Aspects of Mathematical Knowledge for Teaching[1]

Analysing mathematical thinking: A problem-centred approach

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[1] Slightly adapted from Olivier, A., 2002, **Study Guide: Mathematics for Teachers I**, Unisa Publishers: Pretoria.

1. INTRODUCTION AND COURSE OVERVIEW

Mathematics Education has failed our children!

We merely have to refer to the poor national matriculation results, the high failure rate in mathematics in all grades, the fact that too few learners take mathematics as subject in the FET phase (on HG), and too few study mathematical sciences at tertiary level.

Urgent and radical change in all aspects of education, and specifically Mathematics Education is necessary if the promise of a better future for our children in our new democracy is ever to become a reality.

What we need is
- a high quality mathematics curriculum (*excellence*), and
- *equity* for all, so that children do not drop out along the way, but the doors to higher levels of education are kept open to all children.

However, it is not easy, and not enough, to simply improve the mathematics curriculum. Oakes (1985) warns us that

> *… the unquestioned assumptions that drive school practice and the basic features of schools may themselves lock schools into patterns that make it difficult to achieve either excellence or equality.*

Some of these "unquestioned assumptions" that makes it impossible to improve the teaching and learning of Mathematics includes outdated perspectives and beliefs about the nature of mathematics and therefore the mathematical diet we offer our children, how children learn mathematics, and the role of the teacher in teaching (teaching is not learning!).

We distinguish between
- *micro problems*: Problems internal to mathematics education, e.g. curriculum, teacher development, textbooks, the use of calculators, problem solving, etc., and
- *macro problems:* Problems affecting mathematics education because of external pressures from other sectors of society, economy, politics, culture, language, ...

Both macro and micro problems must be addressed and solved!

Curriculum 2005 is a good start! It is rightly a political statement formulating different outcomes, i.e. different products of schooling – this means delivering a different kind of *person!* This view of different outcomes is again based on
- different views on children
- different views on the nature of mathematics
- different views on how children learn mathematics.

These fundamentally different assumptions require that teachers change radically in fundamental ways.

These beliefs underlying traditional teaching, and the new views underlying Curriculum 2005 are broadly summarised in the following table:

Traditional	Curriculum 2005
VIEW OF MATHEMATICS	
A finished body of facts, rules and skills	Dynamic continually expanding field
It can be discovered	A human social invention
VIEW OF TEACHING	
Instructor/Explainer	**Facilitator/organiser**
Break the maths into small logical pieces	Present tasks and problems that lead to
Explain, drill each piece in sequence	learners inventing mathematics
VIEW OF LEARNING	
Passive reception of knowledge	Active construction of knowledge
Learn by imitation, practice and repetition	Learn through social interaction and reflection
OBJECTIVES / OUTCOMES	
Submissiveness	Intellectual independence
Obedience	Understanding
Compliance	Problem solving and exploration
Following rules	Communication and reasoning
⇩	⇩
DISEMPOWERMENT	**EMPOWERMENT**

A teacher preparation course such as this that wants to contribute to the state of mathematics education in the country, should to our mind

- address macro and micro issues undermining and sabotaging an improvement in mathematics education
- unpack the underlying theoretical assumptions of Curriculum 2005
- offer teachers encouraging alternative approaches to the teaching of different topics
- help teachers in their new roles in the classroom.
- Etc.

Taylor and Vinjevold (1999: 159-161) point to a culture of rote learning, as evidenced by the recent President's Education Initiative research:

> *The most unequivocal finding about teachers is that a poor grasp on the part of teachers of the fundamental concepts in the knowledge area they are responsible for is a major problem in disadvantaged classrooms. . . . reform initiatives aimed at revitalising teacher education and classroom practices must not only create a new ideological orientation consonant with the goals of the new South Africa. They also need to get to grips with what is likely to be a far more intractable problem: the massive upgrading and scaffolding of teachers' conceptual knowledge and skills.*
>
> *... the fundamental mechanism for its propagation [the vicious cycle of rote learning] is the lack of conceptual knowledge, reading skills and spirit of enquiry amongst teachers.*

This is an echo of a previous report (ANC, 1994):

> *Science and mathematics … is characterised by a "cycle of mediocrity". … Under-qualified and poorly prepared teachers in turn produce weak and poorly prepared school students, and they cannot be expected to teach the subject with enthusiasm.*

Surely we have made major progress in the years since 1994, especially in the improvement of macro problems. But it is this "cycle of mediocrity", which at the micro level is propagated by *rote learning*, that must be broken.

We believe the most pressing content, and therefore the focus of this course, is teachers' *mathematical content knowledge* (what to teach) and *pedagogical content knowledge* (how to teach it).

What mathematical content is worth teaching and learning? Beeby (1935: 10) draws our attention to the importance of revisiting our assumptions about the content of mathematics:

> *The black doubt that lurks in the bottom of every honest pedagogue's heart is not so much whether he is teaching correctly as whether what he is teaching is worth teaching at all. The real danger is not that we shall teach the right things inefficiently, but that we shall teach the wrong things more and more efficiently.*

Schoenfeld describes traditional teaching:

> *All too often we focus on a narrow collection of well-defined tasks and train students to execute those tasks in a routine, if not algorithmic fashion. Then we test the students on tasks that are very close to the ones they have been taught. If they succeed on those problems, we and they congratulate each other on the fact that they have learned some powerful mathematical techniques. In fact, they may be able to use such techniques mechanically while lacking some rudimentary thinking skills. To allow them, and ourselves, to believe that they "understand" the mathematics is deceptive and fraudulent.* (p. 30)

The point to make is that while the mastery of techniques may have been the major objective of mathematics teaching, the goals have changed! The South African Curriculum 2005 and the Revised Curriculum Statements set ambitious goals for Mathematics (Department of Education, 2001: 17):

The teaching and learning of Mathematics aims to develop in learners:

- *A critical awareness of how mathematical relationships are used in social, environmental, cultural and economic relations.*
- *The necessary confidence to deal with any mathematical situation without being hindered by the fear of mathematics.*
- *An appreciation for the beauty and elegance of Mathematics.*
- *A spirit of curiosity.*
- *A love for the Learning Area.*

According to the Revised Curriculum Statements (Department of Education, 2001: 16), the Mathematics Learning Area includes interrelated knowledge and skills:

Knowledge	**Skills**
Numbers, operations and relationships	Representation and interpretation
Patterns, functions and algebra	Estimation and calculation
Shape and space (geometry)	Reasoning and communication
Measurement	Problem-posing
Data handling	Problem-solving and investigation
	Describing and analysing

Table 1

In describing the content, the Revised Curriculum Statements (p. 18) use *verbs* like describing, representing, interpreting, analysing, synthesising, conjecturing, inferring, deducing, reflecting, generalising, predicting, refuting, explaining, specialising, defining, modelling, validating, justifying and to collect, summarise, display and critically analyse data to draw conclusions and make predictions.

Our changing society requires a changed perspective on the nature of mathematics and what is worth teaching and learning. This is eloquently described by the National Research Council (1989):

Mathematics is a living subject which seeks to understand patterns that permeate both the world around us and the mind within us. It is important that students move beyond rules to be able to express things in the language of mathematics. This suggests changes both in curricular content and instructional style. It involves renewed effort to focus on:

- *Seeking solutions, not just memorizing procedures;*
- *Exploring patterns, not just memorizing formulas;*
- *Formulating conjectures, not just doing exercises.*

Students should have opportunities to study mathematics as an exploratory, dynamic, evolving discipline rather than as a rigid, absolute, closed body of laws to be memorized. They should be encouraged to see that mathematics is really about patterns and not merely about numbers.

The National Council of Teachers of Mathematics (1989) advocates *mathematical power* for all and describes "mathematical power" as follows:

Mathematical power includes the ability to explore, conjecture, and reason logically; to solve nonroutine problems; to communicate about and through mathematics; and to connect ideas within mathematics and between mathematics and other intellectual activity. Mathematical power also involves the development of personal self-confidence and a disposition to seek, evaluate, and use quantitative and spatial information in solving problems and in making decisions. Students' flexibility, perseverance, interest, curiosity, and inventiveness also affect the realization of mathematical power.

For this course we adhere to the following view of Schoenfeld:

Mathematics is an inherently social activity, in which a community of trained practitioners (mathematical scientists) engages in the science of patterns – systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically ("pure mathematics") or models of systems abstracted from real world objects ("applied mathematics"). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation. However, being trained in the use of these tools no more means that one thinks mathematically than knowing how to use shop tools makes one a craftsman. Learning to think mathematically means (a) developing a mathematical point of view -- valuing the processes of mathematization and abstraction and having the predilection to apply them, and (b) developing competence with the tools of the trade, and using those tools in the service of the goal of understanding structure – mathematical sense-making.

This vision not only challenges teachers' assumptions about mathematics and mathematics teaching and learning, but also asks them to teach a mathematics that they may never have experienced themselves. Teachers are themselves victims of their own previous education and are likely to continue to teach the way they were taught unless a way is found to interrupt this self-perpetuating cycle. This course is an attempt to help students to break out of this cycle by experiencing *mathematical power* as described above.

Theory and research shows that we develop habits and skills of interpretation and meaning construction though a process of *socialization* or *enculturation* rather than through *instruction:*

… becoming a good mathematical problem solver -- becoming a good thinker in any domain -- may be as much a matter of acquiring the habits and dispositions of interpretation and sense-making as of acquiring any particular set of skills, strategies, or

knowledge. If this is so, we may do well to conceive of mathematics education less as an instructional process (in the traditional sense of teaching specific, well-defined skills or items of knowledge), than as a socialization process. In this conception, people develop points of view and behavior patterns associated with gender roles, ethnic and familial cultures, and other socially defined traits. When we describe the processes by which children are socialized into these patterns of thought, affect, and action, we describe long-term patterns of interaction and engagement in a social environment. (Resnick, 1989: 58)

This view of *enculturation* highlights the importance of *perspective* and *point of view* as core aspects of knowledge. The case can be made that a fundamental component of thinking mathematically is *having a mathematical point of view,* or having a mathematical *attitude of mind* -- seeing the world in ways like mathematicians do.

This course is about developing a mathematical attitude of mind, and the way to do it is to immerse participants in a typical *mathematical culture*. This course is about *mathematical thinking*, emphasising the “skills” mentioned in Table 1: Representation and interpretation; Reasoning and communication; Problem-posing; Problem-solving and investigation; Describing and analysing.

The medium through which the course is presented is through *problem solving*, i.e. non-routine problem solving – either through illustration of the process of problem solving, or through students' own engagement with problems.

One aspect of the course is about *our own mathematical activity* – we will reflect on the process and product of developing mathematics and we will reflect on our own experiences in doing mathematics (mathematical content knowledge). The other aspect of the course is, through our engagement with the content, to reflect on *learning obstacles* for our school learners, and how we can adapt our *teaching* to address their needs (pedagogical content knowledge).

We trust that you find the course worthwhile, empowering, and enjoyable! However, mathematics is not a spectator sport! You will only really *learn* if you *actively* engage the activities and *reflect* on what you are doing. The famous mathematician Paul Halmos says this about reading mathematics:

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

HIERDIE NOTAS

Die notas probeer 'n *tipiese wiskundige werkwyse* bevorder – wat dit beteken om Wiskunde te *doen!*

Die notas bevat probleme, bespreking van sommige klasaktiwiteite, ander oplossings, en nog probleme vir jou.

Ek moet egter weer herhaal: *Mathematics is not a spectator sport!*

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? Paul Halmos

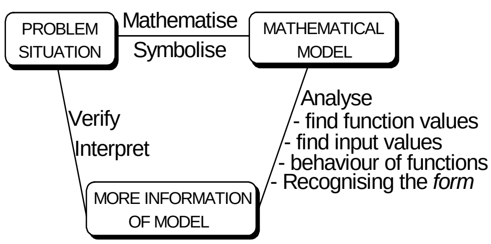
Jy sal dus hier slegs iets leer as jy *aktief* lees … met potlood en papier, die probleme *doen* … En beskikbare tegnologiese werktuie kan help, veral Excel, Sketchpad en die grafiese sakrekenaar. Gebruik dit as *ondersoekmiddel* en *kontrolemiddel* – dit behoort 'n natuurlike deel van jou gereedskap te wees, net soos potlood en papier! Die notas verwys soms na ander rekenhulpmiddels, bv. hierdie faktormasjien – gebruik dit gerus!

Let op: Die *hiperskakels* na ander dokumente is natuurlik slegs op die web aktief! So, as jy hierdie dokument print, sal jy ook moet kyk na gekoppelde dokumente!
Skakels buite SUNET word aangedui met die notasie 🗗.

Jy moet die *Wiskunde* verstaan, maar terselftertyd aktief betekenis gee aan ons *Wiskunde-didaktiek konsepte* wat ons nodig het om oor wiskunde-onderwys te kan kommunikeer. Ek merk onder andere die volgende in hierdie stuk:

induksie, matematiese induksie	abstraheer
deduksie, struktuur	conjecture = vermoede/hipotese
funksie, fuksionele formule	stelling
rekursie. rekursiewe formule	analities/algebraïes
regressie, regressie-formule	heuristiek
modelleer	databasis
spesialiseer	veranderlike
veralgemeen	Onbekende
Teenvoorbeeld (counter example)	parameter

Die breë wiskundige fokus van al die probleme is *wiskundige modellering*, en dit is belangrike dat ons die *proses* en die *probleemtipes* begryp (Kyk MALATI Algebra rasionaal):



There are many situations involving two variables where the one variable is *dependent* on the other variable, i.e. where a *change* in the value of one (the independent) variable causes a deterministic change in the value of the other (the dependent) variable.

Algebra is a language and a tool to study the nature of the relationship between specific variables in a situation. The power of Algebra is that it provides us with *models* to describe and analyse such situations and that it provides us with the analytical tools to obtain additional, unknown information about the situation. We often need such information as a basis for *reasoning* about problem situations and as a basis for *decision-making.*

….. The additional information we need to generate is mostly of the following five *types*:

1. *finding values of the dependent variable* (finding function values)
2. *finding values of the independent variable* (solving equations)
3. *describing and using the behaviour of function values* (increasing and decreasing functions, rate of change, gradient, derivative, maxima and minima, periodicity, . . .)
4. *finding a function rule* (formula)
5. *transforming to an equivalent expression* ("manipulation" of algebraic expressions)

Terwyl jy "lees" is dit belangrik dat jy identifiseer watter van hierdie vyf probleemtipes gebruik word …

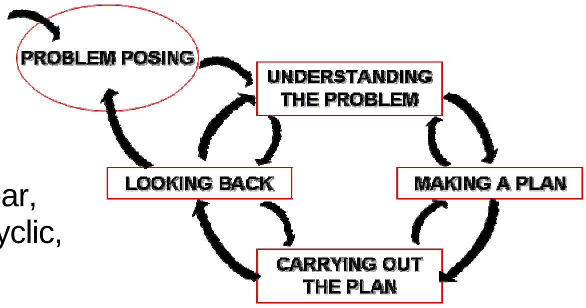
2. LEARNING *ABOUT* PROBLEM SOLVING[2]

Problem-solving strategies (heuristics), specialisation and generalisation, induction and deduction

In this section we illustrate some processes of problem solving and you will have ample opportunity to implement processes.

The major contributions on problem solving was made by George Polya (1954, 1957). He formulated the following four-phase approach to problem solving:

1. Understanding the problem
2. Making a plan
3. Carrying out the plan
4. Looking back



These phases should not be interpreted as a linear, step-by-step recipe for problem solving – it is a cyclic, dynamic process as illustrated here.

Together with these phases, Polya formulated several *problem solving strategies* or *heuristics* to help us in problem solving. These phases and heuristics are summarised in Table 1 overleaf. Much of our work will be about using and reflecting on these phases and strategies.

2.1 SPECIALISING AND GENERALISING

Specialising means looking at special or particular cases of some general statement. It is passing from the consideration of a given set of objects to that of a smaller set containing the given one. For example, we specialise we specialise when we pass from considering polygons to that of a regular polygon, and we specialise further when we pass from regular polygons with n sides to a regular quadrilateral, i.e. a square ($n = 4$ is a special case).

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[2] Verskillende mense gebruik dikwels dieselfde woorde soos "probleemoplossing", maar gee heel verskillende betekenisse daaraan! Mense kan dan maklik dink hulle praat dieselfde taal of praat oor dieselfde ding, maar eintlik praat hulle oor heel verskillende goed!

Wat is 'n probleem?
Vir iets om 'n *probleem* te wees, moet daar een of ander blokkasie of struikelblok wees, d.w.s 'n wiskundige probleem is 'n taak wat jy nie weet hoe om op te los nie. Anders is dit mos nie 'n probleem vir jou nie! Ek sal dit effens aanpas: 'n Probleem is 'n taak waarvoor jy nie 'n geroetineerde (outomatiese) oplossingsmetode het nie.

Wat is probleemoplossing?
Wanneer 'n onderwyser of 'n handboek *eers* die nodige wiskundige konsepte en metodes vir 'n probleemtipe ontwikkel en dit met 5 uitgewerkte voorbeelde illustreer, en dan leerlinge *toepassings* aan die einde van die hoofstuk laat doen, sê hulle dit is probleemoplossing. Dit kan tog nie wees nie – leerlinge weet dan reeds hoe om dit te doen! Ons noem hierdie soort take bloot *oefeninge!*

Ons sou in plaas van *probleme* vs *oefeninge*, die terminologie *nie-roetine probleme* en *roetine probleme* kon gebruik. Maar wat ek met *probleme* bedoel is dus nie-roetine probleme en *nie* oefeninge of roetine probleme nie!

Die volgende is 'n nuttige raamwerk om tussen verskillende benaderings te onderskei:

- *Onderrig vir probleemoplossing:* Die onderwyser onderrig *eers* vooraf en apart die "tools" – die nodige wiskundige konsepte en metodes in die abstrak, en dan word dit *agterna* toegepas.
- *Onderrig via (deur) probleemoplossing:* Die onderrig *begin* met relevante probleme en die wiskundige konsepte en metodes word ontwikkel *terwyl* leerlinge die probleme oplos.
- *Onderrig omtrent/van probleemoplossing:* Die onderwyser illustreer in die algemeen probleemoplossingsmetodes.

Met probleemoplossing in die klaskamer bedoel ek dus *onderrig via (deur) probleemoplossing.*

Die onderliggende perspektiewe
Die twee benaderings *onderrig vir probleemoplossing* en *onderrig via (deur) probleemoplossing*

- Is gebaseer op heel verskillende onderliggende teorieë oor die aard van kennis (epistemologieë), in besonder die aard van Wiskunde, verskillende leerteorieë (insluitend die aard van geheue) en verskillende perspektiewe oor die betekenis van "verstaan", en
- dit lei tot totaal verskillende klaskamerkulture (die onderlinge verwagtings en verantwoordelikhede van onderwysers en leerlinge), en
- dit bepaal op sy beurt wat (en of) leerlinge leer.

How to Solve It

A summary of George Polya's (1957) four phases of problem solving and heuristics.

1. UNDERSTANDING THE PROBLEM

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

2. DEVISING A PLAN

- **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- *Have you seen it before?* Have you seen the same problem in a slightly different form?
- *Do you know a related problem? Do you know a theorem that could be useful?*
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it?* Could you use its result? Could you use its method?
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you *prove* that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result?* Can you *check the argument?*
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

Table 1

Generalising is the reverse process of specialising. Generalising is passing from considering a given set of objects to that of a larger set, containing the given one. For example, we generalise when we pass from considering any square to that of considering any quadrilateral, in which we do not restrict the quadrilaterals to having equal sides. And we generalise when we move from any quadrilateral to any polygon, relaxing the restriction that it should have four sides.

Generalisation is the process of formulating a statement that is more general or inclusive, i.e. enlarging the number of cases for which a statement is true. For example, the cosine formula

$$a^2 = b^2 + c^2 - 2bc\cos \mathrm{A}$$

is a generalisation of the Theorem of Pythagoras

$$a^2 = b^2 + c^2$$

because it applies to *any* triangle, removing the restriction that angle A = 90°.

Of course, the Theorem of Pythagoras is a special case (a specialisation) of the cosine formula!

Specialisation serves at least the following purposes in mathematical activity:

1. It gives us a feeling for *what the statement is saying*
2. It gives us a sense *if the statement may be true*.

PROBLEM 1: DISCOUNT AND VAT

In a transaction a customer receives 10% discount for cash, but must pay 14% Value Added Tax (VAT). Which is cheaper for the customer: that the tax be calculated first or that the discount be calculated first?

It helps to look at a *special case* to get a feeling for what is happening.
So suppose the purchase price is R100 and let's calculate the final price:

<u>VAT first:</u>
VAT: R100 + 14% of R100 = R114
Discount: R114 – 10% of R114
= R114 – R11,40
= R102,60

<u>Discount first:</u>
Discount: R100 – 10% of R100 = R90
VAT: R90 + 14% of R90
= R90 + R12,60
= R102,60

So, you pay the same! Are you surprised? But are you *convinced?*

Do you think the reasoning above will *convince* a friend? Should it convince the shopkeeper?

If it is correct, can you explain *why* the results are the same?

PROBLEM 2: ALL THE SAME

Jane says it is obvious that it does not matter if you calculate the VAT or the discount first, because you have to pay a net 4% extra in both cases, e.g:
R100 + 14% – 10% = R100 – 10% + 14% = R100 + 4%.
Is Jane correct? *Explain!*

PROBLEM 3: PETROL PRICE

In January the petrol price is increased by 10%. Then, in February the petrol price is reduced by 10%. John says that the petrol price is now the same as it was before the first increase. Is this correct? *Explain!*

PROBLEM 4: RENTING A CAR

You want to rent a car for one day. Imperial and Avis both charge a basic amount per day plus a rate per kilometre for the distance driven, as shown in the table below. Which company is the cheaper?

