

HIERDIE NOTAS

Hierdie is opsommende notas/bespreking van sommige probleme wat ons (amper) in die klas gedoen het, wat ek op my rekenaar het ...

Die notas probeer 'n *tipiese wiskundige werkwyse* bevorder – wat dit beteken om Wiskunde te *doen!* Onthou: *Mathematics is not a spectator sport!*

Don't just read it; fight it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? What about the degenerate cases? Where does the proof use the hypothesis? Paul Halmos

Jy sal dus hier slegs iets leer as jy *aktief* lees ... met potlood en papier, die probleme *doen* ...

En beskikbare tegnologiese werktuie kan help, veral Excel, Sketchpad. Gebruik dit as *ondersoekmiddel* en *kontrolemiddel* – dit behoort 'n natuurlike deel van jou gereedskap te wees, net soos potlood en papier!

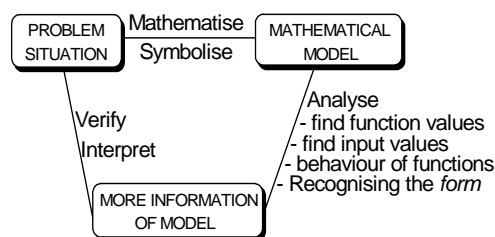
Jy moet die *Wiskunde* verstaan, maar terselfertyd aktief betekenis gee aan ons *Wiskunde-didaktiek konsepte* wat ons nodig het om oor wiskunde-onderwys te kan kommunikeer. Ek merk onder andere die volgende in hierdie stuk:

induksie , matematiese induksie	abstraheer
deduksie , struktuur	conjecture = vermoede/hipotese
funksie, funksionele formule	stelling
rekursie, rekursiewe formule	analities/algebraïes
regressie, regressie-formule	heuristiek
modelleer	databasis
spesialiseer	veranderlike
veralgemeen	Onbekende
Teenvoorbeeld (counter example)	parameter

Die breë wiskundige fokus van al die probleme is *wiskundige modellering*, en dit is belangrike dat ons die *proses* en die *probleemtipes* begryp:

There are many situations involving two variables where the one variable is *dependent* on the other variable, i.e. where a *change* in the value of one (the independent) variable causes a deterministic change in the value of the other (the dependent) variable.

Algebra is a language and a tool to study the nature of the relationship between specific variables in a situation. The power of Algebra is that it provides us with *models* to describe and analyse such situations and that it provides us with the analytical tools to obtain additional, unknown information about the situation. We often need such information as a basis for *reasoning* about problem situations and as a basis for *decision-making*.



..... The additional information we need to generate is mostly of the following five *types*:

1. *finding values of the dependent variable* (finding function values)
2. *finding values of the independent variable* (solving equations)
3. *describing and using the behaviour of function values* (increasing and decreasing functions, rate of change, gradient, derivative, maxima and minima, periodicity, . . .)
4. *finding a function rule* (formula)
5. *transforming to an equivalent expression* ("manipulation" of algebraic expressions)

Terwyl jy "lees" is dit belangrik dat jy identifiseer watter van hierdie vyf probleemtipes gebruik word ...

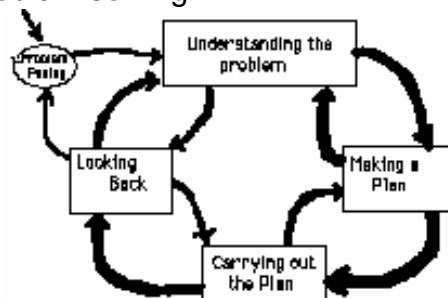
LEARNING ABOUT PROBLEM SOLVING¹

Problem-solving strategies (heuristics), specialisation and generalisation, induction and deduction

In this section we illustrate some processes of problem solving and you will have ample opportunity to implement processes.

The major contributions on problem solving was made by George Polya (1954, 1957). He formulated the following four-phase approach to problem solving:

1. Understanding the problem
2. Making a plan
3. Carrying out the plan
4. Looking back



These phases should not be interpreted as a linear, step-by-step recipe for problem solving – it is a cyclic, dynamic process as illustrated here.

Together with these phases, Polya formulated several *problem solving strategies* or *heuristics* to help us in problem solving. These phases and heuristics are summarised in Table 1 overleaf. Much of our work will be about using and reflecting on these phases and strategies.

