehow flat and one-sided, the student's do-it-yourself algebra is not devoid of certain consistency. The problem is that those who adopt the pseudostructural approach and confuse the powerful abstract objects with their representations do not realize that the symbols themselves cannot perform the magic their referents are able to do: they cannot glue together lots of detailed pieces of knowledge into one powerful whole. Thus, when talking about the mechanistic, pseudostructural outlook, one may certainly say "though this be method, yet there is madness in it".

The confrontation between the developmental model we promoted in this paper and the structural way of teaching algebra hints at a possible reason for the unsatisfactory results of schooling. The curriculum literally reverses the order in which algebraic notions seem to be related to each other, the order in which they developed through ages. The advanced structural approach is assumed at the outset even though the student is evidently not ready yet to grasp the idea of process-object duality, let alone to cope with the functional outlook. From the very beginning, the letter is supposed to play the role of a variable and not just of an unknown, although we have seen that the former is more advanced than the latter. The stand-alone algebraic formula is introduced before it is incorporated into an equation or inequality, even though it is only in the context of simple ('arithmetical') equations that it may be interpreted in the operational way, more accessible for the young student. Finally, equations and inequalities are taught simultaneously although the latter put heavier demands on student's understanding. The conceptual pyramid is put on its head, so it is only natural that it has a tendency for falling. Revision of the order in which the pupils are exposed to the central ideas of algebra is likely to bring some improvement. It seems reasonable that we can capitalize on the students' natural propensity for an operational approach by beginning with processes rather than with ready-made algebraic objects. As to the latter, there is some evidence that their construction in a student's mind may be helped by the computer (Dreyfus and Halevi, 1988; Waits and Demana, 1988; Schwartz et al., 1990; Breidenbach et al., 1992). (This does not suggest that the early introduction of the functional approach is the only culprit, or that teaching functional algebra from the very start cannot be successful in any circumstances. For example, it is fairly possible that the massive use of computer graphics in teaching functions will reverse the 'natural' order of learning so that the structural approach to algebra will become accessible even to young children. Also, one should remember that no alterations in the organization or presentation of subject matter can suffice to significantly enhance learning. A real change for the better will not come until the teachers find ways to boost students' willingness to struggle for meaning.)

We hope that there is much potential in the suggested didactic ideas and we are now engaged in a teaching experiment in which they are systematically put to test. Notwithstanding our belief that the method may work, there is a question that keeps bothering us: even if our students succeed in acquiring a versatile and adaptable assortment of perspectives, how durable and robust will their flexible knowledge be in the long run? The chances are that pseudostructural thinking may sometimes be compared to secondary illiteracy. After all, the mechanical mode has so much to offer to intellectual sluggards: it exempts the problem-solver from the need for constant alertness, from the strain which inevitably accompanies going back and forth from one perspective to another. As testified by Souriau, mathematicians themselves yield to the charm of mechanized symbolic manipulations:

> Does [the algebraist] follow [the original meaning of symbols] through every stage of the operations he performs? Undoubtedly not: he immediately loses sight of them. His only concern is to put in order and combine, according to known rules, the signs which he has before him; and he accepts with confidence the results thus obtained. [quoted by Hadamard, 1949, p. 64]

The problem is that unlike the mathematician, the student may easily become addicted to the automatic symbolic manipulations. If not challenged, the pupil may soon reach the point of no return, beyond which what is acceptable only as a temporary way of looking at things will freeze into a permanent perspective. When it happens, there is not much chance that the student will be able to explain his or her decisions. If asked for justification, he or she may become as confused as a centipede who has been required to tell how it moves its legs. Thus, to fight pseudostructural conceptions it may be not enough to reform the teaching method in the ways proposed above. It seems very important that we try to motivate our students to actively struggle for meaning at every stage of the learning. We must make them active sense-seekers who, as Davis (1988, p. 10) put it, would 'habitually' "*interpret* situations, *interpret their actions*, think of the *meanings* of symbols and *meanings* of symbols and operations".

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