Car:	Per day	Per km
Toyota Corolla 1.6	R133	R1,08

Car:	Per day	Per km
VW Citi Golf 1600	R85	R1,48

One approach is to specialise. John does it like this: he takes the distance as 100 km and works out the cost for each company:

Imperial: Cost = 133 + 1,08 $\times$ 100 = R241
Avis: Cost = 85 + 1,48 $\times$ 100 = R233

John's calculation show that Avis is the cheaper. *Do you agree?*

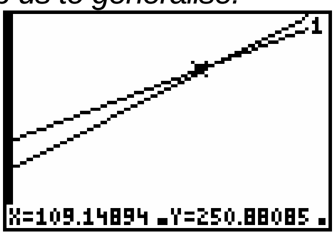
The calculations show that *for a distance of 100 km* Avis is cheaper, but it says nothing about any other distance! Surely the costs are *dependent* on the distance travelled. The distance is a variable that can *change (vary)*. If we calculate and compare the costs for *different distances*, Avis is not *always* cheaper, as is shown in this table:

Distance	**90**	**95**	**100**	**105**	**110**	**115**	**120**	**125**	**130**	**135**
Imperial	230,20	235,60	241	246,40	251,80	257,20	262,60	268	273,40	278,80
Avis	218,20	225,60	233	240,40	247,80	255,20	262,60	270	277,40	284,80
Difference	12	10	8	6	4	2	0	-2	-4	-6

A distance of 120 km is the *break-even* point, where the costs for the cars are the same, and for distances greater than 120 km, Imperial is the cheaper. (We note that as a *real* problem the differences are too small to really make a difference!)

We can also easily find the break-even point *algebraically* or *graphically*, by expressing the cost for each car for *any distance* of x km and equalling the costs. *Note that the algebraic formulation is merely a generalisation of exactly the procedure we used for our numeric calculations. So, first working numerically (specialising), can help us to generalise!*

$$133 + 1{,}08x = 85 + 1{,}48x$$
$$\Rightarrow 0{,}4x = 48$$
$$\Rightarrow x = 120$$



PROBLEM 5: SAVING

In the Imperial-Avis table above:

- Are you surprised that the differences left and right of 120 km is "symmetrical"? Can you explain *why* this is so?
- Find a formula for the row of differences, i.e. a formula to calculate the difference in cost for *any* distance directly, without first calculating the costs for Imperial and Avis. Use it to calculate how much you will save by renting the cheaper option if you drive 400 km.

Now let's return to Problem 1. How do we know that the *one* value of R100 that we chose, is not a special case? Will it also be the same for R80, and R75 and R217,83? The point we are trying to make is that in Renting a car, which car was cheaper depended on the distance travelled – for some distances Avis was cheaper and for other distances Imperial was cheaper. Similarly, will in our VAT-Discount problem the answer not depend on the purchase price, so that for some prices it is better to add the VAT first and for other prices it is better to deduct the discount first? Are the two situations not the same?

One approach would be to calculate the final price for several purchase prices. Another approach is to work *generally*, i.e. with *all* purchase prices simultaneously. How can this be done? By introducing some *symbol* to represent *any* purchase price, e.g. let the purchase price by "PRICE" or p. When we now use this symbol, it refers to *any* price, but not to any *particular* price.

<u>VAT first:</u>
VAT: PRICE + 14% of PRICE
= $1{,}14 \times$ PRICE
Discount: $0{,}9 \times (1{,}14 \times$ PRICE)

<u>Discount first:</u>
Discount: PRICE – 10% of PRICE
= $0{,}9 \times$ PRICE
VAT: $1{,}14 \times (0{,}9 \times$ PRICE)

This general statement now without doubt *proves* that the final prices are *always* equal, *independent* of the price of the purchase, because our variable "PRICE" represents *any* price. The structure of the general statement also clearly *explains why* the final price is the same: multiplication is *associative* (the grouping does not matter) and *commutative* (the order does not matter)!

Always true, sometimes true, never true

It is vital that we should understand how the mathematics in our two problems (the VAT problem and the Renting a Car problem) are the same and how they are different.

In our VAT problem the structure of the situation is symbolised by

$$0{,}9 \times 1{,}14 \times x = 1{,}14 \times 0{,}9 \times x$$

This statement is *always true* for *all values of the variable.*

In Renting a Car the structure of the situation is symbolised by

$$133 + 1{,}08x = 85 + 1{,}48x$$

This statement is *sometimes true* for *some values of the variable* (here only one value).

This distinction and the different meanings of the symbol x and the equal sign in these two statements are some of the most powerful and simultaneously some of the most difficult concepts in learning and using algebra, as formulated by William Betz (1930):

The symbolism of algebra is its glory. But it is also its curse.

Let's give examples of statements that are always true, sometimes true and never true in algebra:

Two algebraic expressions like $2x + 3x$ and $5x$ are ***equivalent***, because they have the same values for ***any*** value of the ***variable*** x.
We can say that $2x + 3x$ = $5x$ is ***always true*** for ***all*** values of x.
We call an algebraic statement like $2x + 3x$ = $5x$ which is true for all values of the variables an ***algebraic identity***.

The algebraic expressions $4x + 12$ and $7x + 3$ are not equivalent expressions, because they have different values for all values of x, except for x = 3.
We can say that the statement $4x + 12$ = $7x + 3$ is ***sometimes true*** for ***some*** values of x.
We call an algebraic statement like $4x + 12$ = $7x + 3$ which is ***sometimes true*** for ***some*** values of the variables an ***algebraic equation***.

$10x + 40$ = $10x + 50$ for ***no values*** of x. We call this an ***algebraic impossibility.***

Remark: Exceptions
It is merely for making some distinctions that we here talk about *always true*, *sometimes true*, and *never true* (*false*). In the process we are using the word "true" rather loosely and that creates a slight semantic problem, because in logic and in mathematics, if we say a statement is true, we definitely mean that it is *always* true. A statement that is sometimes true is not true!

It is for this reason that mathematicians often add exceptions to "save" a beautiful theorem, so that it is true (i.e. always true). For example:

Any prime number can be written in the form 6n – 1 or 6n + 1, *n* $\in$ *N.*

You may test it through specialisation, e.g. 29 = 6 $\times$ 5 – 1 and 19 = 6 $\times$ 3 + 1. We will give a general proof and explanation in section 4. But you may notice that if you take the special case of prime numbers 2 and 3, they cannot be expressed in this form (i.e. 2 = 6*n* – 1 or 2 = 6*n* + 1 have no solution for n $\in$ N). They are *counter-examples* showing that the statement is *not true* (i.e. not *always* true). But they are the *only two* exceptions. So, rather than discarding this useful theorem, we "save" it by rather changing the statement:

Any prime number greater than 3 can be written in the form 6n – 1 or 6n + 1, n $\in$ *N.*

Now it is a true statement, without exception!

Mmm … How do you *know, for sure,* it is *true?*

Some conceptual difficulties

Let's briefly illustrate how the above concepts often are learning obstacles for children. Carefully study the following episode observed in a typical Grade 8 classroom and try to explain the source of Ame's struggle.

The pupils are working on "simplification" exercises, including:
$x + x$
$2x + 3x^2$
$3a + 4a$

Ame produces the following:
$x + x = x^2$
$2x + 3x^2 = 5x^3$

However, she is unsure about her answers. So she asks the teacher.

Ame: *Is it right, miss?*

The teacher wants the pupils to take responsibility for their own learning, to validate answers for themselves on logical ground. So she helps Ame by suggesting a strategy that would enable her to judge for herself whether her "simplifications" are correct.

Teacher: *Listen everyone*. *Check your answers! Choose any value for x and check whether the left-hand side is equal to the right-hand side.*

This seems like good advice. But Ame does not immediately follow her advice. She keeps brooding on the problem and does not seem to make any progress at all.

The researcher (R) now joins Ame to observe what is happening as she broods on her $x + x = x^2$ result.

R: *So why do you not choose a value for x like the teacher said and check if $x + x = x^2$?*
Ame: *Yes, I know <u>she</u> does it, but I cannot ... I do not know what x is ...*
(silence)
R: *But just choose <u>any</u> value ... Let's choose $x = 3$. Then what is the value of the left-hand side?*
(silence ... Ame continues staring at the exercise, but does not substitute a value for x.)
Ame: *But how do you know that x is 3? Is it?*
R: *It does not <u>have</u> to be 3. We can choose <u>any</u> value for x. Choose 5 if you like ...*
Ame: *But how can we say x is 5 if we have not yet worked it out?*
(silence ...)
Ame: *Ah! I worked it out! It is 2! See, if x is 2, then the left-hand side is $2 + 2 = 4$ and the right-hand side is $2^2 = 2 \times 2 = 4$. So its correct!*
R: *So? Does that mean that it is correct to say that $x + x = x^2$?*
Ame: *Yes!*

In the same class it is observed that Kenneth interprets the letter symbols in the exercises concretely as letters of the alphabet. He “simplifies” $3a + 4a$ by reasoning that “3 a’s and 4 a’s give 7 a’s”. So these exercises are really very easy for Kenneth. He is getting all his simplifications correct and he consequently does not pay any attention to the teacher’s advice of checking by substituting arbitrary values for the letters. However, one cannot but wonder how his conception of the letter *a* as a real letter of the alphabet will help him to give meaning to $a \times a$ or what sense he will make of solving an equation like $a + 3 = 5$.

Clearly, here is a disastrous mismatch between what the teacher intends to convey to the children, and the children’s conception of the meaning of the letters and the nature of the task. Ame and Kenneth simply did not, and at this stage *cannot* understand the teacher’s notion of “choose any value for x”.

For the teacher, the task to “simplify” means to construct an *algebraic identity,* for example $x + x = 2x$. $x + x$ and $2x$ are two *equivalent algebraic expressions* that, although different, nevertheless yield the same value for the same value of x. This is true for any value of x. So for the teacher x represents an *arbitrary* but *unspecified* number, and therefore one can illustrate the validity of the identity with any *particular* number or set of numbers.

Ame on the other hand, is interpreting the letter x in the context of solving *equations*: In an equation like $2x + 3 = 15$, x is an *unknown* and the task is to find out what *specific* value of x will make the open sentence true. It is an unique solution, in this case $x = 6$. Therefore, the teacher’s suggestion that she chooses *any* value for x is not understandable to her at all because her interpretation of the task is that she must *first* solve the *equation* $x + x = x^2$ in order to find out the unknown value of x. So how can she simply choose a value for x if she does not know what the value of x is, that is, she is concerned that she may choose an incorrect value of x, that is not a solution of the equation. Ame's struggle with the ambivalent x echoes that of the great philosopher Bertrand Russell:

> *When it comes to algebra we have to operate with x and y. There is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me.*

It is important to realise that even if Ame had produced the correct identity $x + x = 2x$, she would interpret it as an equation, and therefore a question to find what value of x makes the equation true and she still would not be able to choose any value to check it, because she would not know what x is.

However, her “simplification” $x + x = x^2$ is *not* generally true (is *not* an *identity*). She therefore has now produced an *equation*, which is true only for *some* values of x. She then proceeds to actually find a solution for this equation, but her confusion between the meaning of an identity and the meaning of an equation, leads her to now wrongly conclude that $x + x = x^2$ is an identity (is generally true)[3].

For Kenneth the letters do not represent *numbers*, but real *concrete letters of the alphabet*. So the teacher's suggestion to replace the letters with numbers does not make sense to him at all. His interpretation of the task is not to construct an equivalent algebraic expression, but he (and we suspect Ame too) is doing *calculations with letters* to get an *answer* in the same way he did calculations with numbers in the primary school.

Situations like this are commonplace in classrooms throughout the country and, indeed, throughout the world. It is well-documented that children have severe difficulties in mastering the basic notions of algebra. Yet, Ame and Kenneth are average, hard-working pupils who pay attention in class and do their homework diligently. They have excelled at mathematics in the primary school. How come they are suddenly having trouble with algebra? How do we explain it? This question of course underlies all our work. We will return to it in a later section. We also attach three research papers about children's thinking at the end of this section.

Variables have explanatory power

PROBLEM 6: THINK OF ANY NUMBER
Think of any number Double it. Add 6. Halve the result. Subtract your original number. What is your result?

It may come as a surprise to us that, although different people thought of different numbers, they all obtain the same result, namely 3. The mathematical attitude, and the attitude in an inquiry classroom, is that we naturally want to *explain* this phenomenon. There is no need to prove that it is true – we know it is true. But we wonder *why* the result is always 3, even when we all started with different numbers. Why 3, why not 5? To try to understand what is going on, we could write down different people's results (special cases), for example:

	Thandi	Sam	Thuli	Vusi
Think of any number	3	7	8	12
Double it	6	14	16	24
Add 6	12	20	22	30
Halve it	6	10	11	15
Subtract original number	3	3	3	3

[3] It is not at all simple! We may think that the transformation $x + x = x^2$ is "wrong". But it is *true* for $x = 0$ and $x = 2$! Yet, these are *special cases*, and we cannot conclude, like Ame does, that the statement is generally true.
To understand this, learners must understand that:

- $x + x = x^2$ is a quadratic *equation* which is true for *only* two values of x (compare the Avis-Imperial problem)
- $x + x = 2x$ is an *identity* which is true for *all* values of x (compare the VAT-tax problem)

We are here making the same distinction mathematicians make when they use quantifiers:
$\exists\, x$ so that $x + x = x^2$ (there *exist some x such that …*)
$\forall\, x,\ x + x = 2x$ (for *all x* ...)

To *check* a transformation by checking with arbitrary values – the teacher's advice – one must realise that

- if you choose a value and the resulting *numerical* sentence is *false*, you know for sure that the transformation is wrong, e.g. for $x = 1$, $x + x = x^2$ is false and $x + x = x^2$ is an incorrect simplification (transformation).
- if you choose a value and the resulting *numerical* sentence is *true*, you cannot be sure the transformation is correct, e.g. for $x = 2$, $x + x = 2x$ is true, but remember, $x + x = x^2$ is also true for $x = 2$!

It is difficult, if not impossible to deduce the underlying structure from such specific examples. It is exactly in such situations where the advantages of a “generalised number” becomes clear: In stead of taking any specific number, select a *symbol* such as ♥ or x to stand for *any* number. Then we have:

	♥ notation	**x notation**
Think of any number	♥	x
Double it	♥♥	$2x$
Add 6	♥♥ 6	$2x + 6$
Halve it	♥ 3	$x + 3$
Subtract original number	3	3

The *structure* is now quite clear. $\frac{2x+6}{2} - x = 3$ is an *identity*, so *for any (all) values* of x the result is always 3. Choosing, manipulating and interpreting such generalised numbers (variables) is an essential part of mathematical reasoning and of the language of algebra.

PROBLEM 7: THINK OF ANY NUMBER[4]

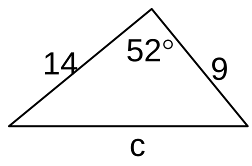
Think of any secret number Multiply by 4. Add 6. Halve the result. Subtract your original number. What is your result?

Do the calculations using Thandi, Sam, Thuli and Vusi's numbers above. Explain! How is this different from the situation in Problem 12?

Explain how, if someone told you their final answer, you can immediately tell them their secret number (e.g. if Thandi tells you her answer is 6, you can surprise her by telling her that her secret number is 3!).

Substitution

An obvious example of specialisation is when we substitute values for variables in a formula, identity or equation. This is such a standard practice in our everyday mathematical activity that we probably seldomly think of it as specialisation or explicitly think of the logical relationships involved. Yet, many problems in “doing” mathematics are due to our failure to specialise appropriately, as you will realise when you look at the following examples.



For example, once we decide to calculate the length of c in the figure using the cosine formula, we first write down the cosine formula in general form, and then substitute the specific known values:

$$c^2 = a^2 + b^2 - 2bc\cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$\Rightarrow c^2 = 14^2 + 9^2 - 2\times 14\times 9\times \cos 52 \quad \ldots\ldots\ldots(2)$$

The logical reasoning here is that if the general formula (1) is true, then the specific case (2) is true. (See section 3 on Modus Tollens: P $\Rightarrow$ Q, P is true, so Q is true.)

Of course we do not only substitute *numbers* for variables, but often also substitute variables for variables. There are two important cases to consider.

[4] See also this birthday problem and another birthday problem

First, we substitute *algebraic expressions* into general statements to manipulate general expressions. This is the thought process behind most manipulations in algebra. For example, if we want to factorise $x^4 - 4y^2$, we first transform the expression into a recognisable form. In this case we can write the expression as $(x^2)^2 - (2y)^2$. Once we recognise this form as a *special case* of the general statement $a^2 - b^2 = (a + b)(a - b)$, we merely substitute $a = x^2$ and $b = 2y$ into the general statement:

$$a^2 - b^2 = (a + b)(a - b) \text{ ……………(3)}$$
$$\Rightarrow x^4 - 4y^2 = (x^2)^2 - (2y)^2 = (x^2 + 2y)(x^2 - 2y) \text{ …………(4)}$$

The logical reasoning here is that the general formula (3) is true, then the specific statement (4) is true.

Second, we often specialise to deduce general statements that are specific cases of more general statements. For example, to find or recall a formula for cos 2θ, we start with the general statement

$$\cos (A + B) = \cos A \cos B - \sin A \sin B \text{ ………………(5)}$$

and then merely substitute A = B = θ into (5):

$$\cos 2\theta = \cos (\theta + \theta) = \cos\theta \cos\theta - \sin\theta \sin\theta \text{ ………..(6)}$$
$$\Rightarrow \cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

Statement (6) is a special case, or specialisation of statement (5).
And because the general statement (5) is true, it follows that the specific case (6) is true.

Of course, we also substitute *values* into general statements to prove that the statement is *false!* This is disproving or rejecting a statement with a *counter-example*. For example, if we think that $x + x = x^2$ is always true, we can check it by testing a specific case, say $x = 1$:

If $x + x = x^2$ is true, *then* $1 + 1 = 1^2$ is true.
But $1 + 1 = 1^2$ is false, therefore $x + x = x^2$ is false!

Of course, we must be careful, for if we choose $x = 2$ we have:

If $x + x = x^2$ is true, *then* $2 + 2 = 2^2$ is true.
But because $2 + 2 = 2^2$ is true, we cannot conclude that $x + x = x^2$ is true!
This specific invalid logic is a big problem in mathematical activity! In section 3 we will later analyse the situation in more detail[5].

---

[5] To check if the manipulation $x + x = x^2$ is correct, if we check for $x = 1$:
$x + x = x^2$ (P) $\Rightarrow$ $1 + 1 = 1^2$, i.e. 2 = 1 (Q). Q is false, so P is false by *Modus Ponens*.

However, if we check for $x = 2$:
$x + x = x^2$ (P) $\Rightarrow$ $2 + 2 = 2^2$, i.e. 4 = 4 (Q). Q is true. But we can't deduce that P is true – it is the *Converse trap!*

PROBLEM 8: GENERALISING PYTHAGORAS

On the worksheet below, different *similar* figures are drawn on the sides of a right-angled triangle. Find the areas of these similar figures (you can *count* the number of units, or you can *calculate* the areas using known formulae). Organise your data in the table below, then look for a pattern and *generalise*. Formulate a *conjecture*. Try to *prove* your conjecture.

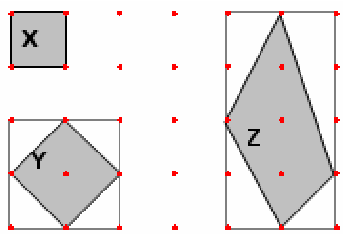
In what sense is your conjecture a *generalisation* of the Theorem of Pythagoras (why is it *more general)*?

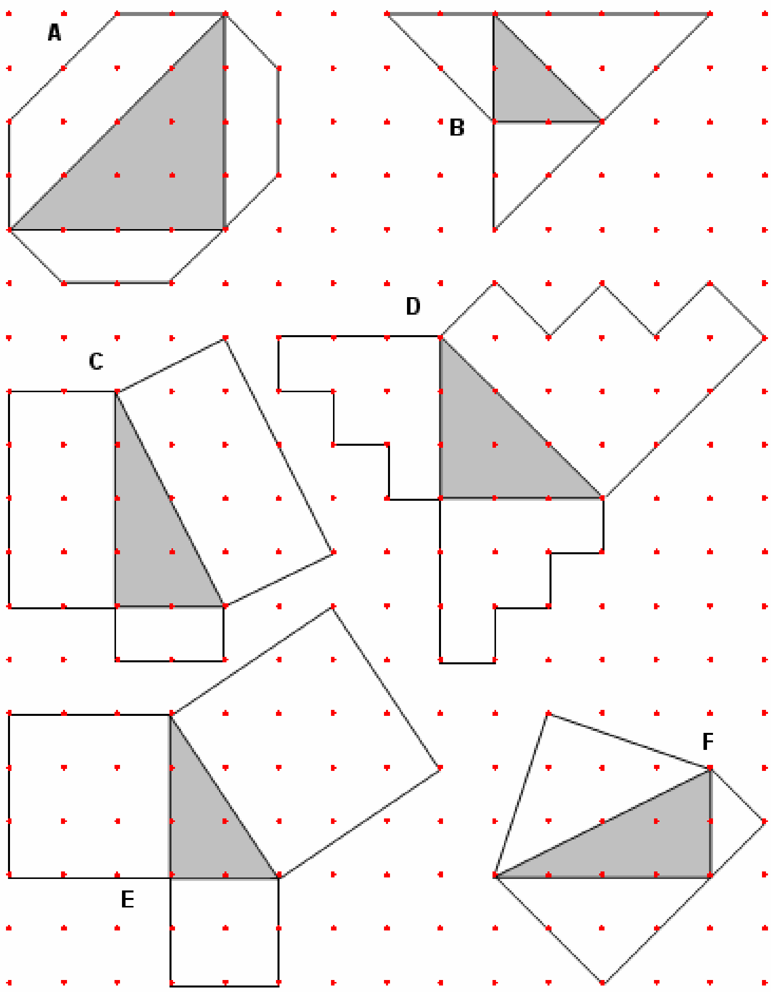
Figure	Area of figure on shortest side	Area of figure on other side	Area of figure on hypotenuse
A			
B			
C			
D			
E			
F			

On this dotty paper, the unit area is the area of the 1×1 square X. The area of other figures can be found by dissecting or surrounding the figure and adding or subtracting known areas (the whole is equal to the sum of its parts). For example:

Area of Y $= 2 \times 2 - 4 \times \frac{1}{2} = 2$ or $4 \times \frac{1}{2} = 2$

Area of Z $= 2 \times 4 - (1 + 1 + \frac{1}{2} + 1\frac{1}{2}) = 4$



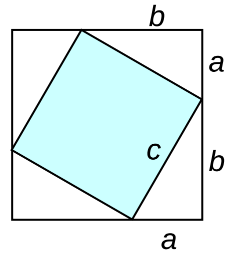


PROBLEM 9: SIMILAR FIGURES

Prove for each of the cases A-F on the worksheet that the three figures on the sides of the triangle are similar.