Specialising and generalising

Specialising means looking at special or particular cases of some general statement. It is passing from the consideration of a given set of objects to that of a smaller set containing the given one. For example, we specialise when we pass from considering polygons to that of a regular polygon, and we specialise further when we pass from regular polygons with n sides to a regular quadrilateral, i.e. a square ($n = 4$ is a special case).

¹ Verskillende mense gebruik dikwels dieselfde woorde soos "probleemoplossing", maar gee heel verskillende betekenis daaraan! Mense kan dan maklik dink hulle praat dieselfde taal of praat oor dieselfde ding, maar eintlik praat hulle oor heel verskillende goed!

Wat is 'n probleem?

Vir iets om 'n *probleem* te wees, moet daar een of ander blokkasie of struikelblok wees, d.w.s 'n wiskundige probleem is 'n taak wat jy nie weet hoe om op te los nie. Anders is dit mos nie 'n probleem vir jou nie! Ek sal dit effens aanpas: 'n Probleem is 'n taak waarvoor jy nie 'n geroetineerde (outomatiese) oplossingsmetode het nie.

Wat is probleemoplossing?

Wanneer 'n onderwyser of 'n handboek *eers* die nodige wiskundige konsepte en metodes vir 'n probleem tipe ontwikkel en dit met 5 uitgewerkte voorbeelde illustreer, en dan leerlinge *toepassings* aan die einde van die hoofstuk laat doen, sê hulle dit is probleemoplossing. Dit kan tog nie wees nie – leerlinge weet dan reeds hoe om dit te doen! Ons noem hierdie soort take bloot *oefeninge!*

Ons sou in plaas van *probleme* vs *oefeninge*, die terminologie *nie-roetine probleme* en *roetine probleme* kon gebruik. Maar wat ek met *probleme* bedoel is dus *nie-roetine probleme* en *nie oefeninge* of *roetine probleme* nie!

Die volgende is 'n nuttige raamwerk om tussen verskillende benaderings te onderskei:

- *Onderrig vir probleemoplossing*: Die onderwyser onderrig *eers* vooraf en apart die "tools" – die nodige wiskundige konsepte en metodes in die abstrak, en dan word dit *agterna* toegepas.
- *Onderrig via (deur) probleemoplossing*: Die onderrig *begin* met relevante probleme en die wiskundige konsepte en metodes word ontwikkel *terwyl* leerlinge die probleme oplos.
- *Onderrig omtrent/van probleemoplossing*: Die onderwyser illustreer in die algemeen probleemoplossingsmetodes.

Met probleemoplossing in die klaskamer bedoel ek dus *onderrig via (deur) probleemoplossing*.

Die onderliggende perspektiewe

Die twee benaderings *onderrig vir probleemoplossing* en *onderrig via (deur) probleemoplossing*

- Is gebaseer op heel verskillende onderliggende teorieë oor die aard van kennis (epistemologieë), in besonder die aard van Wiskunde, verskillende leerteorieë (insluitend die aard van geheue) en verskillende perspektiewe oor die betekenis van "verstaan", en
- dit lei tot totaal verskillende klaskamer kulture die onderlinge verwagtings en verantwoordelikhede van onderwysers en leerlinge), en
- dit bepaal op sy beurt wat (en of) leerlinge leer.

How to Solve It

A summary of George Polya's (1957) four phases of problem solving and heuristics.

1. UNDERSTANDING THE PROBLEM

- **First.** You have to *understand* the problem.
- What is the unknown? What are the data? What is the condition?
- Is it possible to satisfy the condition? Is the condition sufficient to determine the unknown? Or is it insufficient? Or redundant? Or contradictory?
- Draw a figure. Introduce suitable notation.
- Separate the various parts of the condition. Can you write them down?

2. DEVISING A PLAN

- **Second.** Find the connection between the data and the unknown. You may be obliged to consider auxiliary problems if an immediate connection cannot be found. You should obtain eventually a *plan* of the solution.
- *Have you seen it before?* Have you seen the same problem in a slightly different form?
- *Do you know a related problem? Do you know a theorem that could be useful?*
- *Look at the unknown!* And try to think of a familiar problem having the same or a similar unknown.
- *Here is a problem related to yours and solved before. Could you use it? Could you use its result? Could you use its method?*
- Could you restate the problem? Could you restate it still differently? Go back to definitions.
- If you cannot solve the proposed problem try to solve first some related problem. Could you imagine a more accessible related problem? A more general problem? A more special problem? An analogous problem? Could you solve a part of the problem? Keep only a part of the condition, drop the other part; how far is the unknown then determined, how can it vary? Could you derive something useful from the data? Could you think of other data appropriate to determine the unknown? Could you change the unknown or data, or both if necessary, so that the new unknown and the new data are nearer to each other?
- Did you use all the data? Did you use the whole condition? Have you taken into account all essential notions involved in the problem?