Formulate an argument to prove the following statements:

- Any two squares are similar
- Any two circles are similar



PROBLEM 10: PROVING PYTHAGORAS

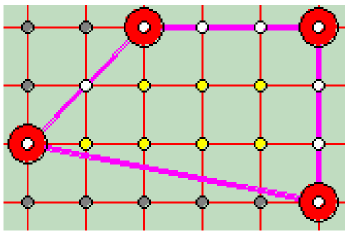
Use this figure to *prove* the Theorem of Pythagoras, i.e. prove that $c^2 = a^2 + b^2$.

(Four congruent triangles are inscribed in the corners of a square.)

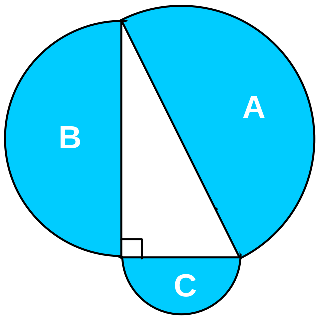
PROBLEM 11: PIX THEOREM

There is a theorem called Pix theorem which gives the relationship between the area of a figure and the number of points on square dotty paper:

$$\text{Area} = \tfrac{1}{2}b + i - 1$$



where b is the number of points on the *boundary* of the figure, and i is the number of points *inside* the figure. Make a table of the areas of the figures on the worksheet to illustrate the theorem numerically (you are therefore *specialising*).



PROBLEM 12: GENERALISING PYTHAGORAS

A, B and C are semi-circles drawn on the sides of a right-angled triangle.

Prove the following generalisation of the Theorem of Pythagoras:

$$\text{Area A} = \text{Area B} + \text{Area C}$$

PROBLEM 13: PYTHAGORAS TRIPLES

A triple (a, b, c) of whole numbers that satisfies the equation $a^2 + b^2 = c^2$ is called a Pythagorean triple. Examples of such triples are (3, 4, 5) and (5, 12, 13).

Check whether these specific cases are true. Generalise!

$3^2 + 4^2 = 5^2$	$4^2 + 3^2 = 5^2$
$5^2 + 12^2 = 13^2$	$8^2 + 15^2 = 17^2$
$7^2 + 24^2 = 25^2$	$12^2 + 35^2 = 37^2$
$9^2 + 40^2 = 41^2$	
$11^2 + 60^2 = 61^2$	

Prove that there is an infinite number of relative prime[6] Pythagoras triples of the form (a, b, b + 1) and of the form (a, b, b + 2), but none of the form (a, b, b + 3).

[6] Relatively prime triplets have no common factor other than 1. So (3, 4, 5) is a relative prime triplet, but (6, 8, 10) is not. You can use this applet to generate Pythagoras triples. Formulate some conjectures?

PROBLEM 14: ARITHMETIC SEQUENCES

Prove that the sum of two Arithmetic Sequences is again an Arithmetic Sequence.

We can attack the problem inductively or deductively, depending on whether we see a plan. We have emphasised that many times it is a little of both. For example, to make sure we understand what the problem is about, Let's specialise by taking two specific Arithmetic Sequences:

4, 7, 10, 13, 16, …
6, 10, 14, 18, 22, …

The sum of the sequences means that we add the two sequences term by term to form a new sequence:

10, 17, 24, 31, 38, …

This certainly looks like an arithmetic sequence, because there is a common difference of 7, but this is an inductive observation, so unless we can show structurally why the difference will always be 7, we cannot be sure.

Also, we will want to know exactly how this new sequence is derived from the original two sequences, i.e. can you look at the two original sequences and write down their sum without actually adding them? There are different ways to do this. We can look at the *numbers* in the three sequences and try to recognise a pattern. Or we can look at their generating formulae and try to recognise a pattern: In this case we have $3n + 1$, $4n + 2$ and $7n + 3$. Can you see how the third is formed from the first two?

For the moment we are going to pretend we cannot immediately see the patterns in the special cases. We suggest that it is many times probably easier to tackle the problem deductively. If we start inductively we have to still do it deductively to prove and explain our conjectures. But if we *first* do it deductively, we not only prove that it is true, we also show why it has this form, plus we have the added bonus that the result actually provides a formula for the new sequence. Let's try.

Symbolise *any* two Arithmetic Sequence with

$a, a + d, a + 2d, a + 3d, \dots a + (n - 1)d$ …………..(1)
$b, b + e, b + 2e, b + 3e, \dots b + (n - 1)e$ …………..(2)

Are you sure you understand why it is necessary to use different symbols for the two sequence? Of course b can be equal to a and e equal to d – that would be the special case of adding a sequence to itself. But if we write both sequences in terms of a and d, then we are proving *only* that a sequence plus itself is an arithmetic sequence and not that the sum of *any* two sequences is an Arithmetic Sequence.

So the sum of these two new sequences is

$(a + b), (a + b) + (d + e), (a + b) + 2(d + e), + \dots + (a + b) + (n - 1)(d + e)$ ……(3)

It is important that we see the *structure* in this representation: If we put $a + b = c$ and $d + e = f$, sequence 3 can be written as

$c, c + f, c + 2f, c + 3f, \dots c + (n - 1)f$

which is clearly an Arithmetic Sequence with first term $c = a + b$ and common difference $f = d + e$.

If we did not in the inductive phase recognise how the new Arithmetic Sequence derives from the first two, it will now be easy to use sequence 3 as a formula and then to *specialise*.

Looking back, our result, and the *form* of our result is not surprising – this is exactly how we add polynomials, by adding corresponding like terms, for example:

$(3x + 1) + (4x + 2) = 7x + 3$

In general, $(ax + b) + (cx + d) = (a + b)x + (b + d)$, which really is a special case of the additions of vectors: $(a, b) + (c, d) = (a + c, b + d)$.

We should make the connections between the addition of Arithmetic Sequences and the addition of functions. Maybe this table will help?

x	1	2	3	4	10	11,1	20
$3x + 1$	4	7	10	13	31	34,3	61
$4x + 2$	6	10	14	18	42	46,4	82
$7x + 3$	10	17	24	31	73	80,7	143

Still looking back:
What if? What will happen if we add two Geometric Sequences?
What if? What will happen if we multiply two Geometric Sequences?

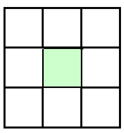
PROBLEM 15: LAST DIGIT
If 3^{2005} is written in normal Hindu-Arabic notation, what is its units (ones) digit?

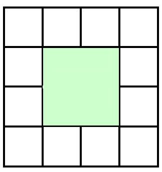
Specialise to understand the problem: 3^4 in normal notation is 81, so the units digit is 1.

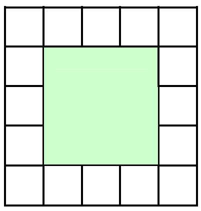
Note: Reflection, looking back, means that the intention is not that you should merely directly answer the question. Rather, the typical mathematical attitude is to make sure *that* it is true and *why* it is true. So you should develop a full theory of all the possible units digits of 3^n. And you must try to *explain* it, which necessarily means that you must unravel the *structure* of the situation, so it means that you must dig deeper than the surface patterns …

Click here for a brief discussion.

PROBLEM 16: FLOWER BEDS
The town council of *Tendele* decides to beautify the town. They build square flower beds of different sizes and surround them with square tiles as shown in these sketches:

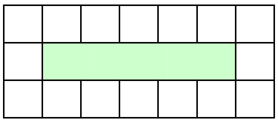

1 by 1 bed

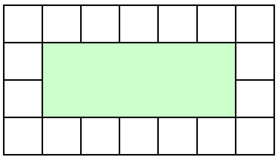

2 by 2 bed

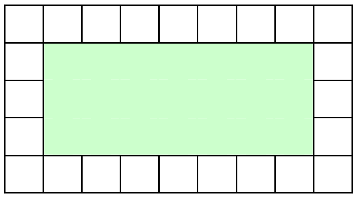

3 by 3 bed

1. How many tiles are needed for a
   (a) 10 by 10 bed (b) 100 by 100 bed (c) n by n bed?
2. 500 tiles are available for a new bed. What is the largest bed that can be built?
3. After the tiles for a new bed have already been delivered, they decide to make the side of the bed one tile larger. How many extra tiles must be ordered?

PROBLEM 17: MORE FLOWER BEDS
The town council decides to also build rectangular beds, as shown below:


5 by 1 bed

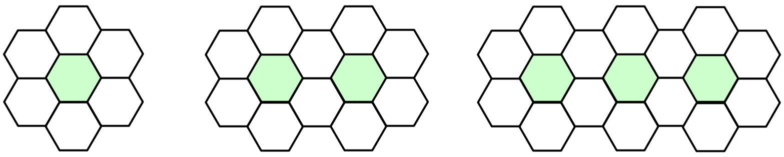

5 by 2 bed


7 by 3 bed

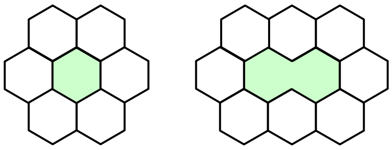
1. How many tiles are needed for a
   (a) 6 by 5 bed (b) 100 by 90 bed? (c) m by n bed?
2. Show how your formula for an n by n bed can be deduced as a special case of your formula for an m by n bed.
3. What is the largest bed that can be built with 500 tiles?

PROBLEM 18: SMART FLOWER BEDS
The city council wants to use hexagonal paving tiles to build flower beds according to the different designs shown in the sketches below.
In each case, how many tiles do the council need for a 100-bed? And for a n-bed?
In each case, what is the largest bed that can be built with 500 tiles?
In each case, how many extra tiles are needed to change an n-bed into an (n+1)-bed?

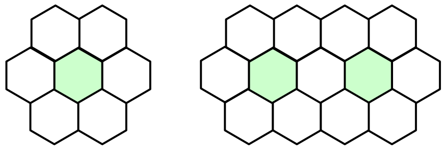
Design 1


1-bed 2-bed 3-bed

Design 2


1-bed 2-bed

Design 3

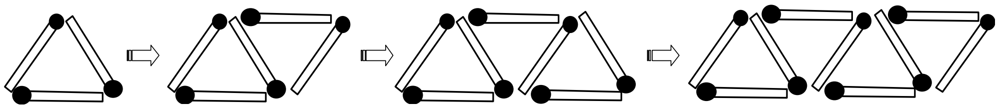

1-bed 2-bed

PROBLEM 19: NUMBERS
How many different 3-digit numbers can you make using only the three digits 2, 3 and 4? (Example: 234, 423, but not 422.)

What if? How many 4-digit number scan be made using the digits 3, 6, 8 and 9?
Generalise: How many n-digit numbers can be made using n different digits?

PROBLEM 20: PLAYING WITH MATCHES

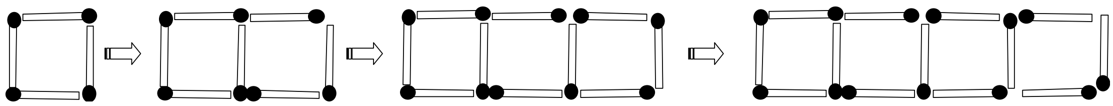
1. Sylvia forms **triangle** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of triangles	1	2	3	4	5	6	7	8	9	10	100
No of matches	3	5	7	9							

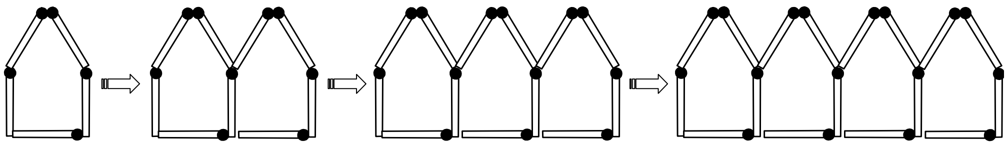
2. On another Sylvia forms **square** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of squares	1	2	3	4	5	6	7	8	9	10	100
No of matches	4	7	10	13							

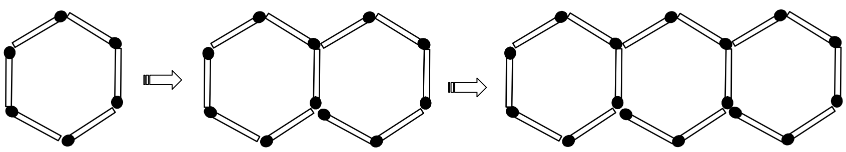
3. On another day Sylvia forms **pentagon** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of pentagons	1	2	3	4	5	6	7	8	9	10	100
No of matches	5	9									

4. On another day Sylvia forms **hexagon** patterns with matches as shown below:



How many matches does Sylvia need to build 100 **hexagons**?

5. What is the same and what is different in the situations in 1, 2, 3 and 4?

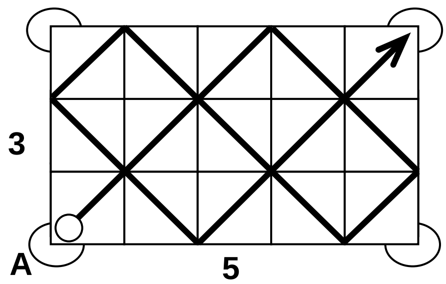
6. How many matches will Sylvia need to build 100 **decagons** (a polygon with 10 sides) in the same way?

7. How many matches will Sylvia need to build 100 n**-gons** (a polygon with n sides) in the same way?

For a discussion, see Fireworks notes.

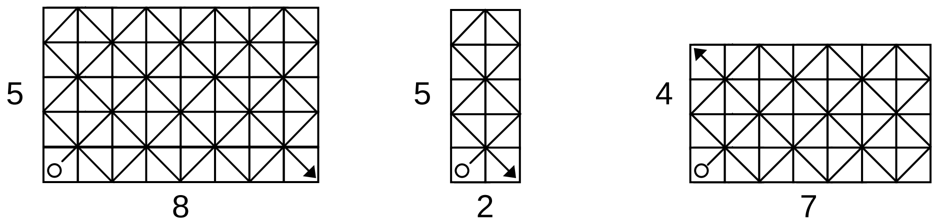
PROBLEM 21: SNOOKER

The special snooker table in the sketch has dimensions of 3 by 5 and has 4 pockets, one at each corner. A ball is placed at pocket A, and then hit away from the pocket at an angle of 45° to the sides of the table. The ball rebounds from each side of the table at an angle of 45° until it falls into the top right pocket. On this table, the ball rebounds form the sides 6 times before it falls into the pocket. (We do not count the start and end positions as rebounds – a "rebound" is when the ball bounces from a *side*). Predict:



(a) the number of rebounds for tables of different dimensions.
(b) In which pocket the ball will fall.

One can generate some special cases and make a table of the results, for example:



WIDTH	LENGTH	# REBOUNDS
3	5	6
5	8	11
5	2	5
4	7	9

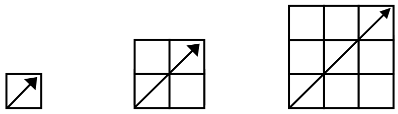
We were not very systematic, with the result that it probably will not be that easy to recognise a pattern in the above table. Can you?

In this problem the dependent variable (the number of rebounds) depends on *two* variables (the width and the length), so it is not so clear how to approach the problem systematically. A good approach is to keep the one variable constant and systematically varying the other, e.g. 1×1, 1×2, 1×3, … then 2×1, 2×2, 2×3, … then 3×1, 3×2, 3×3, … etc.

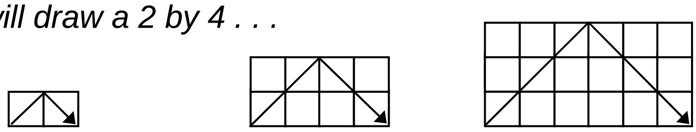
Here we will show a different approach by describing how a group of teachers at an in-service workshop attacked the problem, not by specialising systematically to numbers (1, 2, 3, …), but by systematically considering *different classes of special cases*. There is much to discuss about the process and benefits of learning through group activity, but we concentrate here only on the mathematics.

Vusi: Let's try a square …
Jane: What do you mean a square? How big?
Thandi: Well, let's start with a 1 by 1.

Between the children in the group, they quickly draw different square tables:

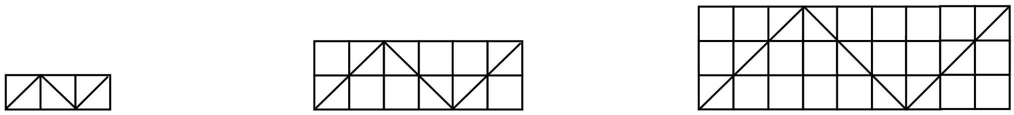


Vusi: It's the same for any square!
Jane: That was easy! What next?
Vusi: For a square, the length is equal to the width, let's see what if the length is twice the width
Mary: OK, let's try a 1 by 2.
Thabo: I will draw a 2 by 4 . . .



Mary: They are the same, just bigger.
Vusi: Now let's check a 1 by 3.

Again Vusi draws a 1 by 3 table, but Mary also draws a 2 by 6 and a 3 by 9:



Thabo: OK, let's put it in a table!

Vusi makes this table:

WIDTH	LENGTH	# REBOUNDS
1	1	0
1	2	1
1	3	2
1	4	3

However, Mary makes this table:

WIDTH	LENGTH	# REBOUNDS
1	1	0
1	2	1
2	4	1
3	6	1

Vusi is very quick to notice a pattern and to formulate a conjecture:

Vusi: It is always 1 less than the length.
Mary: You must first simplify the fraction.

We notice that Vusi's generalisation is *wrong*. Or rather, as happens many times, his generalisation is based on too few cases, or rather, on only a special kind of number, namely only for tables with a width of 1. Vusi's "generalisation" is true for all the numbers in his table, but Mary quickly sees that it is not true for some of her tables.

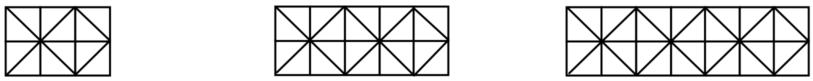
In essence, Mary is *refuting* Vusi's generalisation! She has noticed that his generalisation "it is always 1 less than the length" is not true for a 2 by 4 table – this simple case is a *counter-example* which *disproves* Vusi's conjecture.

At the same time, however, instead of searching for a completely new conjecture, Mary "saves" Vus's conjecture by *adapting* it – Vusi's conjecture is still true, provided that the "fraction" is first simplified.

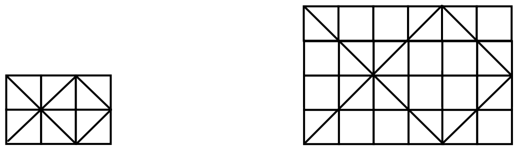
We make two remarks: First, Mary has constructed a new lens of looking at the situation, namely *fractions*, probably triggered by her previous knowledge that $\frac{1}{2}$, $\frac{2}{4}$ and $\frac{3}{6}$ are equivalent fractions. So Mary is *making connections* between this knew situation and her previous knowledge. This is a powerful mode of learning! Second, the new conjecture, i.e. "Simplify the faction if it can be simplified, then the number of rebounds is always 1 less than the length", is of course still a special case applicable only to 1 by *n* tables or multiples of such tables.

*Vusi (*confirming Mary's adaption)*: A 1 by n is n – 1, if it is not 1 you must first simplify it.*
Jane: But what if it cannot be simplified?
Thabo: OK, lets take a 2 by 3:

Vusi draws a 2 by 3, a 2 by 5 and a 2 by 7, i.e. he *specialises* by taking specific cases of tables that "cannot be simplified":



Mary draws a 2 by 3 and a 4 by 6 table:



Mary: Its always a fish, just larger!

We notice that Vusi and Mary are thinking differently. Vusi is systematically working with the numbers 1, 2, 3, … while Mary is generalising towards "equivalent fractions" and probably thinking *geometrically* – her remark "its always a fish, just larger" probably expresses a good understanding of *similarity!*

Vusi now completes the following table:

WIDTH	**LENGTH**	**# BOUNCES**
1	1	0
1	2	1
1	3	2
1	4	3
2	3	3
2	5	5
2	7	7

Thabo (pointing to the table): *If its 1, its one less, if its 2, its equal, so if it is 3 its one more!*

Thabo is here making an *inductive generalisation*. From the pattern he observes for tables with widths of 1 and 2 (for 1 the number of bounces is one less than the length, i.e. Vusi's conjecture, and for 2 it is equal to the length), he *extends* the pattern (for 3 it is one more than the length). He adds the following rows to the table, which *confirms* his conjecture. (Note that he obtained these values from interchanging the widths and lengths in the table above, which is of course based on yet another conjecture, namely that an m by n table and an n by m table give the same number of bounces!)

3	1	2
3	2	3
3	4	5

Vusi now has the following table:

WIDTH	LENGTH	# BOUNCES
1	1	0
1	2	1
1	3	2
1	4	3
2	3	3
2	5	5
2	7	7
3	4	5

Vusi: Look here (pointing)*, the 1 by 4 and the 2 by 3 are both 3!*
So 1 + 4 = 5 and 2 + 3 = 5, so if the sum is the same, the answer is the same!
Also here (pointing): *2 + 5 = 7 and 3 + 4 = 7 are the same, the answer for both is 5. And it's the same as 1 by 6 which we know is one less than 6.*
So if you have say 4 by 7, you change it to 1 by 10 and then the answer is 9!

This is a powerful conjecture. Do you agree? But the others do not listen to Vusi!

Thabo: I see a new pattern: You take the width plus the length and then you minus two!

Mary: Yes, Thabo is right, if you take 3 by 4, then it is 3 + 4 = 7, minus 2 gives 5 which is right!

We can generalise Thabo's conjecture in symbols: For an m by n table, the number of bounces is $m + n - 2$. Of course, this is only true if m and n have no common factor ("if it cannot be simplified"), otherwise it must first be simplified and then the formula applied.

They did not stop to look back. They accepted that Vusi's initial conjecture ("it is one less") was wrong. Of course he was wrong. But what they, and we should see, is that Vusi was correct for the special cases that $m = 1$. So he formulated a special case, which is not generally true. The interesting thing is that such a correct special case can be deduced from the general case by specialisation:

If $m = 1$, the formula $m + n - 2$ becomes $1 + n - 2 = n - 1$

So Vusi was correct (for $m = 1$)! They also did not give credit to Vusi's last conjecture, which meant that he could bypass Thabo's formula: he can reduce any m by n table to a 1 by $m + n - 1$, and then use his own conjecture ("it is one less")! Do you agree?

PROBLEM 22: Design a short method to calculate the square of a two-digit number ending in 5, e.g. to calculate 65^2.

2.2 INDUCTIVE AND DEDUCTIVE REASONING

After having worked through some problems, we can now make our approach to problem solving more explicit. In all our work, we are basically using two different kinds of reasoning:

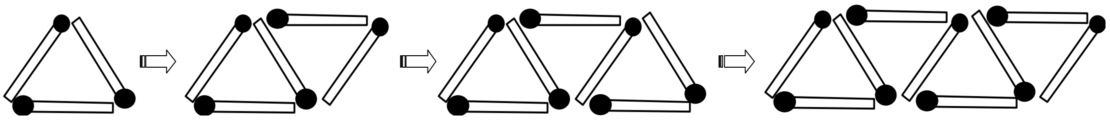
Inductive reasoning is a method of using *numerical pattern recognition* to draw general conclusions. Although inductive reasoning is of great importance in developing ideas, it cannot prove that they are correct, because it is based on a limited set of observations.

Deductive reasoning is a method of using *structural analysis* to draw conclusions from ideas we accept as true by using logic (Jacobs, 1982). We need deductive reasoning to prove that our ideas are correct.

In the examples that follow, we will further analyse the processes of inductive and deductive reasoning. Let's return to Problem 20:

PROBLEM 20: PLAYING WITH MATCHES

1. Sylvia forms triangle patterns with matches as shown below:

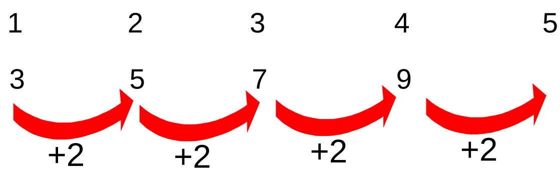


Complete the following table. Describe any number patterns!

No of triangles	1	2	3	4	5	6	7	8	9	10	100
No of matches	3	5	7	9							

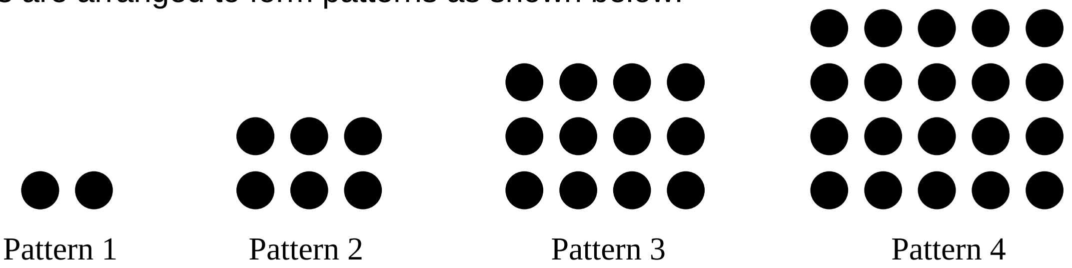
Inductive reasoning consists of two sub-processes:
1. pattern *recognition* in a finite set of data (*abstraction*)
2. pattern *extension* to cases not in the present set (*generalisation*)

One can focus on the *numbers* given in the table (we call this the *database)* and recognise a vertical (functional) relationship $m = 2t + 1$ which easily yields all the solutions. Or one can recognise a horizontal (recursive) pattern $f(t + 1) = f(t) + 2$, which can serve as a model to generate additional information about the situation:



While the inductive approach above looked at the *numbers* and ignored the *matches* (picture), a deductive approach focuses on the process of packing the *matches* and ignore the numbers. For example, the structure of the matches can be formulated in words as "you start off with 3 matches and then add another 2 matches for every additional triangle that you build". So, for 100 triangles we will need $3 + 99 \times 2 = 201$ matches. This can be *generalised* to $3 + 2(t – 1)$ matches for t triangles.

PROBLEM 23: DOTS
Dots are arranged to form patterns as shown below:



How many dots are there in:

- Pattern 200?
- Pattern *n*?

Many people prefer an *inductive approach*, and organise the information in a table and then try to identify *patterns in the numbers* in the table to solve the problem:

Pattern no	1	2	3	4	5	6	200	n
# dots (D)	2	6	12	20				

Do you see any patterns in the table? Most people recognise a recursive (horizontal) pattern +4; + 6; +8; … but it is not so useful in this case. It is not so easy to recognise a functional (vertical) relationship in the table!

A *deductive approach* focus on the *structure* of the sketches. The figures are rectangles, and the number of dots can be seen as the area of the rectangle. So, in stead of *counting* the dots and working with the numerical *answers* 2, 6, 12, 20, …, when we work with the *structure*, we describe the *method* for calculating the number of dots without calculating it, i.e. 1×2, 2×3, 3×4, etc. Then we essentially have our functional rule!

Pattern no	1	2	3	4	5	6	200	n
# dots (D)	1×2	2×3	3×4	4×5	5×6	6×7		

Now the pattern in the numbers in the table is easy to see! This shows the inter-relationship between induction and deduction – deduction actually helped to make induction easier! We often do not use just one approach, but both.

There is a tradition of opposition between adherents of induction and of deduction. In my view it would be just as sensible for the two ends of a worm to quarrel.

Alfred North Whitehead

Let's now return to Problem 22 above:

PROBLEM 22:
Design a short method to calculate the square of a two-digit number ending in 5.

To make sense of the question, i.e. to *understand the problem*, you must connect it to your previous knowledge and experiences. What is meant with a "short method"? Do you know the term? Maybe you recall these methods you learned in primary school:

- To multiply a number by 25, first multiply by 100 and then divide by 4.
- To multiply a number by 125, first multiply by 1000 and then divide by 8.

A short method therefore means a method for getting the answer by not doing the given problem, but rather some other process which you can basically do mentally without much use of paper and pencil.

"A two-digit number" of course is a general statement, not meaning any specific two-digit number, but really *any* two-digit number. So the problem really wants us to be able to *look* at a calculation like 65^2 and write down the answer without really calculating 65×65 by long multiplication.

How to get started? We emphasise again that there are two basic routes: induction or deduction. One should develop a feeling for when the one could be more accessible than the other. Let's try the inductive approach: Try some special cases, be systematic, organise your data. Use a calculator to generate the following answers:

$$
\begin{aligned}
15^2 &= 225\\
25^2 &= 625\\
35^2 &= 1225\\
45^2 &= 2025\\
55^2 &= 3025
\end{aligned}
$$

So it is all about seeing useful patterns that will enable us to write down such answers. It is easy to recognise/abstract and to conjecture that the answer will always end in 25. There is enough structural backing ($5^2 = 25$) to accept and believe that this is always true. So we already know that $85^2 = ??25$

We need more. We suggest that we ignore the 25s in the answer as *noise* which distracts us from seeing other patterns. Many times when working on this problem, students get very excited and say that they see

$$
\begin{array}{rl}
2 & \\
 & \mathbf{+4}\downarrow\\
6 & \\
 & \mathbf{+6}\downarrow\\
12 & \\
 & \mathbf{+8}\downarrow\\
20 & \\
 & \mathbf{+10}\downarrow\\
30 &
\end{array}
$$

This is a recursive pattern: $T_n = T_{n-1} + 2n$. It is very interesting, but not very useful. If we want to use this pattern to calculate 85^2 we will first have to know the answer of 75^2 , i.e. we will have to write down the whole pattern from the beginning. Hardly a short method!

What would be more useful, would be a functional relationship. Let's remove more noise – the 5-part probably takes care of the 25-part on the right, so we expect

Input	**Output**
1	2
2	6
3	12
4	20
5	30
n	?

Maybe you recognise the following pattern:

Input		**Output**
1	× **2**	2
2	× **3**	6
3	× **4**	12
4	× **5**	20
5	× **6**	30
n	**?**	

To see the pattern, ignore the rest and concentrate only on the relevant data:

Input	
1	**2**
2	**3**
3	**4**
4	**5**
5	**6**
n	$\boldsymbol{n+1}$

It should be clear that we multiply the ones-digit by the next whole number. So our conjecture is:

To square a two-digit number ending in 5: The number ends in 25 and the first digits are given by multiplying the tens-digit by the next whole number.

Is it true? We can easily check all 9 cases, so we know it is true. But still we want to know *why* it is true. Deduction will help:

Any two-digit number is of the form 10a + b. We have a special case where the units-digit is 5, so b = 5:

$$
\begin{aligned}
(10a + 5)^2 &= 100a^2 + 100a + 25 \\
&= 100a(a + 1) + 25
\end{aligned}
$$

That pretty much explains it, provided we can interpret the symbolism. The purpose of the 100 is to guarantee that the answer ends in 25. Understanding the function of the 100 means that we can also understand why this method works for numbers ending in 5 – we only get 100 in the middle term of the expression because of the 5!

PROBLEM 24:
Calculate each of the following. What patterns do you notice? Make a conjecture. Test your conjecture with further special cases and then prove your conjecture.

$23 \times 27 =$	$31 \times 39 =$
$24 \times 26 =$	$57 \times 53 =$
$28 \times 22 =$	$87 \times 83 =$

PROBLEM 25: WIMBLEDON
You are the sports secretary of your sports club and you need to know how many games will be played in your knock-out competition. The competition works like the Wimbledon tennis championship: two players are drawn to play against each other in the first round. The loser falls out and the winner plays against a winner of another game. This continues, with losers being eliminated and winners playing against winners until the champion is crowned. If there are 60 players, how many games are played in total? And if there were 35, or 28 or 44 players, i.e. can you develop an easy method (formula) to calculate how many games are played for any number of players?

A diagram will help us to understand the situation. For example, Figure 1 is a special case with 4 players: Player P_1 plays against P_2 and the winner, W_1, plays against W_2, the winner of the game between P_3 and P_4.

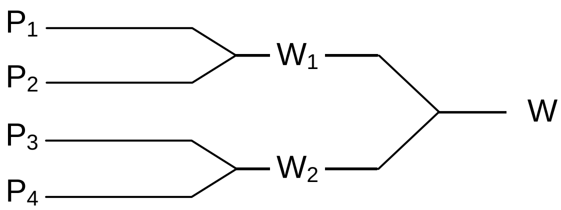


Figure 1

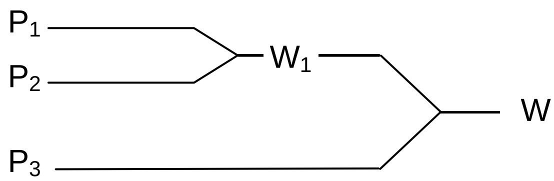


Figure 2

We also need to understand some finer details of the structure of the competition. For example, Figure 2 shows what happens when there is an odd number of players: one player gets a bye, i.e. does not play and proceeds automatically to the next round.

Thinking inductively, we can now organise our data of special cases into a table:

# *players*	1	2	3	4	5	6
# *games*	0	1	2	3	4	5

We easily recognise the pattern in the table and generalise it: The number of games is one less than the number of players, or written in symbols:

For n players there are $n – 1$ games.

Sometimes a *deductive explanation* can be deceptively simple. See if you follow this reasoning:

There can be only one champion, i.e. one player who did not lose. So, with 60 players, to get *1 winner*, we need *59 losers*. To get 59 losers we need *59 games*. That is all!

The reasoning is completely general: For n players, to get $n – 1$ losers, we need $n – 1$ games.

PROBLEM 26: NETBAL

In a netball game the final score was 47-38. How many possible half-time scores were there?

Try some *special cases*, to get to understand the structure of the situation. Let's look at a final score of 3-2 and make a *systematic list* of all the possible half-time scores. Let's do it also for a few other special cases:

<u>3-2</u>

3-0	2-0	1-0	0-0
3-1	2-1	1-1	0-1
3-2	2-2	1-2	0-2

<u>4-2</u>

4-0	3-0	2-0	1-0	0-0
1	1	1	1	1
2	2	2	2	2

<u>3-4</u>

3-0	2-0	1-0	0-0
1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4

A good strategy is to count the number of scores and organise the information in a table like this:

Final score	**# half-time scores**
3-2	12
4-2	15
3-4	20
m-n	?

The strategy is now to recognise a pattern in this database. That is not always easy! Can you see the pattern and then extend (generalise) it to answer the original question?

Mathematical thinking is not about numerical *answers*, but about relationships between numbers, about *structure*. Now lets look at the list of scores again. It is not a good idea to count the scores one by one, because that does not draw out the structure of the situation. However, any systematic way of counting will bring out some aspect of the structure. For example, looking at the 3-2 score, a "clever" way of counting would be to see equal groups. For example, counting vertically: 4 + 4 + 4. Or counting horizontally: 3 + 3 + 3 + 3. Both of these involve repeated addition and can be written as 4×3. The art of abstraction lies in recognising the same *structure* in the other examples as well.

We have not yet sufficiently unravelled the structure to can generalise it. We can see the structure of the *operation*, i.e. why it is multiplication or repeated addition. But what about the numbers? How do the numbers 3 and 2 in the score of 3-2 lead to the numbers 4 and 3 in the structure 4×3? We can inductively recognise the pattern $4 = 3 + 1$ and $3 = 2 + 1$, but *why* is it one more? The *explanation* lies in the introduction of 0 as a score – a score of 4 includes the possibilities of 0, 1, 2, 3 and 4 as scores. So once we understand the role of 0 as a score, the structure is clear, and we can formulate a generalisation: For a final score of m-n, there are $(m + 1)(n + 1)$ possible half-time scores.

PROBLEM 27: ROUND ROBIN

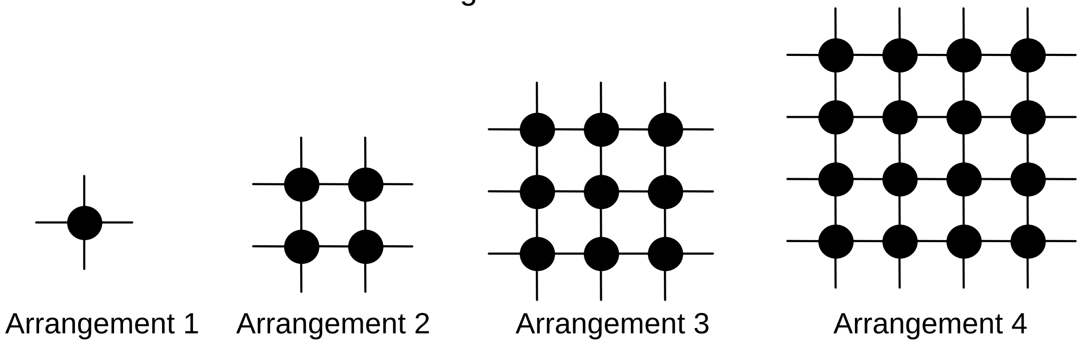
Mr Daniels is the match secretary for the Mpumalanga soccer league. He must arrange the soccer schedule for next year. Each team plays each other team twice – one match at home, and one match away.

How many league matches will be played if there are 9 soccer teams in the league?

Find a formula to work out how many matches must be played if there are *n* teams.

PROBLEM 28: BALLS AND RODS

Thabo links balls with rods in arrangements like this:



Find formulae for the number of balls and the number of rods in Arrangement *n*.

PROBLEM 29: SQUARE ROOT

Calculate: $\sqrt{12345678987654321}$

PROBLEM 30: ANGLES OF A REGULAR N-GON

You know that in a regular triangle (i.e. an equilateral triangle) each angle is 60°. You also know that in a regular quadrilateral (i.e. a square), each angle is 90°, and you probably know that in a regular hexagon (i.e. a polygon with 6 sides) each angle is 120°. Investigate and find a formula for the relationship between the number of sides of a regular polygon and the size of one of its angles.

Draw a graph of the relationship.

PROBLEM 31: PAINTING A ROOM

A man paints a room in 1 day en uses 4 tins of paint for the job.

How long will it take him, and how much paint is needed, to paint a room twice as long, twice as wide and twice as high?

PROBLEM 32: CONSECUTIVE NUMBERS

Some numbers can be written as the sum of two or more *consecutive* whole numbers, e.g.

13 = 6 + 7

14 = 2 + 3 + 4 + 5

15 = 7 + 8 = 1 + 2 + 3 + 4 + 5

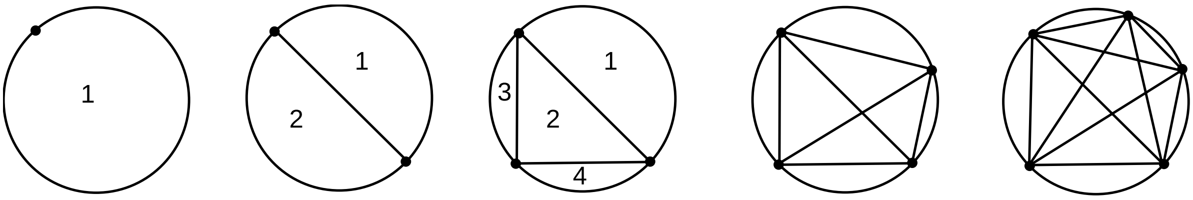
Some numbers *cannot* be written as the sum of consecutive whole numbers. Try writing 16 as the sum of consecutive numbers …

Investigate: Which numbers can and which numbers cannot be written as the sum of consecutive whole numbers. Try to develop a method so that you:

- can immediately decide *if* any given number can be written as the sum of consecutive numbers.
- can easily write the number as the sum of consecutive numbers.

2.3 THE PITFALLS OF INDUCTION

PROBLEM 33: REGIONS



If 3 *points* on a circle are joined, 4 *regions* are formed, as shown above.
Complete the table for 4 and 5 points by using the given sketches above.
What is the *maximum* number of regions into which 6 points on a circle will divide the circle if the points are joined? And 20 points?

# points (*p*)	1	2	3	4	5	6	20
# regions (*R*)	1	2	4				

You should really first solve the problem before continuing reading!

You probably noticed the recursive pattern 1, 2, 4, 8, 16, … in the numbers in the table and then continued the pattern to *predict* *R*(6) = 32. Or you may have inductively deduced the formula $R(p) = 2^{p-1}$, which also yields *R*(6) = 32.

Unfortunately our expectation that the pattern is continued beyond the fifth case is wrong! In fact *R*(6) = 31! You are encouraged to check for yourself by drawing a nice big circle and physically counting the number of regions.

We will return to this problem later (see page 58).

This example serves to remind us that inductive reasoning, powerful as it may be in discovering new patterns, is prone to error:

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks. Kasner & Newman, 1940

PROBLEM 34: ONES

Check these patterns:

x	x^2	Digit sum of x^2
1	1	1
11	121	4
111	12321	9
1111	1234321	16
11111	123454321	25

Now predict the digit sum of 11111111111^2 (11 ones squared).

Most of us will recognise the pattern in the digit sum of x^2 in the last column as squares, and then extend beyond the given five examples to predict that the digit sum of 11111111111^2 is $11^2 = 121$. However, if you actually calculated it, you will see that the predicted answer is incorrect!

Our abstraction that the numbers in the given examples are squares is correct. But this pattern is not continued, because the *structure* breaks down after 9 ones because of carrying. If you have not yet done it, you will want to check that this is correct and think about the reasons why the structure breaks down! This example again emphasises the danger of *only* focussing on the *numbers* in a situation and to generalise through induction without considering the *structure* of the situation.

$$\begin{array}{ccccc} & & 1 & 1 & 1 \\ & 1 & 1 & 1 & \\ 1 & 1 & 1 & & \\ \hline 1 & 2 & 3 & 2 & 1 \end{array}$$

PROBLEM 34A: ONES AGAIN

The pattern above breaks down because of our place value notation. But if we look at the situation differently, we find that the pattern is indeed continued! Check:

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 10^2$

$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 + 11 + 10 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 11^2$

Generalise! Explain! Prove!