3. CARRYING OUT THE PLAN

- **Third.** *Carry out* your plan.
- Carrying out your plan of the solution, *check each step*. Can you see clearly that the step is correct? Can you *prove* that it is correct?

4. LOOKING BACK

- **Fourth.** *Examine* the solution obtained.
- Can you *check the result*? Can you *check the argument*?
- Can you derive the solution differently? Can you see it at a glance?
- Can you use the result, or the method, for some other problem?

Table 1

Generalising is the reverse process of specialising. Generalising is passing from considering a given set of objects to that of a larger set, containing the given one. For example, we generalise when we pass from considering any square to that of considering any quadrilateral, in which we do not restrict the quadrilaterals to having equal sides. And we generalise when we move from any quadrilateral to any polygon, relaxing the restriction that it should have four sides.

Generalisation is the process of formulating a statement that is more general or inclusive, i.e. enlarging the number of cases for which a statement is true. For example, the cosine formula

$$a^2 = b^2 + c^2 - 2bc \cos A$$

is a generalisation of the Theorem of Pythagoras

$$a^2 = b^2 + c^2$$

because it applies to *any* triangle, removing the restriction that angle $A = 90^\circ$.

Of course, the Theorem of Pythagoras is a special case (a specialisation) of the cosine formula!

Specialisation serves at least the following purposes in mathematical activity:

1. It gives us a feeling for *what the statement is saying*
2. It gives us a sense *if the statement may be true*.

PROBLEM 1: DISCOUNT AND VAT

In a transaction a customer receives 10% discount for cash, but must pay 14% Value Added Tax (VAT). Which is cheaper for the customer: that the tax be calculated first or that the discount be calculated first?



It helps to look at a special case to get a feeling for what is happening. So suppose the purchase price is R100 and let's calculate the final price:

VAT first:

VAT: $R100 + 14\% \text{ of } R100 = R114$

Discount: $R114 - 10\% \text{ of } R114$

$$= R114 - R11,40$$

$$= R102,60$$

Discount first:

Discount: $R100 - 10\% \text{ of } R100 = R90$

VAT: $R90 + 14\% \text{ of } R90$

$$= R90 + R12,60$$

$$= R102,60$$

So, you pay the same! Are you surprised? But are you *convinced*?

Do you think the reasoning above will convince a friend? Should it convince the shopkeeper?

If it is correct, can you explain *why* the result is the same?

PROBLEM 2: ALL THE SAME

Jane says it is obvious that it does not matter if you calculate the VAT or the discount first, because you have to pay a net 4% extra in both cases, e.g:

$$R100 + 14\% - 10\% = R100 - 10\% + 14\% = R100 + 4\%.$$

Is Jane correct? Explain!

PROBLEM 3: PETROL PRICE

In January the petrol price is increased by 10%. Then, in February the petrol price is reduced by 10%. John says that the petrol price is now the same as it was before the first increase. Is this correct? Explain!

PROBLEM 4: RENTING A CAR²

You want to rent a car for one day. Imperial and Avis both charge a basic amount per day plus a rate per kilometre for the distance driven, as shown in the table below. Which company is the cheaper?



Car:	Per day	Per km
Toyota Corolla 1.6	R133	R1,08



Car:	Per day	Per km
VW Citi Golf 1600	R85	R1,48

One approach is to specialise. John does it like this: he takes the distance as 100 km and works out the cost for each company:

Imperial: Cost = 133 + 1,08 × 100 = R241

Avis: Cost = 85 + 1,48 × 100 = R233

So this shows that Avis is the cheaper.