PROBLEM 35: PRIMES

Investigate the nature of $P(n) = n^2 - n + 11,\ n \in N$

Let's approach the problem inductively by generating the following special cases:

$P(1) = 11$

$P(2) = 13$

$P(3) = 17$

$P(4) = 23$

$P(5) = 31$

$P(6) = 41$

$P(7) = 53$

$P(8) = 67$

$P(9) = 83$

$P(10) = 101$

These numbers are all prime. Would you agree that we can conclude that the answer is *always* a prime number?

Unfortunately, the very next case, i.e. $P(11) = 121 = 11^2$ is *not* prime! Although our observed pattern of primes is true for the given 10 cases, it is not correct to extend (generalise) the pattern beyond these 10 cases.

PROBLEM 36: ODD NUMBERS

The examples of numbers $P(n)$ in Problem 35 are all odd. Would you agree that we can conclude that the answer is always an odd number? How can you be *sure?*

PROBLEM 37: EVEN NUMBERS

The difference between consecutive numbers $P(n)$ in Problem 35 is even. Would you agree that we can conclude that the difference is *always* an even number? How can you be *sure?*

PROBLEM 38: MORE PRIMES

Show that $P(n) = n^2 - n + 41$ is prime for n = 0 to 40 but not prime for n = 41.

Show that $P(n) = n^2 - 79n + 1601$ is prime for n = 0 to 79 but not prime for n = 80.

Show that $n^2 - pn + q$ cannot be prime for $n = q$.

PROBLEM 39: IQ PROBLEM

What is the next number in this sequence: 2; 4; 6; 8; ?

We would probably all opt for 10. Are you *sure?* This is based on an implicit function rule $2n$. But as the table below shows, the next value may as well be 34, based on another valid function rule:

n	1	2	3	4	5	6
$2n$	2	4	6	8	10	12
$2n + (n-1)(n-2)(n-3)(n-4)$	2	4	6	8	34	132

PROBLEM 40: IQ TEST

What is the next number in this number pattern: 1, 4, 9, 16, ?

Surely, we will all answer 25! It is a sequence we all know very well, namely the sequence of square numbers, i.e. n^2. But other formulae also generate these values, but then complete the pattern differently:

n	1	2	3	4	5
n^2	1	4	9	16	25
$n^2 + (n-1)(n-2)(n-3)(n-4)$	1	4	9	16	?
$2^n - \frac{n(n-2)(n-4)}{3}$	1	4	9	16	?

PROBLEM 41: WHOLE NUMBER

Check some special cases for the conjecture that for $y \in \mathrm{N}$, $x = \sqrt{1141y^2 + 1}$ is never a whole number[7].

The conjecture is true for all values of y up to 30 693 385 322 765 657 197 397 207, but for the very next y, x *is* a whole number! Can you find the value of x?

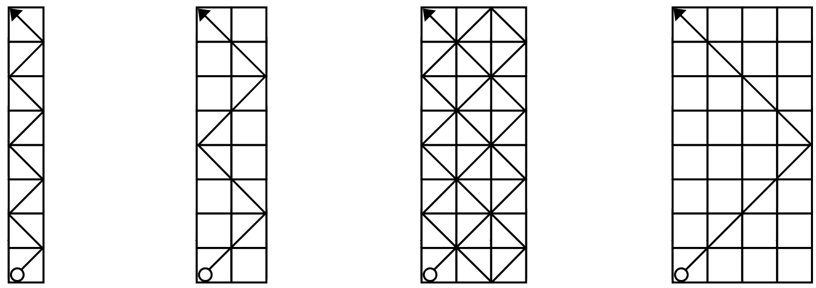
This means that the conjecture is true for 30 693 385 322 765 657 197 397 207 consecutive cases, yet the conjecture turns out to be false!

[7] This is a *Pell equation*, i.e. whole number solutions (x, y) of equations of the form $x^2 - \mathrm{N}y^2 = 1$, with N a non-square. natural number. Why the fine print that N must not be a perfect square? Make a conjecture and prove it.

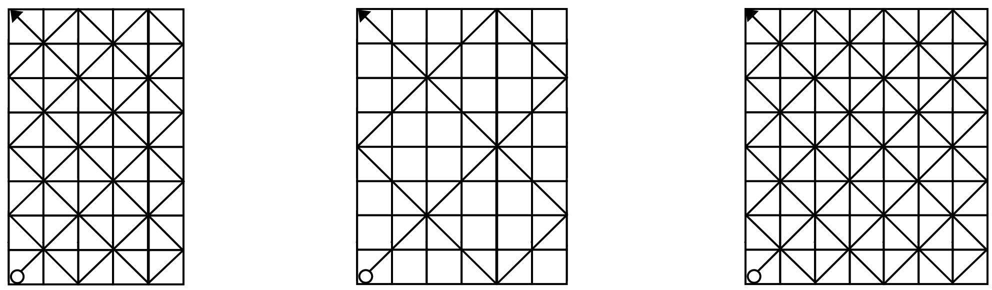
Our conjecture here is that there are *no* whole number solutions (x, y) for the equation $x^2 = 1141y^2 + 1$.
You can explore using this Excel worksheet, but the numbers are beyond Excel's range of 15 digits!
You can use this factorisation tool to find the factors of 1141*30 693 385 322 765 657 197 397 208^2+1 and deduce the value of x. Or try this excellent online Pell equation solver.

PROBLEM 21: SNOOKER
(b) In which pocket will the ball fall.

Let's return to the Snooker problem. Did you work out in which pocket the ball will fall? Here is an inductive approach – lets *systematically* look at different tables with a width of 8, as shown below:



8 by 1 8 by 2 8 by 3 8 by 4



8 by 5 8 by 6 8 by 7

From this data we make the following conjectures:

- The ball will always fall in the top-left corner.
- If the width of the table is 8, the ball will always fall in the top-left pocket.
- If the width of the table is even, the ball will always fall in the top-left pocket.

Do you agree with these conjectures?

We have 7 systematic cases for which all three conjectures are true. But not one of these conjectures can be true, because for an 8 by 8 table the ball will fall into the top-right pocket! *One counter-example disproves a conjecture!*

What can we learn from this? Well, of course in general, in an inductive approach we can *never* take enough examples, unless we take *all* the examples! We should vary the data – we should be systematic, but taking only special cases (e.g. only tables with a width of 8) increases the chances that we may recognise and extend a pattern which is actually only applicable to special cases.

PROBLEM 42: MULTIPLES
Is the following statement true?
If x and y are even numbers, then $x + y$ is divisible by 4.

One can check the validity of the statement by specialising, for example:

6 + 2 = 8	6 + 10 = 16
2 + 10 = 12	4 + 16 = 20
8 + 12 = 20	14 + 10 = 24

These are all pairs of even numbers whose sum is divisible by 4, so the statement seems reasonable. But the statements says that this should be true for *any* even numbers and it is easy to find even numbers for which the sum is *not* divisible by 4, for example:

4 + 2 = 6
2 + 8 = 10
4 + 10 = 14

PROBLEM 43: INDUCTION ERROR FROM HISTORY

In 1640 the French mathematician Pierre Fermat conjectured that *all* numbers of the form $F(n) = 2^{2^n} + 1$, $n \in \mathbb{N}$ (these are now known as Fermat numbers) were prime on the basis of investigating only the first few numbers in the sequence. For example, F(1) = 5 and F(2) = 17.
This conjecture was disproved only in 1732 by the Swiss Leonard Euler, who showed that F(5) was not prime. F(6) was factorised in 1885, and F(7) only in 1970! No further Fermat primes are known beyond the first four terms.

Calculate[8] the first eight Fermat numbers with the formula $F(n) = 2^{2^n} + 1$.
Show that $F(n)$, $n > 4$ are *not* prime.

PROBLEM 44: IN THE CLASSROOM

As teacher, you set the following question in an examination, counting 10 marks:
Prove that $\sin^2 \theta + \cos^2 \theta = 1$.

John answers it like this:

If $\theta = 30°$: $\sin^2 30 + \cos^2 30 = (\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2 = \frac{1}{4} + \frac{3}{4} = 1$

If $\theta = 60°$: $\sin^2 60 + \cos^2 60 = (\frac{\sqrt{3}}{2})^2 + (\frac{1}{2})^2 = \frac{3}{4} + \frac{1}{4} = 1$

So $\sin^2 \theta + \cos^2 \theta = 1$

How many marks would you award to the answer (explain why!), and what feedback will you give to John?

PROBLEM 45: REMAINDER

Investigate the remainder when the square of an odd number is divided by 8. Are you *sure*?

We emphasise again that mathematics is not a spectator sport! So it is necessary that you first work intensively on the problems yourself before reading on! The famous mathematician Paul Halmos says this about reading mathematics:

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis?

---

8 You can use this factorisation tool. You should appreciate that in the 17th century Fermat did not have nearly your calculation power!

An inductive approach yields:

$3^2 = 9$ and $9 \div 8 = 1$ rem 1
$5^2 = 25$ and $25 \div 8 = 3$ rem 1
$7^2 = 49$ and $49 \div 8 = 6$ rem 1
$11^2 = 121$ and $121 \div 8 = 15$ rem 1

A conjecture that the remainder is always 1 seems reasonable. But how can we be sure that it will *always* be 1? Considering our previous examples of the pitfalls of induction, how can you be sure that suddenly after a million cases a remainder of 2 will not appear? As someone said:

Absence of evidence does not mean evidence of absence!

In order to be sure the conjecture is *always* true, we must check *all* the cases, not just a few, and not just a few million! It is impossible to check all because the natural numbers are infinite! But we can check all by using the power of algebraic symbols, representing *any* case or *all* cases.

In order to explain *why* the remainder is only 1, it is necessary that we analyse the *structure* of the situation. As we have mentioned before (see, for example page 15), we can only explain the structure of a situation when we work with the general case, i.e. use *deductive reasoning*, as follows:

We can express *any* odd number and *all* odd number as $2n + 1$, $n \in N_0$[9]. Make sure that you understand the meaning and structure of this statement! Check it for special cases, e.g. can 731 be written in the form $2n + 1$, $n \in N_0$?

Now we have $\dfrac{(2n+1)^2}{8} = \dfrac{4n^2+4n+1}{8} = \dfrac{4n(n+1)+1}{8}$

Now it is necessary to know that if n is odd, $n + 1$ is even and the other way around, so $n(n + 1)$ is even and therefore $4n(n + 1)$ is divisible by 4 and divisible by 2, i.e. divisible by 8. From the right distributive property of division over addition, i.e.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

we can now deduce that the remainder is 1.

You should note the power of deductive reasoning: While in Problem 41 we may have felt sure that for $y \in \mathrm{N}$, $\sqrt{1141y^2+1}$ is never a whole number, it turned out to be false for y equal to 30 693 385 322 765 657 197 397 208. But in problem 45, after our deductive analysis, we are *absolutely certain* that no matter what odd number we take – take 30 693 385 322 765 657 197 397 209 if you like! – if we square it and we then divide by 8, the remainder will *definitely* be 1. In fact, we can predict it with certainty, without having to actually do the calculations. That is the power of mathematics, and *that power lies in the generality of algebraic symbols and in deductive thinking!*

---

9 Note that we are using the notation N for the natural numbers, i.e. {1, 2, 3, 4, …} and N_0 for the whole numbers. i.e. {0, 1, 2, 3, …}. If we wrote any odd number as 2n + 1, n ∈ N, i.e. started with n = 1, the first odd number would be 3, but starting with n = 0, the first odd number is 1. Of course we can also write any odd number as 2n – 1, n ∈ N, but the form 2n + 1 is often more convenient.

Looking back

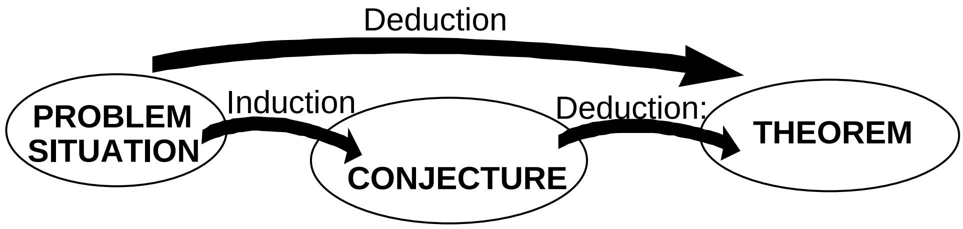
Let us repeat: inductive reasoning is a powerful method to discover new relationships. But we can never be sure that an inductive pattern will not somewhere break down, even after millions of cases. To be *sure that* it is always true (validity), and to *explain why* it is true – why the pattern has this form and not another, we must use *deductive reasoning,* i.e. reason using the structure of the situation.

This means that in mathematical activity there are two approaches:

- One can work deductively, or
- One can work inductively, but to make sure, it should be *followed by deduction*.

But just induction on its own is not adequate! When we use induction we cannot be sure our conclusion is correct, no matter how many cases we check.

The following diagram depicts the relationship between induction and deduction and the status of knowledge (a conjecture is not yet proved; a theorem is a proved conjecture):



2.4 MORE INDUCTION AND DEDUCTION

After seeing the pitfalls of inductive reasoning, our attitude towards problem solving should now be different. We should be *sceptical* about each result! It does not mean that we should not use inductive reasoning. To the contrary! Inductive reasoning is very powerful – most discoveries in mathematics are probably made through inductive reasoning. But we should be sceptical, doubt every result and insist on proof before we are convinced. But we must realise that certainty, proof, can only be reached through deductive reasoning, and it should become part and parcel of all our work.

Let's return to Problems 35-37. We again give the first 10 cases of $P(n) = n^2 - n + 11,\ n \in N$:

$$P(1) = 11$$
$$P(2) = 13$$
$$P(3) = 17$$
$$P(4) = 23$$
$$P(5) = 31$$
$$P(6) = 41$$
$$P(7) = 53$$
$$P(8) = 67$$
$$P(9) = 83$$
$$P(10) = 101$$

Thabo says that he thinks the answer is *always* an *odd number*. Is this correct? Surely, it is correct for all the cases above. If we try more cases, we find that they are all also odd. No matter what number we choose for *n*, we get an odd number every time. Can we conclude that the answer is always odd for all $n \in$N?

No ways! Not until we have checked *all* cases, and that we can only do through deduction. Look at this:

$$\begin{aligned} P(n) &= n^2 - n + 11 \\ &= n(n-1) + 11 \end{aligned}$$

(*n*-1) and n are consecutive natural numbers. So if one is even, the other one is odd. So $n(n-1)$ is the product of an even and an odd number. Are you happy that it always is an even number? How do you know? So $P(n)$ is an even number plus 11, i.e. an even number plus an odd number, which always is an odd number. Are you happy with that?

Note that we have in this deduction assumed several statements without trying to prove them (e.g. any even number plus any odd number is always an odd), assuming that it is accepted background knowledge. But if someone does not agree, one should prove it. So, is an even number plus an odd number always odd? How do we know? We probably developed this knowledge through generalisation from special cases:

2 + 5 = 7
4 + 7 = 11
8 + 9 = 17

A deductive proof and explanation would be something like:
If $n \in N_0$, $2n$ is any even number; if $k \in N_0$, $2k + 1$ is an odd number
$2n + 2k + 1 = 2(n + k) + 1$, which is an even number plus one, which is odd.
Now we have *proved* it for *any* even number and *any* odd number!

In Problem 37 we noted that the difference between consecutive numbers is even. How can we be sure this is correct? The following argument works generally with *any* number of the form $n^2 - n + 11$, i.e. $P(n)$ and the next number, i.e. $P(n+1)$ and therefore leads to a general result for all such pairs:

$$\begin{aligned} & P(n+1) - P(n) \\ &= (n+1)^2 - (n+1) + 11 - (n^2 - n + 11) \\ &= n^2 + 2n + 1 - n - 1 + 11 - n^2 + n - 11 \\ &= 2n \end{aligned}$$

$2n$, $n \in N$ is a general description for any even number …

PROBLEM 46: f(a) – f(b)
If $f(x) = x^2 - 3x + 4$, show that:
f(3) – f(1) is divisible by 2
f(5) – f(1) is divisible by 4
f(7) – f(4) is divisible by 3
Generalise: f(a) – f(b) is divisible by ___
Prove it!

The enquiring mind will immediately ask:
Is this a special property of f(x) or is it also true for other quadratic functions?
Is this a special property of quadratic functions, or is it also true for cubic functions?
Is it true for *any* polynomial function $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n$?

PROBLEM 47:
Find the value of $(1-\frac{1}{4})(1-\frac{1}{9})(1-\frac{1}{16})(1-\frac{1}{25})(1-\frac{1}{36}) \ldots (1-\frac{1}{10000})$.
Generalise.

Many problems can be attacked through either induction or deduction. If you do not see a way to do it through deduction, try induction. One advantage is that you do not have to activate any special relevant knowledge – you simply have to do the calculations of the special cases and then look for a pattern. A second advantage is that working inductively often helps our understanding of the structure of the situation, therefore helping us with the deductive phase.

If we decide to work inductively, we must calculate the first few special cases. Let's use the notation P(n) for the product of n factors:

$P(1) = \frac{3}{4}$

$P(2) = \frac{3}{4} \times \frac{8}{9} = \frac{2}{3}$

$P(3) = \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} = \frac{5}{8}$

$P(4) = \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{24}{25} = \frac{3}{5}$

$P(5) = \frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \frac{24}{25} \times \frac{35}{36} = \frac{7}{12}$

Do you see a functional rule? Try again before continuing!
Let's strip some of the noise:

n	P(n)
1	$\frac{3}{4}$
2	$\frac{2}{3}$
3	$\frac{5}{8}$
4	$\frac{3}{5}$
5	$\frac{7}{12}$

I first saw the relationship $\frac{n+2}{2(n+1)}$ in these cases, then realised making equivalent fractions would fit this structure, as shown on the right.

n	P(n)
1	$\frac{3}{4}$
2	$\frac{4}{6}$
3	$\frac{5}{8}$
4	$\frac{6}{10}$
5	$\frac{7}{12}$

One does not just "look" and "see" patterns! Patterns do not come flying from the paper to your eyes and *into* your mind! We think it is the other way around – you first have a *conjecture*, then you *test* it against the data to see if it satisfies your conjecture. *So you first have a pattern in your head!* In this case, I looked at n = 1 and asked "What must I do to with the 1 to get 3 (the numerator)". Of course there are many possibilities and these all flash through your mind: +2, ×3, × 4 – 1, etc. But simultaneously you are looking at the other pairs (2,2), (3, 5), (4,3), (5,7) looking for a relationship that is the same for all of them. But lets repeat – I ask questions like "OK, for (1, 3) the relationship is × 3, is it also true for (2,2) and (3,5)? No, so this idea is abandoned. Now I try something else: "OK, for (1, 3) the pattern is +2. Is it also true for (2, 2) and (3, 5)? No, not for both. Usually one would then dismiss the conjecture. But sometimes (here, and also previously in Mystic Rose) you notice something constant in a *subset* of the data – here (1,3), (3, 5), 5,7) all satisfy the rule +2, while (2,2) and (4, 3) does not. I probably thought that there were different patterns for odd n and even n, but I could not see a constant pattern for these even cases. So I jumped to looking at the denominators. I noticed different constant patterns for odds and evens: (2, 3) and (4, 5) could be +1. For n odd I noticed in (1, 4), (3, 8), (5, 12) the even numbers 4, 8, 12 jumping with 4 instead of 2, and I thought it was a pity that the values in between were not 6 and 10. That is when it hit me that I could *make* them 6 and 10 if I used equivalent fractions! So everything fell in place.

So the formula is $P(n) = \frac{n+2}{2(n+1)}$. To solve the original problem we just have to *specialise*: n = 99, so $P(99) = \frac{101}{200}$.

It again emphasises that in algebra, it is all about *structure* and not calculations or numerical *answers*. While in context free exercises we insist that $\frac{4}{6}$ should always be "simplified" to $\frac{2}{3}$, in problem solving it often is the other way around. Which one is "simpler" depends on the context, and in this case $\frac{4}{6}$ describes the *structure* much better!

Proud as we may be for finding a pattern, the mathematical mind knows that such induction may be wrong – we based our solution on only 6 examples. To be sure that it is *true* and to explain *why it is true*, we must follow this process with deduction. We tried to show above that it is not always easy to see an inductive pattern, and it is often difficult to explain to others exactly how you found the pattern. In deduction we have a similar problem – if you can activate and recall an appropriate knowledge schema, the problem is often almost solved. But there is no "method" to help you to activate an appropriate knowledge schema! The activation of a schema is *triggered* by some *cue* (does it remind me of something I know?) and this depends on your available knowledge structures, the connections between knowledge and previous experience of similar problems (have I seen it before? Have I seen a similar problem before?).

If we are looking at the *structure* in

$$(1-\tfrac{1}{4})(1-\tfrac{1}{9})(1-\tfrac{1}{16})(1-\tfrac{1}{25})(1-\tfrac{1}{36})\ldots(1-\tfrac{1}{10000}).$$

we see the *invariant* ones and subtraction. The only "variable" is the 4, 9, 16, 25, … part. Most of us will probably recognise these as perfect squares. Maybe "squares" and "subtraction", together with the fact that the first 1 in every factor is of course also a square act as clues to trigger and let you think of "the difference between two squares", i.e. the identity $a^2 - b^2 = (a + b)(a - b)$. Now we can *transform* the problem into something else and see if we make any progress:

$$(1-\tfrac{1}{2})(1+\tfrac{1}{2})(1-\tfrac{1}{3})(1+\tfrac{1}{3})\,(1-\tfrac{1}{4})(1+\tfrac{1}{4})(1-\tfrac{1}{5})(1+\tfrac{1}{5})\ \ldots\ (1-\tfrac{1}{100})(1+\tfrac{1}{100})$$

$$= \frac{1}{2}\times\frac{3}{2}\times\frac{2}{3}\times\frac{4}{3}\times\frac{3}{4}\times\frac{5}{4}\times\frac{4}{5}\times\frac{6}{5}\times\ldots\times\frac{100}{99}\times\frac{99}{100}\times\frac{101}{100}$$

Still looking at structure, we realise that we have multiplicative inverses (reciprocals) – you may call it "cancelling", but the real *structure* is that $\frac{a}{b}\times\frac{b}{a} = 1$:

$$= \frac{1}{2}\times(\frac{3}{2}\times\frac{2}{3})\times(\frac{4}{3}\times\frac{3}{4})\times(\frac{5}{4}\times\frac{4}{5})\times\ldots\times(\frac{100}{99}\times\frac{99}{100})\times\frac{101}{100}$$

$$= \frac{1}{2}\times\frac{101}{100} = \frac{101}{200}$$

The structure of pairs of reciprocals are guaranteed by the fact that the consecutive factors $(1+\frac{1}{n})(1-\frac{1}{n+1}) = \frac{n+1}{n}\times\frac{n}{n+1}$. Note: Here n means *any* factor.

To write down the generalisation for n factors, we must again be careful: Because the first factor begins with $1 - \frac{1}{4} = 1 - \frac{1}{2^2}$, the nth factor (the last factor) is given by $1-\frac{1}{(n+1)^2}$. The general expression is therefore:

$$\frac{1}{2}\times(\frac{3}{2}\times\frac{2}{3})\times(\frac{4}{3}\times\frac{3}{4})\times(\frac{5}{4}\times\frac{4}{5})\times\ldots\times(\frac{n+1}{n}\times\frac{n}{n+1})\times\frac{n+2}{n+1} = \frac{n+2}{2(n+1)}$$

PROBLEM 48:
Take any two-digit number, then subtract the sum of the digits.
What do you find?

First specialise, for two reasons.
First, to understand what is going on.
So take 34. What we must do is 34 – (3 + 4) = 27
Second, to get a sense of pattern – there seems nothing special in this number, so if there is an underlying pattern we need to list several cases:

34 – (3 + 4) = 27
35 – (3 + 5) = 27
36 – (3 + 6) = 27
37 – (3 + 7) = 27

Can it be that the answer is always 27? Hardly. Maybe it is true that for numbers in the *thirties* the answer is always 27. As a matter of fact the statement is definitely true, because we can easily test it for *all* the cases – if we also 30, 31, 32, 33, 38 and 39 we know for sure that the answer is always 27. But what about other decades? Let's try the sixties:

64 – (6 + 4) = 54
65 – (6 + 5) = 54
64 – (6 + 6) = 54

So, for the sixties, the answer seems to be always 54. We make up conjectures about the structure as we go along – we *guess* an underlying pattern and then we check it. . Different people will guess different patterns. For example, I notice that 54 is double 27, and then I get excited as I notice that the sixties is double the thirties. So I can now *predict* that , for example, the answer in the forties will be double the answer in the twenties. I check:

21 – (2 + 1) = 18   44 – (4 + 4) = 36
25 – (2 + 5) = 18   42 – (4 + 2) = 36
28 – (2 + 8) = 18   47 – (4 + 7) = 36

So I was correct! I even see a "bigger" pattern: the sixties is three times the twenties, and $3 \times 18 = 54$.

I also notice that the sum of the digits of the answer is *always* 9! See what I mean? So far we have the following answers:

18 and 1 + 8 = 9
27 and 2 + 7 = 9
36 and 3 + 6 = 9
54 and 5 + 4 = 9

But *why* will this be?

I also begin to notice that the answer is always a multiple of 9. We have already tested nearly all 99 cases and we can if we want to. There can be no doubt – I am confident that the answer always is a multiple of 9. But *why* is it true, meaning why a multiple of 9, why not 7 or 8 or something else?

Can you explain *why* by looking at all our special cases above? I cannot. So let's go general.

Any two-digit number can be written as 10a + b, where a, b $\in N_0$ and a, b < 10.

So the number subtract the sum of its digits is 10a + b – (a + b)

Again we have no clear idea where we are going. We want to prove that it is a multiple of 9. What is the structure of a multiple of 9? How will we recognise it? Let's for the moment simply simplify:

$$\begin{aligned} & 10a + b - (a + b) \\ &= 10a + b - a - b \\ &= 9a \end{aligned}$$

Surely, 9a is a multiple of 9! Do you agree? If not, you can understand it by *specialising:*

a	1	2	3	4	5	6	7	8	9
9a	9	18	27	36	45	54	63	72	81

We get more out of the deduction. The answer is not only a multiple of 9, it also tells us which multiple of 9! How did I miss it? We must *interpret* the *meaning* of the expression and of the variable. There is the syntactical meaning: 9a means $9\times a$. a is not just any number. It is the tens-digit in 10a + b. Therefore, if I have 57, I can predict that the answer is $9 \times 5 = 45$ without doing the calculation.

PROBLEM 49:

Try to generalise the result in the previous activity:
Take any *three*-digit number, then subtract the sum of the digits …
What about a four-digit number?
What about an *n*-digit number?

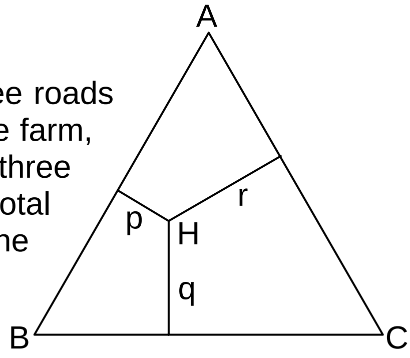
PROBLEM 50: ROOTS

The two polynomial equations below have the same coefficients, but in "backwards order":

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_{n-2}x^2 + a_{n-1}x + a_n = 0 \quad \ldots\ldots.(1)$$
$$a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0 = 0 \quad \ldots\ldots.(2)$$

What is the relationship between the roots of equation (1) and the roots of equation (2)?
Prove your answer!

PROBLEM 51: ON THE FARM



You set the following question as homework for Grade 8:
A farm is in the shape of an equilateral triangle, with three roads along the sides. The farmer wants to build a house on the farm, and then build connecting roads from the house to the three roads. Where should he build the house so that the total length of the roads is a minimum, i.e. so that p + q + r in the sketch is a minimum?

Some children have constructed the triangle and compared some distances through measurement. They are totally convinced that the farm house should be built at point B. Actually John is quite vehement about it!

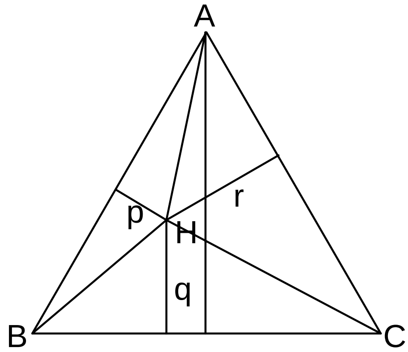
First, it is important that everyone understands that arguments based on geometric construction and measurement is not acceptable as proof in mathematics. Construction is a form of inductive reasoning – one cannot construct *any* triangle, one must necessarily construct a *specific* triangle which is a *special case*, and that can lead to problems, just like special numbers can cause problems in algebra. And of course, construction and measurement can never be exact, so we cannot base our arguments on data generated through construction and measurement.

It is a very important mathematical perspective that learners will understand the power of construction and measurement as an investigatory tool (just like numerical calculation in algebra), but simultaneously realise its limitations in mathematical proof (just like induction in algebra). The question is how do learners come to these perspectives? We suggest through appreciating logical argument, not through the authority of the teacher!

We can make drawings to help us visualise, but we need a general deductive argument. George Polyá said: “Geometry is the science of correct reasoning on incorrect figures.”

What knowledge, can we bring to the situation to make a start? Let’s try working with the area of the triangle. If we connect H to the vertices of ΔABC, we can express the area of ΔABC in two different ways – the “plan” is that the whole is equal to the sum of its parts. Let’s now carry out our plan:

$\Delta ABC = \Delta ABH + \Delta BCH + \Delta ACH$
$\quad = \frac{1}{2}AB.p + \frac{1}{2}BC.q + \frac{1}{2}AC.r$
$\quad = \frac{1}{2}BC(p + q + r)$ [AB = BC = AC] ………… (1)



But we also know
$\Delta ABC = \frac{1}{2}BC.h$ [h is height of ΔABC on base BC] (2)

From equations 1 and 2 we have:
$\frac{1}{2}BC.h = \frac{1}{2}BC(p + q + r)$

from which it follows that $p + q + r = h$.

What does this mean? p, q and r are *variables* that change as the position of H changes. But h is fixed (constant) for our specific triangle, therefore p + q + r is constant. What does this mean? It means that the value of p + q + r is *independent* of the position of H, so no matter where the house is built, the length p + q + r is the same.

In looking back, we first make the point that the deductive argument *proves* that the total distance of roads is constant for any position of the house. Now if we accept the validity of this logical argument, then it *proves* that John must have made some construction or measurement or recording mistake! In an inquiry classroom we use rational argumentation to convince ourselves and others.

Second, even when we are convinced of the validity of our reasoning, it is useful and interesting to check special cases:

PROBLEM 52: SPECIAL CASE
Look at special limiting cases, e.g. what happens if H is "on" one of the borders (sides), e.g. if H is on BC? Or if H is at a corner, e.g. at B?

Third, it is always useful to re-check and re-think our arguments. We notice that the plan is based on the *transitive property of equality*: if a = c and b = c, then a = b. This is the symbolic form of the well-known proof scheme for proving identities, namely working separately with the left-hand side and the right-hand side, and proving them equal to the same thing.

Fourth. The mathematical habit of mind always wonders if we can generalise the result. *What if not?* What are the variables that we can vary?
1. A triangle
2. An equilateral triangle.

Take the second first: What is the situation if it is not an equilateral triangle, but any triangle? An important mathematical habit of mind is not try to prove a result if you do not have a strong conviction that it is true or possible. It is part of *control*, not going on a wild goose chase! We must answer the question: Why are equal sides crucial or necessary for this result? Well, just looking at the *method*, we can see that the interesting result *depends* on taking out the side as a common factor in the expression of the sum of the areas of the triangles in equation 1. If the sides are not equal, we will be stuck at this point, so it is clear that we cannot get the same kind of interesting result if the sides are not equal. This is deductive reasoning!

Now lets look at the first variable. What if it is not a triangle? What if we have a regular quadrilateral (a square!), or regular pentagon, or hexagon? And if the result is interesting enough, can we generalise to *any* regular quadrilateral?

An important habit of mind when trying to generalise, is to try to use the method of the special case also for the general case. And if it does not apply, the problem is to try to make some small adjustment. It is clear that we can still use the idea of the dissection of the polygon into triangles. Figure 1 shows a regular *n*-gon dissected into triangles.

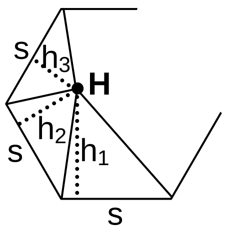


Figure 1

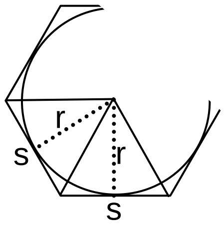


Figure 2

It is clear that we can express the area of the *n*-gon as

Area $= \Delta_1 + \Delta_2 + \ldots + \Delta_n$
$= \frac{1}{2}s.h_1 + \frac{1}{2}s.h_2 + \ldots + \frac{1}{2}s.h_3$
$= \frac{1}{2}s(h_1 + h_2 + \ldots + h_n)$ …………………..(3)

However, the second part of the triangle proof, i.e. expressing the area of the whole in terms of the height, seems not to apply to the polygon. How can we adapt it, i.e. how can we express the area of the polygon in a different way so that we can use our proof structure of transitivity? There is a well known formula expressing the area of a regular *n*-gon in terms of the radius of the inscribed circle of the polygon that we may be able to use. So in figure 2, we can express the area of the *n*-gon as:

Area $= \Delta_1 + \Delta_2 + \ldots + \Delta_n$
$= \frac{1}{2}s.r + \frac{1}{2}s.r + \ldots + \frac{1}{2}s.r$
$= \frac{1}{2}nsr$ ……………………………………….(4)

From equations 3 and 4 we can deduce that $h_1 + h_2 + \ldots + h_n = nr$.

This again shows that the sum of the lengths of the roads is equal to a constant – a *different* constant for *differen*t *n*-gons.

In looking back, we check on a very important mathematical perspective: a generalisation should include previous special cases. Of course the deduction above applies also to the special case of a regular triangle. So for a triangle, the constant length of the roads is 3r. Recalling our earlier result, the constant length of the roads was h. That will immediately imply that in an equilateral triangle, h = 3r! Now two perspectives are possible: If we have confidence in our arguments, we without hesitation accept h = 3r as a *proven* interesting new relationship in an equilateral triangle of which we maybe were not aware. If we do not have much confidence in our arguments, we mistrust the result and investigate whether it is indeed true that in an equilateral triangle h = 3r. If we manage to show that it is true, it serves as important back-up for the validly of our general result.

PROBLEM 53:
Show that in an isosceles triangle, h = 3r

PROBLEM 54: INSCRIBED CIRCLE

Show that in *any* ΔABC, the radius r of the inscribed circle is given by

$$r = \frac{ab\sin C}{a + b + c}$$

Now deduce as special cases of the general result:

In right-angled ΔABC, with $\angle$C= 90^0, $r = \frac{ab}{a + b + c}$

In equilateral triangle ΔABC, $r = \frac{h}{3}$

PROBLEM 55: MEDIANS

In triangle ΔABC, medians AD and BE intersect in M.

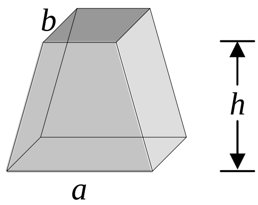
Prove that $MD = \frac{AD}{3}$

Can you use this result to deduce that $r = \frac{h}{3}$ in Problem 52?

PROBLEM 56: AREA OF CIRCLE

Use the formula Area = $\frac{1}{2}$nsr for a regular n-gon in Problem 52 to deduce the formula for the area of a circle.

PROBLEM 57: THE TRUNCATED PYRAMID

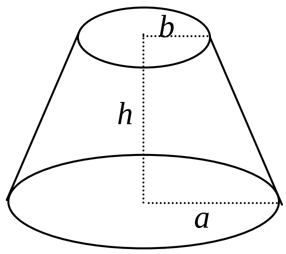


1. The Moscow Papyrus (written about 1850 BC) shows that the ancient Egyptians knew the formula for the volume of a truncated pyramid (called a frustum) with square bases, having sides of length a and b, and height h:

   $$V = \frac{h(a^2 + ab + b^2)}{3}$$

   Show how to derive this formula.
   Show that the frustum is a generalisation of a cube.
   Show that the frustum is a generalisation of a pyramid with square basis.

2. Show that the volume of the truncated cone in the sketch is:

   $$V = \frac{\pi h(a^2 + ab + b^2)}{3}$$

   What happens in the special case when $a = b$?

3. Show that both formulas can be written as

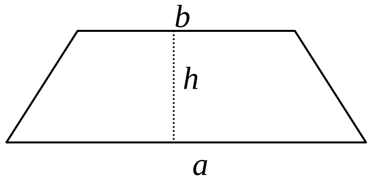
   $$V = \frac{h(A + B + \sqrt{AB}}{3}$$

   where A and B are the areas of the two bases of the frustum.

   Important information:

   The formula for the volume of a pyramid and a cone is $\frac{1}{3} \times$ *area of base* $\times$ *height*.

PROBLEM 58: TRAPEZIUM AND TRIANGLE



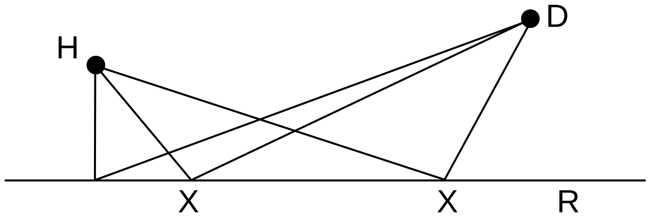
Part of the power of mathematics, and one aspect of generalisation, is that we can often use the same method in different contexts, if we recognise an *analogy* between the contexts.

The trapezium in the sketch can be seen as the two-dimensional analogue of the three-dimensional pyramid in the previous question.
Deduce the formula for the area of this trapezium, by using the same plan (method) you used for the frustum.

Show that in the special case when b = 0, we get the formula for the area of a triangle.

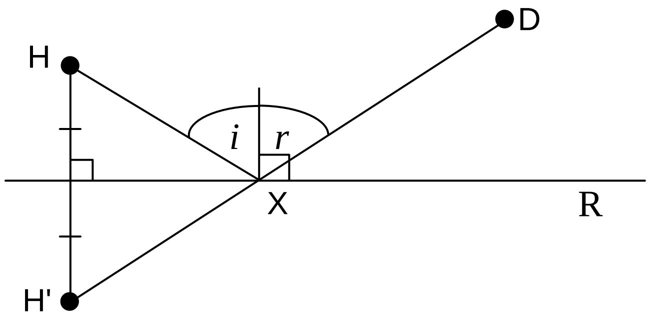
Now show that in the special case when a = b, we get the formula for the area of a parallelogram.

PROBLEM # 1: THANDI THE MILKMAID



Thandi every day walks from her house H to the straight river R to wash her bucket, and then fetches milk at the dairy D. Where along the river (position X) should she wash the bucket so that the total distance HX + XD that she walks is a minimum?

PROBLEM # 2: THE PATH OF LIGHT



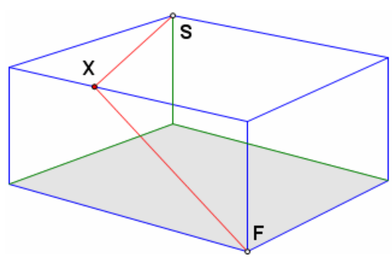
Prove that shortest path travelled by the milkmaid, is in fact also the path travelled by a ray of light from a source of light at H to D, the eye of an observer, when reflected in a mirror R.

What resources (*knowledge*) can we mobilise to solve the problem? In science we learn that if a ray of light is reflected in a mirror, then the angle of incidence is equal to the angle of reflection, i.e. $i = r$ in the sketch. So you can prove that HXD is the path of a light ray if you can *prove* that $i = r$.

Note: In school science this result is obtained *experimentally*, while this is a *mathematical proof!*

Think about the wonder of nature! <u>Why</u> should a ray of light follow this path, and not another?

PROBLEM # 3: THE SPIDER AND THE FLY



A spider sits in the top corner S of a room and a fly at the opposite bottom corner F.
What is the shortest path for the spider to catch the fly?

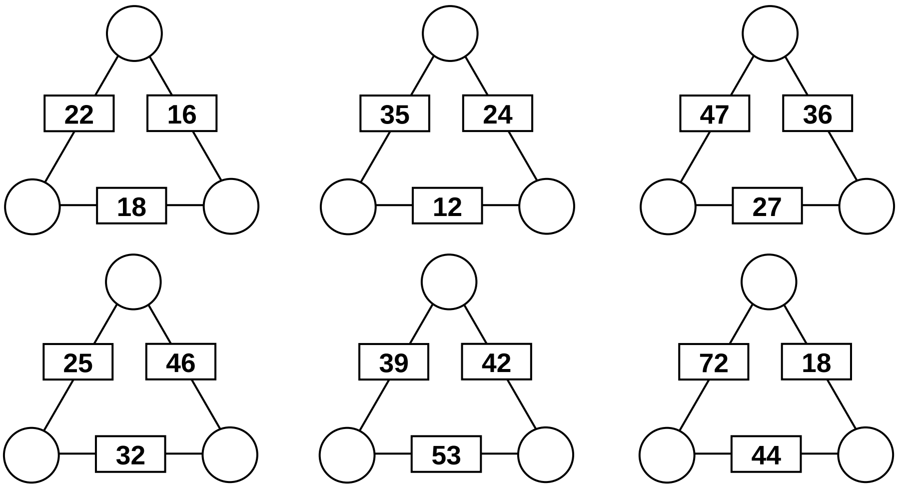
The spider decides to go along the roof up to X, the opposite wall, and then down the wall. Exactly where should she choose X so that the distance is a minimum, if there is a minimum? Can you *prove it*?

PROBLEM 59: TRIANGLES

In the triangles below, the numbers in the rectangles on the sides of the triangle (the "side-numbers"), are equal to the sum of the two numbers in the adjacent circles on the corners of the triangle (the "corner-numbers").

In each of the following cases, find the missing corner-numbers. Show your method and describe any patterns you find.

Develop, describe and illustrate a *general plan* (which can be a computational method or a formula) to easily calculate the corner-numbers in any triangle if the side-numbers are given.



PROBLEM 60: SIMULTANEOUS EQUATIONS

(a) Solve the following pairs of simultaneous equations:

(i) $x + 2y = 3$
$2x + 3y = 4$

(ii) $x + 2y = 3$
$4x + 5y = 6$

(iii) $2x + 3y = 4$
$4x + 5y = 6$

(iv) $3x + 4y = 5$
$2x + 3y = 4$

(v) $5x + 6y = 7$
$x + 2y = 3$

(vi) $8x + 9y = 10$
$4x + 5y = 6$

Do you notice a pattern? Formulate a conjecture and then *prove* your conjecture.