So this shows that *for a distance of 100 km* Avis is cheaper, but it says nothing about any other distance! Surely the costs are *dependent* on the distance travelled. The distance is a variable that can *change*. If we calculate and compare the costs for *different distances*, Avis is not *always* cheaper, as is shown in this table:

Distance	90	95	100	105	110	115	120	125	130	135
Imperial	230,20	235,60	241	246,40	251,80	257,20	262,60	268	273,40	278,80
Avis	218,20	225,60	233	240,40	247,80	255,20	262,60	270	277,40	284,80
Difference	12	10	8	6	4	2	0	-2	-4	-6

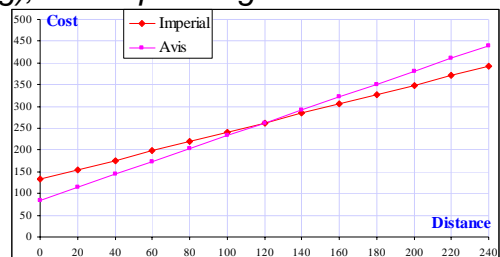
A distance of 120 km is the *break-even* point, where the costs for the cars are the same, and for distances greater than 120 km, Imperial is the cheaper. (We note that as a *real* problem the differences are too small to really make a difference!)

We can also easily find the break-even point *algebraically* or *graphically*, by expressing the cost for each car for *any distance* of *x* km and equalling the costs. *Note that the algebraic formulation is merely a generalisation of exactly the procedure we used for our numeric calculations. So, first working numerically (specialising), can help us to generalise!*

$$133 + 1,08x = 85 + 1,48x$$

$$\Rightarrow 0,4x = 48$$

$$\Rightarrow x = 120$$



² See this [Avis Excel worksheet](#)

PROBLEM 5: SAVING

In the Imperial-Avis table above:

- Are you surprised that the differences left and right of 120 km is “symmetrical”? Can you explain *why* this is so?
- Find a formula for the row of differences, i.e. a formula to calculate the difference in cost for any distance directly, without first calculating the costs for Imperial and Avis. Use it to calculate how much you will save by renting the cheaper option if you drive 400 km.

Now let's return to Problem 1.

How do we know that that the *one* value of R100 that we chose, is not a special case? Will it also be the same for R80, and R75 and R217,83?

The point we are trying to make is that in Renting a car, which car was cheaper depended on the distance travelled – for some distances Avis was cheaper and for other distances Imperial was cheaper. Similarly, will in our VAT-Discount problem the answer not depend on the purchase price, so that for some prices it is better to add the VAT first and for other prices it is better to deduct the discount first? Are the two situations not the same?

One approach would be to calculate the final price for several purchase prices. Another approach is to work *generally*, i.e. with *all* purchase prices simultaneously. How can this be done? By introducing some *symbol* to represent *any* purchase price, e.g. let the purchase price be "PRICE" or p . When we now use this symbol, it refers to *any* price, but not to any *particular* price.

VAT first:

$$\begin{aligned} \text{VAT: } & \text{PRICE} + 14\% \text{ of PRICE} \\ & = 1,14 \times \text{PRICE} \end{aligned}$$

$$\begin{aligned} \text{Discount: } & \text{PRICE} - 10\% \text{ of } 1,14 \times \text{PRICE} \\ & = 0,9 \times 1,14 \times \text{PRICE} \end{aligned}$$

Discount first:

$$\begin{aligned} \text{Discount: } & \text{PRICE} - 10\% \text{ of PRICE} \\ & = 0,9 \times \text{PRICE} \end{aligned}$$

$$\begin{aligned} \text{VAT: } & \text{PRICE} + 0,14\% \text{ of } 0,9 \times \text{PRICE} \\ & = 1,14 \times 0,9 \times \text{PRICE} \end{aligned}$$

This general statement now without doubt *proves* that the final prices are always equal, independent of the price of the purchase, because our variable "PRICE" represents *any* price. The structure of the general statement also clearly *explains why* the final price is the same: multiplication is *commutative*, i.e. the order does not matter!

Always true, sometimes true, never true

It is vital that we should understand how the mathematics in our two problems (the VAT problem and the Renting a Car problem) are the same and how they are different.

In our VAT problem the structure of the situation is symbolised by

$$0,9 \times 1,14 \times x = 1,14 \times 0,9 \times x$$

This statement is *always true* for *all values of the variable*.

In Renting a Car the structure of the situation is symbolised by

$$133 + 1,08x = 85 + 1,48x$$

This statement is *sometimes true* for *some values of the variable* (here only one value).

This distinction and the different meanings of the symbol x and the equal sign in these two statements are some of the most powerful and simultaneously some of the most difficult concepts in learning and using algebra, as formulated by William Betz (1930):

The symbolism of algebra is its glory. But it is also its curse.

Let's give examples of statements that are always true, sometimes true and never true in algebra:

Two algebraic expressions like $2x+3x$ and $5x$ are **equivalent**, because they have the same values for **any** value of the **variable** x .

We can say that $2x+3x = 5x$ is **always true** for **all** values of x .