The mathematician *varies the conditions* of a problem to formulate and investigate *new problems*. For example: Problem (a) involved coefficients that are consecutive increasing *whole numbers*.

(b) *What if* the coefficients were *not* increasing, but *decreasing* consecutive whole numbers (e.g. 3, 2, 1 and 5, 4, 3)?

(c) *What if* the coefficients were consecutive odd numbers (e.g. 1, 3, 5 and 5, 7, 9)?

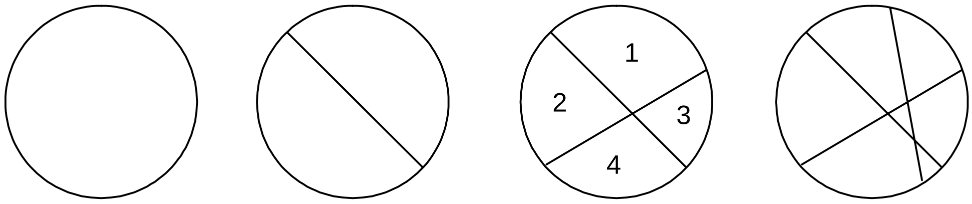
(d) *What if* the coefficients were consecutive even numbers?

(e) Now *generalise: What if* the coefficients were in arithmetic sequence (e.g. 1, 4, 7 and 2, 6, 10)?

(f) *What if* the coefficients were *not* in arithmetic, but in geometric sequence?

PROBLEM 61: CIRCLES, REGIONS AND CORDS

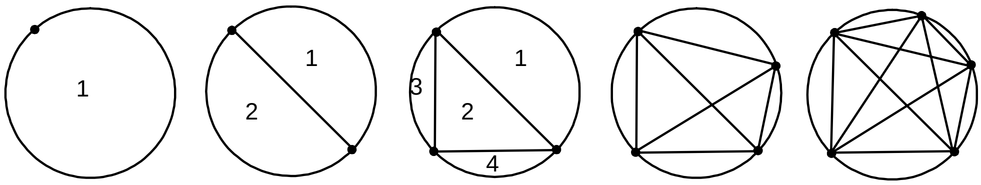
1.



2 chords divide a circle into 4 regions.
What is the *maximum* number of regions into which 6 chords will divide a circle?
And 20 chords?

# chords (*n*)	0	1	2	3	4	5	6	20
# regions (*R*)	1	2	4					

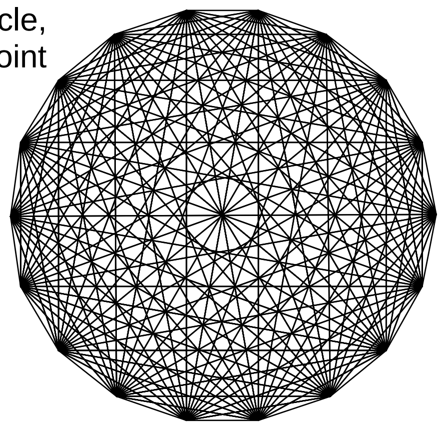
2.



If 3 points on a circle are joined 4 regions are formed.
What is the *maximum* number of regions into which 6 points on a circle will divide the circle if the points are joined?
And 20 points?

# points (*p*)	1	2	3	4	5	6	20
# regions (*R*)	1	2	4				

3. (a) In this figure, there are 18 points on the circle, and every point is connected to every other point on the circle.
How many connecting lines (chords) are there all together?

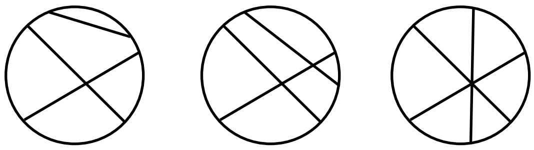


(b) In another circle there are 465 connecting lines. How many points are there on the circle?

We discuss these three problems on the next pages …

1. Chords and regions

First *understand* the situation! *Maximum* number of regions implies that the chords cannot be drawn like these below; each new chord must cut every other chord and no three chords should be concurrent (cut in one point).

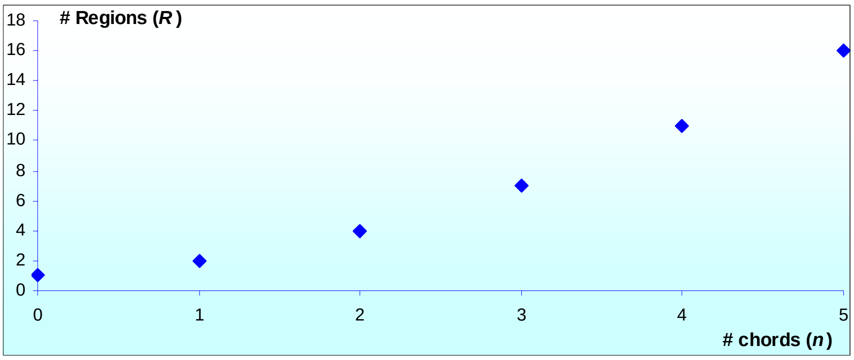


An *inductive* attack tries to find a pattern in the *numbers*.

# chords (*n*)	0	1	2	3	4	5	6	20
# regions (*R*)	1	2	4	7	11	16		

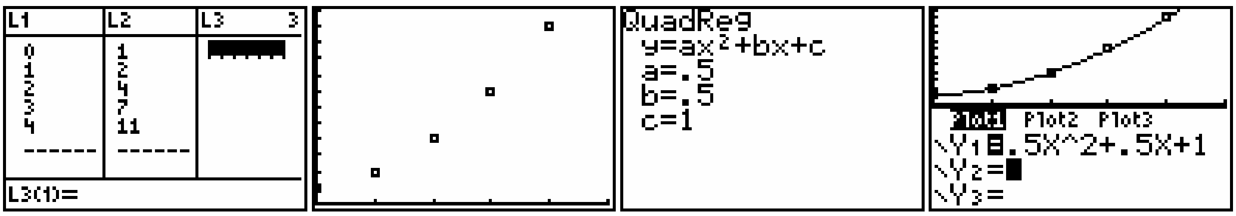
If you want to find a *functional* (here *vertical*) relationship by *inspection* (i.e. by just “looking”) you will have to systematically *ask* – is it maybe +1?, is it ×2 – 2?, is it …??, i.e. the only way of “seeing” a pattern is to first *beforehand* formulate a *conjecture* (an unproven theorem) and use it as a *lense* to look at the data and to *test/check* each conjecture …. and do not stop trying different alternatives until you find one that fits!!

It may help if you represent the data *graphically* and then use your knowledge of the relationship between the shape of a graph and its formula! Here is an Excel *scatterplot* of the data. What kind of a formula do you think may *fit* the points?



If you are using a technology tool like Excel, you may as well go all the way and let it find the *regression formula*[10] (the “curve of best fit”) for you!

You should also be able to plot the points and find the regression equation using the graphing calculator! Here is a sequence of windows using the TI-82 $\boxed{\text{STAT}}$ mode[11]:



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[10] Click here to see Excel’s Trendline. There is also a How to … file.

[11] Here is a TI-82 regression tutorial.

If you want to find a formula *analytically* (i.e. algebraically) you have to bring certain *resources* (knowledge) to the situation. For example, knowledge of *recursive* (horizontal) differences may be helpful here:

# chords (*n*)	0	1	2	3	4	5	6	20
# regions (*R*)	1	2	4	7	11	16		

Differences: +1 +2 +3 +4 +5

Second differences: +1 +1 +1 +1

Maybe you know: *if the second differences are constant, then the formula is quadratic*.

Do you know this theorem? If not, click here for a brief discussion in the appendix.

Now that we know the formula is quadratic, we merely have to solve for the parameters a, b and c in $R(n) = an^2 + bn + c$. The values in the *database* satisfy the equation, so substituting them makes the equation true, and this leads to three equations in three unknowns:

$$R(0) = 0a + 0b + c = 1 \;\; ..... \;\; (1)$$

$$R(1) = 1a + 1b + c = 2 \;\; ...... \;\; (2)$$

$$R(2) = 2a + 4b + c = 4 \;\; .... \;\; (3)$$

Here is yet another *inductive, recursive* method: Look at *R* and the differences *with different eyes*:
$R(0) = 1$
$R(1) = 1 + 1$
$R(2) = 1 + 1 + 2$
$R(3) = 1 + 1 + 2 + 3$
$R(n) = 1 + (1 + 2 + 3 + \ldots + n)$

If you *know* that $1+2+3+...+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ ….

or if you do not, maybe you can *deduce* it from your knowledge of the sum of *an arithmetic series:* $S_n = \frac{n}{2}\{2a + (n-1)d\}$ …

Anyway, $R(n) = 1 + \sum_{i=1}^{n} i = 1 + \frac{n(n+1)}{2} = \frac{n(n+1)+2}{2}, \;\; n \in N_0 = 0, 1, 2, 3, ...$

You should *check it against the known database* so that at least you are sure the formula is valid for *n* = 1 to 5!!

But, of course, although we here used *algebraic reasoning* to deduce the formula, it is nevertheless based on an *inductive analysis* of the *numbers* in the table, *not on the structure of the situation!*

So you can only pray that the generalisation is *valid* after *n* = 5!!!
But how can you be *sure??*

2. Points and regions

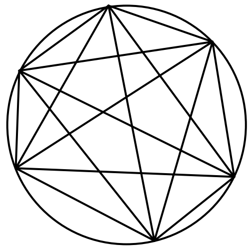
You probably noticed the *recursive pattern* 1, 2, 4, 8, 16, … in the numbers in the table and then continued the *recursive* doubling pattern to *predict* $R(6) = 32$. Or you may have inductively deduced the *functional formula* $R(p) = 2^{p-1}$, which also yields $R(6) = 32$.

A reminder: *Induction* consists of two processes:

1. *Abstraction* (finding the pattern in the *known* set of *numbers* or *database*) – here the pattern in the 6 *known* pairs (1, 1), (2, 2), (3, 4), (4, 8), (5, 16) *is* $R(p) = 2^{p-1},\ 1 \le p \le 5,\ p \in N$.
2. *Generalisation* (*extending* the pattern *beyond* the known database) – here assuming that the next pair is (6, 32) and $R(p) = 2^{p-1},\ \ p \in N$.

In this case the abstraction (1) is correct but the generalisation (2) is not! Unfortunately our expectation that the pattern is continued beyond the fifth case is wrong!

In fact $R(6) = 31$! Check for yourself by physically counting the number of regions in this sketch. *Understand the problem!* The prerequisite of *maximum* number of regions implies that no three chords should be *concurrent* (cut in one point).



This example serves to remind us that inductive reasoning, powerful as it may be in discovering new patterns, is prone to error:

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks. Kasner & Newman, 1940

But can we find the *correct* formula? Let's try differences again:

# points (*p)*	1	2	3	4	5	6		20
# regions (R)	1	2	4	8	16	31		

First differences: 1 2 4 8 15

Second differences: 1 2 4 7

Third differences: 1 2 3

Fourth differences: 1 1

The fourth differences are equal, so the generating formula is a fourth degree polynomial. So the formula is of the form $R(p) = ap^4 + bp^3 + cp^2 + dp + e$. So we must find a, b, c, d and e. But $R(0) = 1$, so $e = 1$. So we must find a, b, c, d, i.e. solve four unknowns and so we need four equations:

$R(1) = 1a + 1b + 1c + 1d = 0$ ……… (1)
$R(2) = 16a + 8b + 4c + 2d = 1$ …….. (2)
$R(3) = 81a + 27b + 9c + 3d = 3$ ……. (3)
$R(4) = 256a + 64b + 16c + 4d = 7$ …... (4)

Solving these four simultaneous equations, we get:

$$R(p) = \frac{1}{24}p^4 - \frac{1}{8}p^3 + \frac{23}{24}p^2 - \frac{2}{3}p + 1$$
$$= \frac{p^4 - 6p^3 + 23p^2 - 18p + 24}{24}$$

Or you could use the known database in a technology software package to easily find the regression formula for you[12]. Do you agree that the technology formula is the same as above?

You may want to check that this formula generates the correct values, i.e. 1, 2, 4, 8, 16, 31.

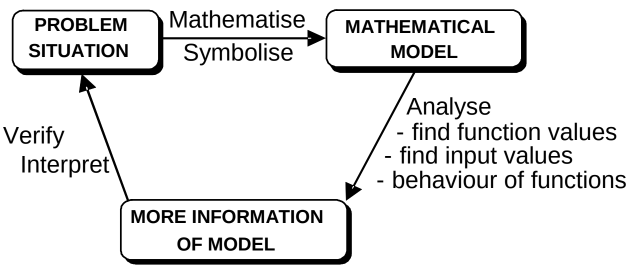
This formula produces the next value *R*(7) = 57. But is it correct? Remember, this formula is generalised from the number pattern, i.e. through *induction*, so we must wonder if induction will get us into trouble yet again! To check *R*(7) = 57, i.e. to check if 7 points on a circle yield 57 regions, you have no choice but to draw and count! You cannot trust this formula as a model of the situation, you cannot be sure!

You can only be sure if you reasoned with the *structure of the situation!*

3. Mystic rose

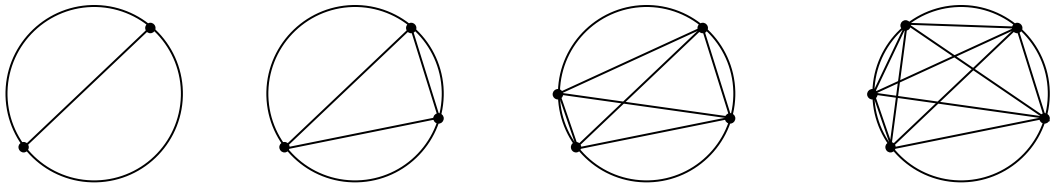
Students often simply formulate a guess like "It's 18 × 18", or "it's 18 × 17" and then cannot *justify* it, expecting the lecturer to tell them if they are right or wrong. This is *not* what mathematical thinking is about! We must learn to see our efforts not as *answers*, but as *conjectures*, as *public statements* that should be discussed, explained, verified and justified through *logical argument*. And we should learn to value logical arguments, and not accept mere *authority* of someone like the lecturer as support or justification for our solutions!

Now let's try again! Of course you can count, but that will be a rather daunting task and it will be prone to error! The essence of mathematics is to construct *mathematical models* that mimic the real situation, and then we manipulate mathematical objects in stead of real-life objects to *predict* unknown information.



Let's begin with an *inductive approach* and let's use some *heuristics*: Let's investigate some special cases, let's do it systematically, let's organise our resulting data in a table, try to find a pattern in the data and then use the pattern as a model to solve the original problems. Here are some special cases, where it is very easy to count the cords:

[12] Use a curve-fitting programme like CurveExpert (on our software page).
Or click here to see the use of the Excel Trendline …



# points (*n*)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10				

What we want is a *functional (vertical) formula* expressing C in terms of *n*. But it is not always so easy to find a formula through *inspection (just by "looking")*. In the table below I identify some easily observed patterns, suggesting a relationship:

# points (*n*)	2	3	4	5	6	7	18	n
		$\downarrow \times 1$		$\downarrow \times 2$		$\downarrow \times 3$		
# chords (C)	1	3	6	10	15	21		

I now "fill in the gaps" using the pattern:

# points (*n*)	2	3	4	5	6	7	18	n
	$\downarrow \times \frac{1}{2}$	$\downarrow \times 1$	$\downarrow \times 1\frac{1}{2}$	$\downarrow \times 2$	$\downarrow \times 2\frac{1}{2}$	$\downarrow \times 3$		
# chords (C)	1	3	6	10	15	21		

I now write the numbers in an *equivalent*, but *more useful* form:

# points (*n*)	2	3	4	5	6	7	18	n
	$\downarrow \times \frac{1}{2}$	$\downarrow \times \frac{2}{2}$	$\downarrow \times \frac{3}{2}$	$\downarrow \times \frac{4}{2}$	$\downarrow \times \frac{5}{2}$	$\downarrow \times \frac{6}{2}$		
# chords (C)	1	3	6	10	15	21		

Please notice that for the purpose of "seeing" the structure in this context, $\frac{3}{2}$ is "simpler" than $1\frac{1}{2}$ and $\frac{6}{2}$ is simpler than 3!!! The conventions you learned in primary school about "always writing in simplest form" are totally irrelevant in context!

To generalise our pattern, we must observe what is unchanging (*invariant*) and what changes. The invariant part is clear: every value is multiplied by something and divided by 2. It is this *invariant structure* that must be continued and *generalised* to C(18) and C(*n*):

# points (*n*)	2	3	4	5	6	7	18	n
	$\downarrow \times \frac{1}{2}$	$\downarrow \times \frac{2}{2}$	$\downarrow \times \frac{3}{2}$	$\downarrow \times \frac{4}{2}$	$\downarrow \times \frac{5}{2}$	$\downarrow \times \frac{6}{2}$	$\times \frac{?}{2}$	$\times \frac{?}{2}$
# chords (C)	1	3	6	10	15	21		

We must now remove the *noise* and concentrate only on the variable part, so that we can more easily *identify the structure*:

# points (n)	2	3	4	5	6	7	18	n
	↓	↓	↓	↓	↓	↓	↓	↓
numerator	1	2	3	4	5	6		

The functional (vertical) relationship is easily seen as –1and extended to 18 and *n*:

# points (n)	2	3	4	5	6	7	18	n
	↓ –1	↓ –1	↓ –1	↓ –1	↓ –1	↓ –1	↓ –1	↓ –1
numerator	1	2	3	4	5	6	17	$n - 1$

We can now answer our original question: C(18) = $\frac{18\times 17}{2}$ and C(n) = $\frac{n\times(n-1)}{2}$.

We have our solution, but it was not so easy to find the functional formula! To emphasize that different people see the same situation differently *because they bring different background knowledge (resources) as lenses to the situation*, let's investigate the *recursive* (horizontal) pattern of differences +2, +3, +4, …

# points (n)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10	15	21		

Differences: 2 3 4 5 6

Second differences: 1 1 1 1

So the second difference is constant, so the formula is quadratic of the form
C(n) = an^2 + bn + c and we merely have to solve for the parameters a, b and c:

$$C(1) = 1a + 1b + c = 0 \quad .... \ (1)$$

$$C(2) = 4a + 2b + c = 1 \quad ... \ (2)$$

$$C(3) = 9a + 3b + c = 3 \quad .... \ (3)$$

$$(2)-(1):\ 3a + b = 1 \ .... \ (4)$$

$$(3)-(2):\ 5a + b = 2 \ .... \ (5)$$

$$(5)-(4):\ 2a = 1 \Rightarrow a = \tfrac{1}{2},\ b = -\tfrac{1}{2},\ c = 0$$

So $C(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2}, \quad n \in N$

This formula generates *C*(18) = 153. But is it correct? You cannot trust this inductively generated formula as a model of the situation, *you cannot be sure!* Do you really want to check by actually physically counting *C*(18)??

Solving the question C(n) = 465 is very difficult without the formula, but with formula it becomes very easy, illustrating the power of Algebra:

$$C(n) = 465$$

$$\Rightarrow \frac{n(n-1)}{2} = 465$$

$$\Rightarrow n^2 - n - 930 = 0$$

$$\Rightarrow (n-31)(n+30) = 0$$

$$\Rightarrow n = 31$$

Deduction

All the previous work was *induction*, i.e. deducing patterns from *numbers*, which are *special cases*. But inductive conclusions may be wrong! And induction does not explain *the form* of the result.

To *prove* a statement (to show *that* it is true) and to explain *why* it is true, we need to reason *deductively*, i.e. *generally*, using the *structure of the situation.*

We can prove *that* a statement is true using **complete mathematical induction**[13]. For example:

Chords and regions[14]*:*

To prove that $R(n) = 1 + \frac{n(n+1)}{2}$, $n \in N_0$ by mathematical induction, we have to prove:

1. $R(k) = 1 + \frac{k(k+1)}{k} \Rightarrow R(k+1) = 1 + \frac{(k+1)(k+2)}{2}$
2. $R(0)$ is true, i.e. $R(0) = 1 + \frac{0(0+1)}{2} = 1$

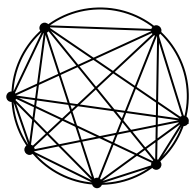
Suppose there are already *k* chords drawn in the circle. Then the next, i.e. the (k+1)st chord will cut each of the previous *k* chords and therefore pass through (k+1) regions, therefore adding an additional (k+1) regions.

So, *if* $R(k) = 1 + \frac{k(k+1)}{2}$

then $R(k+1) = 1 + \frac{k(k+1)}{2} + (k+1) = 1 + \frac{(k+1)[(k+1+1)]}{2}$

which proves the implication in (1) and the rest is obvious …

Mystic rose



A *deductive approach* will use the *structure of the situation, not the number answers for specific cases!* Analyse the *special case* when we have 7 points on the circle, and try to develop some clever way of counting the chords that will bring out the *structure* of the situation.

It should be clear that there should be 6 chords at every point on the circle, *because we are connecting every point with every other point, except itself*. So for 7 points there are 7 $\times$ 6 chords altogether. However, this is still not correct – *we have counted every chord twice*! Therefore, the number of chords in a 7-point Mystic Rose is $\frac{7 \times 6}{2}$, which of course confirms the answer we previously obtained for C(7) through recursion.

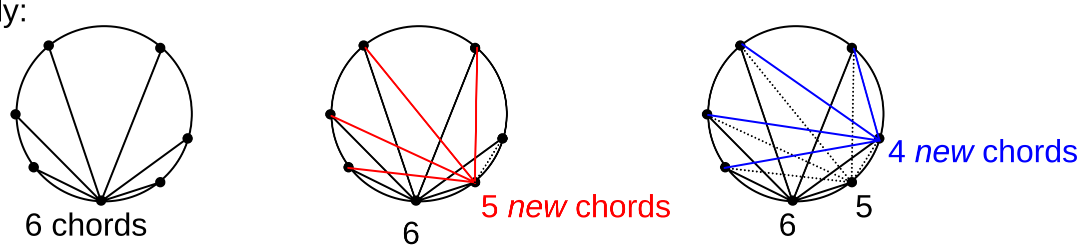
In deduction, and generally in mathematical thinking, we are not concerned with the numerical answer, but with the *method* or *structure*. So if we understand the structure in the one example C(7) = $\frac{7 \times 6}{2}$, we can *generalise* the structure – we must be able to

[13] What we are calling “induction”, i.e. generalising from *a few special cases*, is really better named “incomplete induction”, in contrast to the method of *Mathematical induction*, which is “complete induction”, because it considers *all cases!*

[14] Compare Michael de Villiers

see the general in the particular. So it should be clear that for 18 points, there will be 17 chords at each point, so 18 $\times$ 17 in total, except that we counted each chord twice, so C(18) = $\frac{18 \times 17}{2}$. In *general*, if we have n points, each point is connected to $n - 1$ points, so C(n) = $\frac{n \times (n-1)}{2}$. So deduction confirms and proves our previous result.

Looking back, or discussing the problem with others, we may realise that there are other ways of looking at the structure. Our approach was to count the chords at each point. This counted each chord twice, that is why we divided by 2. Now look at it differently:



It is clear that at the first point there are 6 chords. At the second point there are 5 *new* chords, because the chord to the first point has already been counted at the first point. Similarly, at the third point there are 4 new chords, etc.

So we have C(7) = 6 + 5 + 4 + 3 + 2 + 1

Again we emphasise that we do not want the numerical *answer,* but want to understand the *structure* of the situation. The result is not simply the sum of arbitrary numbers – the structure is clear: it is a decreasing sequence because we are not double-counting chords; 6 is not just any number, but is one less than 7 because we are drawing chords to every other point except itself. *This means that we can see the structure of the situation in this one example. We say we see the general in the particular.*

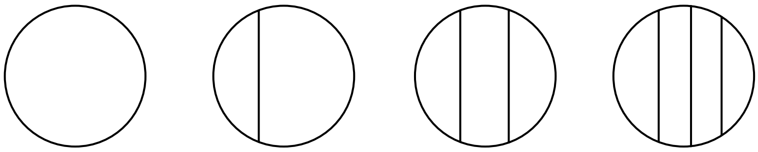
We can now without further ado say that C(18) = 17 + 16 + 15 + 14 + … + 3 + 2 + 1. Please notice that you can find C(18) by adding this sequence manually, but it is nearly impossible to solve (b), i.e. solve *the equation* C(n) = 465 *without a formula!!*

When you *do* Mathematics, problems always lead to new problems!

PROBLEM 61A: SHORT CUT
Develop a short method to calculate 17 + 16 + 15 + 14 + … + 3 + 2 + 1.
Use your method to calculate 1 + 2 + 3 + 4 + 5 + … + 99 + 100
Generalise!
Can you use your method to calculate 1 + 3 + 5 + 7 + 9 + … + 97 + 99?

Chords and regions[15]

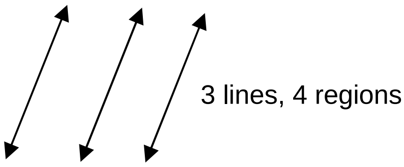
A typical mathematical approach is to *first look at simpler cases*. So let's first understand the simpler case if no chords intersect:



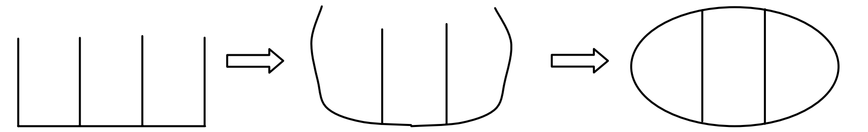
We can say:

$$\boldsymbol{R(n) = 1 + n}, \textbf{ i.e. } \textbf{\# regions = 1 + \# chords}$$

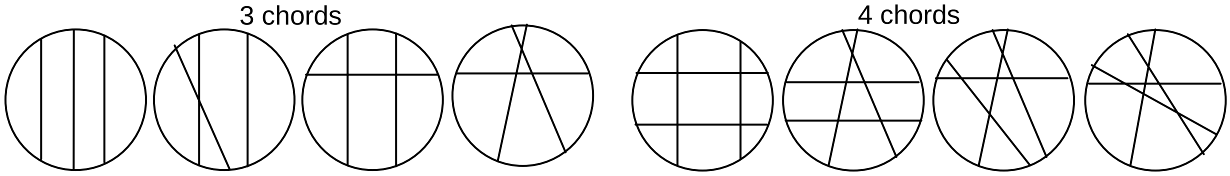
Note that this is not an inductive generalisation from the numbers in the above sketches, but is a *structural statement!* As an analogy, the structure is not different from n parallel lines dissecting the plane into $1 + n$ regions:



Or if we want to think in terms of a boundary: it can be seen as a generalisation of the well-known structure where a fence with n poles has $n - 1$ spaces (regions) between the poles. Now imagine that the two end-poles become the circle …



Now we have to consider the more general case when the chords intersect. Another typical mathematical attitude is not to merely construct a new relationship, but to construct it in such a way that the relationship still applies to the previous special case. So we do not want to change our present formula too much! Now look at these examples:



Convince yourself, structurally, that in all cases (except when 3 chords are concurrent), we have:

$$\textbf{\# regions = 1 + \# chords + \# intersections}$$

This is a generalization of our earlier formula, i.e. this relationship also applies to the case where the chords do not intersect, because then the number of intersections is 0!

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[15] Compare Tickey de Jager at Tickey's Web

How many intersections are there? In the case of the maximum number of regions, if we have *n* chords each intersecting each of the other *n* – 1 chords, the total number of intersections is $\frac{n\times(n-1)}{2}$. So we now have the formula:

$$R(n)=1+n+\frac{n\times(n-1)}{2}$$

If you simplify this formula, you get the *equivalent form* we had before:

$$R(n)=1+\frac{n\times(n+1)}{2}$$

This proves that our inductive result is correct, and it also gives us insight into *why* the result takes this form.

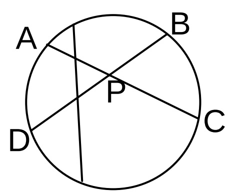
Points and regions

Now we investigate the case when we have *p points* on the circle. Convince yourself that the reasoning is the same as in the previous case:

 # regions = 1 + # chords + # intersections

We already know from the Mystic Rose problem that the number of chords is $\frac{p\times(p-1)}{2}$.

To find the number of intersections, we need to activate knowledge about permutations and combinations:



Every intersection is fixed by four points on the circle. For example, P is fixed by A, C, B, D.
So the number of intersections is the number of ways in which we can choose four points out of p:
We have p choices for the first point, $p-1$ for the second, $p-2$ for the third and $p-3$ for the fourth. So the total number of intersections is $p(p-1)(p-2)(p-3)$.

But the order in which we choose the points does not matters, so we must divide by the number of ways in which we can arrange 4 things in a row: We have 4 choices for the first, 3 for the second, 2 for the third and 1 for the fourth, i.e. 4×3×2×1 choices. So the number of intersections is:

$$\frac{p\times(p-1)(p-2)(p-3)}{4\times3\times2\times1}$$

$$\text{So } R(p)=1 \;+\; \#\text{ chords} \;+\; \#\text{ intersections}$$

$$=1+\frac{p\times(p-1)}{2}+\frac{p\times(p-1)(p-2)(p-3)}{4\times3\times2\times1}.$$

If you “simplify” this, you get the *equivalent form* we had before:

$$R(p)=\frac{p^4-6p^3+23p^2-18p+24}{24}$$

But, because we here reasoned *deductively*, we have now *proved* that the formula is generally valid, so we can *trust* that the value *R*(7)=57 is correct!!

APPENDIX 1: VERSKILLE

Voltooi die tabelle[16] in die volgende *spesiale gevalle* om jou *intuïtief* te oortuig dat:

***As die n^{de} verskil konstant is, is die model 'n polinoom van die n^{de} graad* … (1)**

n	1	2	3	4	5	6		
$y = 2n+3$	5	7	9	11	13	15		

Eerste verskille: 2 2 2 2 2
Tweede verskille:

n	1	2	3	4	5	6		
$y = n^2$	1	4	9	16	25	36		

Eerste verskille: 3 5 7 9 11
Tweede verskille:

n	1	2	3	4	5	6		
$y = n^3$	1	8	27	64	125	216		

Eerste verskille: 7 19 37 61 91
Tweede verskille:
Derde verskille:

x	1	2	3	4	5	6		
$y = 2^n$	2	4	8	16	32	64		

Eerste verskille: 2 4 8 16 32
Tweede verskille:
Derde verskille:

Ons kan byvoorbeeld vir enige kwadratiese funksie f(n) = an^2 + bn + c sê:

$$f(k+1) - f(k) = a(k+1)^2 + b(k+1) + c - (ak^2 + bk + c)$$
$$= 2ak + (a+b)$$

dit wil sê die verskille tussen opeenvolgende terme is 'n eerste-graadse funksie.

Netso is die verskille tussen opeenvolgende terme van 'n 3^{de} graadse funksie 'n 2^{de} graadse funksie, die tweede verskille 'n 1^{ste} graadse funksie en die derde verskille dus konstant.

Wat bostaande toon is dat:

***As die model 'n n^{de} graadse polinoom is, is die n^{de}* verskil konstant … (2)**

Dit is nie wat ons moes bewys nie - (1) is in werklikheid die *omgekeerde* van hierdie stelling! Maar omdat bostaande redenasie omkeerbaar is, het ons dus albei stellings bewys!

Let op:

- Hierdie is 'n eienskap *slegs* van polinoomfunksies. 'n Funskie soos 2^n, d.w.s 1, 2, 4, 8, … kan nooit 'n konstante verskil lewer nie. Dus weet ons dat is die verskil *nie* konstant is nie, die model *nie* 'n polinoom kan wees nie!
- As die eerste verkil konstant is, is die formule 1^{ste}-graads, d.i. y = mx + c.
  Algebraïes is hierdie konstante verskil die *gradient* m van die funksie en dit is die rede waarom die grafiek 'n reguit lyn is !
  Natuurlik is 'n konstante verskil vir diskrete waardes die *definisie* van 'n Rekenkundige ry !
  Maak ons die konneksies tussen rye en funksies, tussen T_n = a + (n – 1)d en f(x) = mx + c?
- Die verskille verteenwoordig natuurlik *die afgeleide*, ons weet dat f'(x^n) = nx^{n-1}, dus is dit nie verrassend dat die verskille-polinoomfunksies van afnemende graad is nie!!
- Hierdie stelling is belangrik in die kurrikulum bv. in modellering en om die vergelyking van die reguit lyn deur twee punte en die vergelyking van die parabool deur drie punte te bepaal!

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[16] See this Excel worksheet

APPENDIX 2: Fireworks Notes

We briefly describe different approaches and different representations that characterise the nature of algebra, specifically the idea of *generality*, the idea of a *model*, the meaning of *algebraic expressions* and the meaning of *equivalent transformations* . . .

It is useful to distinguish two different underlying *thinking strategies (processes)*, which I will call *numerical pattern recognition (induction)* and *structural analysis (deduction).* The processes are not necessarily distinct – there is often an interplay from one to the other. Both processes are important in doing and learning mathematics. (We do not here enter the discussion about the pitfalls of induction and the necessity of proof/explanation through deduction.)

The mathematical relationship between two variables can always be described in terms of either

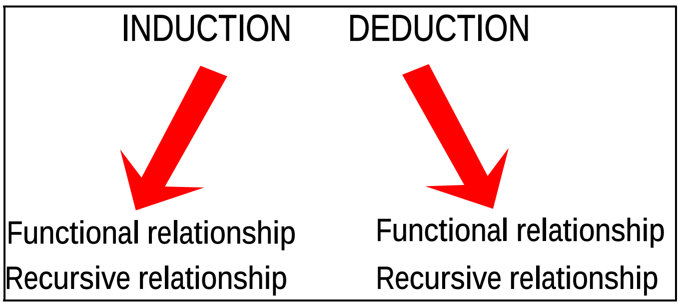
- the *functional relationship* between the variables which can lead to a formula such as, here, $m = f(t) = 2t + 1$, or

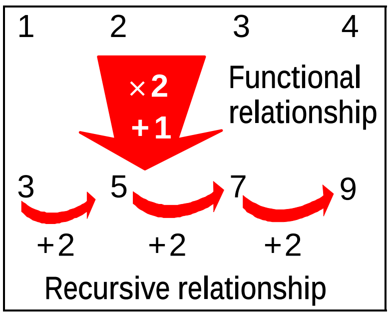
1. ▪ the *recursive relationship* between (successive) function values leading to a formula, as here, $f(t + 1) = f(t) + 2$.

Both types of relationships are important in (the learning of) mathematics. They emphasise different aspects of the relationship:

- a functional formula such as $m = 2t + 1$ makes it easy to find function values, solve equations, ...
- a recursive relationship such as $f(t + 1) = f(t) + 2$ underlies the study of sequences and series, and the important concepts of change, rate of change, gradient and derivative.

2. Both induction and deduction can lead to a functional or a recursive relationship.

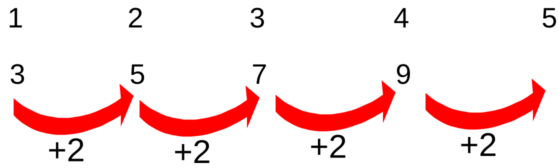




Numerical pattern recognition (induction)
The process of induction consists of two sub-processes:
1. pattern recognition in a finite set of data (abstraction)
2. pattern extension to cases not in the present set (generalisation)

One can focus on the *numbers* given in the table (we call this the *database)* and recognise a vertical (functional) relationship $m = 2t + 1$ which easily yields all the solutions. Or one can recognise a horizontal (recursive) pattern, which can serve as a model to generate additional information about the situation.

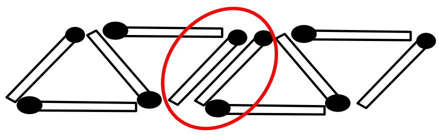
The nature of a *model* is that it *simulates* the physical situation, so that one can generate information by manipulating the mathematical model instead of the practical situation. Therefore, even the simple recursive pattern $f(t + 1) = f(t) + 2$



is a useful model that allows us to determine the number of matches needed for 5, 6, 7, ... triangles *without having to physically build or draw the triangles and count the matches*. (This is what we mean with *predict:* to use a mathematical model, not the physical situation or a physical model of the situation, to generate additional, unknown information about the situation.) However, it is not initially feasible to continue this recursive pattern until $t = 100$. A shorter method is needed!

Efforts at generalising

Most learners recognise the need for a shorter method. However, few learners seem to use the functional relationship – most learners try to adapt the recursive relationship, but this often leads to errors, e.g. $f(10) = 21$, so $f(100) = 10 \times f(10) = 210$. However, this is not a valid property of f. It can be *disproved* by a simple special case, e.g. $f(4) \neq 2 \times f(2)$, or analysing the physical situation and realising that we will be repeating some matches:



$2 \times f(2) = 2 \times 5 = 10$.
But in the database (table), $f(4) = 9$
So $f(4) \neq 2 \times f(2)$

The property that $f(kx) = k \times f(x)$ is a property *only* of the proportionality model $f(x) = mx$.

When children make this proportional multiplication mistake, it is very important that the teacher should help them to compare this situation with situations like in Matches 2, and analyse how they are the same and how they are different, i.e. they should analyse the *properties* of the two functions.

See also the papers Moments of conflict and moments of conviction in generalising and Children's generalisation thinking processes.

A valid "shortcut" can be found by looking at the horizontal pattern with different eyes, i.e. not working with the function *values*, but with the *structure* of the values:

$$
\begin{aligned}
f(1) &= 3 && = 3 \\
f(2) &= 5 && = 3 + 2 && = 3 + 1\times 2 \\
f(3) &= 7 && = 3 + 2 + 2 && = 3 + 2\times 2 \\
f(4) &= 9 && = && 3 + 3\times 2 \\
f(5) &= 11 && = && 3 + 4\times 2
\end{aligned}
$$

We can recognise the pattern in this *structure* and generalise it to

$$f(t) = 3 + (t - 1)\times 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

from which it follows that

$$f(100) = 3 + 99\times 2 = 201$$

Structural analysis (deduction)

While the above came directly from looking at the *numbers* and ignoring the *matches* (picture), some learners focus on the *process* of packing the *matches* and ignore the numbers. They easily formulate "you start off with 3 matches and then add another 2 matches for every additional triangle that you build". This is a model in the form of *words* (the equivalent to (1) in *symbols*) and also easily yields

$$f(100) = 3 + 99 \times 2 = 201$$

Syntactic meaning of algebraic expressions

It is important to note that the models all describe *a computational procedure*, e.g.

- in words: take the *number* of triangles, multiply it by 2 and then add one
- as a flow-diagram: $\longrightarrow \boxed{\times \mathbf{2}} \longrightarrow \boxed{\mathbf{+1}} \longrightarrow$
- as an algebraic expression using symbols: $2 \times t + 1$

Equivalent transformations

It is interesting to compare the recursive formula $3 + (t - 1) \times 2$ with the functional formula $2 \times t + 1$. Of course they yield the same *value* for the same value of *t*. *So they are simply different computational procedures!* Different *methods!* If we now convince ourselves that they are "the same" at the formal/deductive level, when we say

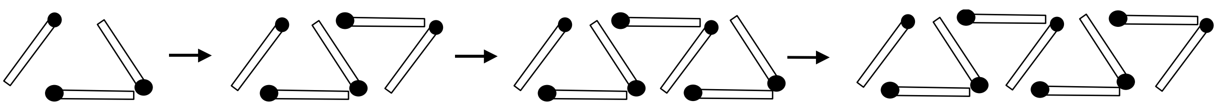
$$\begin{aligned} & 3 + (t - 1) \times 2 \\ &= 3 + 2t - 2 \\ &= 2t + 1 \end{aligned}$$

we are <u>not</u> saying that 2t + 1 is the "answer" of 3 + 2(t – 1)!! We are merely saying that they are different procedures (methods) to calculate the same thing, therefore they give the same numerical result for the same value of the variable. That is what equivalent transformation (i.e. algebraic manipulation) in this context means!

Semantic interpretation of algebraic expressions

The recursers know exactly what their formula $m = 3 + 2\times(t - 1)$ means: you start with 3 matches for the first triangle, and then you add an extra 2 matches for every extra triangle that you make.

But what does the functional relationship $m = 2t + 1$ mean in the physical situation? Well, it is as simple as this:



Is a picture worth a thousand words?

APPENDIX 3: $3^4 = 81$ ends on the digit 1. On what digit does 3^{2005} end?

An inductive attack:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
Units(3^n)	**1**	**3**	**9**	**7**	1	3	9	**1**	**3**	**9**	**7**	1	3	9	7	**1**	**3**	**9**	**7**	1

It should be clear that we have a *cyclic* (i.e. repeating) function: the digits 1, 3, 9, 7 is repeatedly repeated, in that order. So the new problem is simply to decide if 3^{2005} ends in 1, 3, 9 or 7. Of course, the mathematical mind will not be willing to generate this sequence all the way to 2005! Surely there must be some relationship between the two variables – the *exponent n* and the *units digit of* 3^n – that will enable us to *predict* the units digit for $n = 2005$! However, a simple formula like

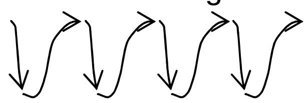
$$\textit{units digit} = 3 \times n + 4$$

is hardly likely. We can expect that some other cyclic concept will feature in the relationship.

Let's organise the data slightly differently:

Units digit	***n***		
1	0, 4, 8, 12, 16, 20, …	*Row 1*	$T_n = 4n$
3	1, 5, 9, 13, 17, 21, …	*Row 2*	$T_n = 4n + 1$
9	2, 6, 10, 14, 18, 22, …	*Row 3*	$T_n = 4n + 2$
7	3, 7, 11, 15, 19, 23, …	*Row 4*	$T_n = 4n + 3$

Can you see any advantage of organising the data in this way?
It is important to see that organised in this way, *all* counting numbers are generated. Can you see how the counting numbers are written as



Although we are working with numbers (so it is inductive), we have a *structure* (so the process is deductive).

Our new problem simply is to decide in which row 2005 will fall. So what patterns do you see that will help you decide in which row 2005 falls? What cues do you identify and what knowledge and schemas can you *impose* on the problem?

It is clear that Rows 1 and 3 consists only of *even numbers*. So 2005, which is *odd,* can fall only in either Row 2 or Row 4, so we know that 3^{2005} can end only in 3 or 7. Which one?

At an informal level, we recognise Row 1 as multiples of 4. At a formal level – have you seen this problem before? Have you seen similar problems? Well, all 4 sequences are *arithmetic sequences!*

Formally, we would say $T_n = 4n + 1$ with $T_n = 2005$ and our problem is to check whether there is a natural number solution for n.

If you did not recognise it as a known problem type, you should reflect on why not??

Another structural approach:
$3^{2004} = (3^4)^{501} = 81^{501}$ ends in 1! *How do we know this?*
So $3^{2005} = 3^{2004} \times 3^1$ ends in 3!

JUMP TO 2.5 PROOF