We call an algebraic statement like $2x+3x = 5x$ which is true for all values of the variables an **algebraic identity**.

The algebraic expressions $4x+12$ and $7x+3$ are not equivalent expressions, because they have different values for all values of x , except for $x = 3$.

We can say that the statement $4x+12 = 7x+3$ is **sometimes true** for **some** values of x .

We call an algebraic statement like $4x+12 = 7x+3$ which is **sometimes true** for **some** values of the variables an **algebraic equation**.

$10x + 40 = 10x + 50$ for **no values** of x . We call this an **algebraic impossibility**.

Remark: Exceptions

It is merely for making some distinctions that we here talk about *always true*, *sometimes true*, and *never true (false)*. In the process we are using the word "true" rather loosely and that creates a slight semantic problem, because in logic and in mathematics, if we say a statement is true, we definitely mean that it is *always true*. A statement that is sometimes true is not true!

It is for this reason that mathematicians often add exceptions to "save" a beautiful theorem, so that it is true (i.e. always true). For example:

Any prime number can be written in the form $6n - 1$ or $6n + 1$, $n \in \mathbb{N}$.

You may test it through specialisation, e.g. $29 = 6 \times 5 - 1$ and $19 = 6 \times 3 + 1$. We will give a general proof and explanation in section 4. But you may notice that if you take the special case of prime numbers 2 and 3, they cannot be expressed in this form (i.e. $2 = 6n - 1$ or $2 = 6n + 1$ have no solution for $n \in \mathbb{N}$). They are counter-examples showing that the statement is *not true* (i.e. not *always true*). But they are the *only two* exceptions. So, rather than discarding this useful theorem, we "save" it by rather changing the statement:

Any prime number greater than 3 can be written in the form $6n - 1$ or $6n + 1$, $n \in \mathbb{N}$.

Now it is a true statement, without exception! Mmm ... How do you know, for sure, it is *true*??

Variables have explanatory power

PROBLEM 6: THINK OF ANY NUMBER

Think of any number Double it. Add 6. Halve the result. Subtract your original number. What is your result?

It may come as a surprise to us that, although different people thought of different numbers, they all obtain the same result, namely 3. The mathematical attitude, and the attitude in an inquiry classroom, is that we naturally want to *explain* this phenomenon. There is no need to prove that it is true – we know it is true. But we wonder *why* is the result always 3, even when we all started with different numbers. To try to understand what is going on, we could write down different people's results (special cases), for example:

	Thandi	Sam	Thuli	Vusi
<i>Think of any number</i>	3	7	8	12
<i>Double it</i>	6	14	16	24
<i>Add 6</i>	12	20	22	30
<i>Halve it</i>	6	10	11	15
<i>Subtract original number</i>	3	3	3	3

It is difficult, if not impossible to deduce the underlying structure from such specific examples. It is exactly in such situations where the advantages of a “generalised number” becomes clear: In stead of taking any specific number, select a *symbol* such as ♥ or *x* to stand for *any* number. Then we have:

	♥ notation	<i>x</i> notation
<i>Think of any number</i>	♥	<i>x</i>
<i>Double it</i>	♥♥	2 <i>x</i>
<i>Add 6</i>	♥♥ 6	2 <i>x</i> + 6
<i>Halve it</i>	♥ 3	<i>x</i> + 3
<i>Subtract original number</i>	3	3

The *structure* is now quite clear. $\frac{2x+6}{2} - x = 3$ is an *identity*, so for *any* (all) values of *x* the result is always 3. Choosing, manipulating and interpreting such generalised numbers (variables) is an essential part of mathematical reasoning and of the language of algebra.

PROBLEM 7: THINK OF ANY NUMBER³

Think of any secret number Multiply by 4. Add 6. Halve the result. Subtract your original number. What is your result?

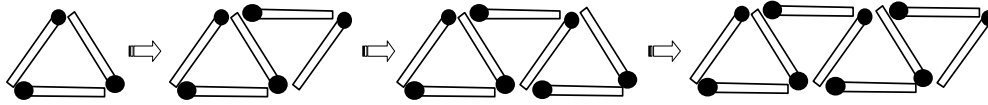
Do the calculations using Thandi, Sam, Thuli and Vusi's numbers above. Explain! How is this different from the situation in Problem 12?

Explain how, if someone told you their final answer, you can immediately tell them their secret number (e.g. if Thandi tells you her answer is 6, you can surprise her by telling her that her secret number is 3!).

³ See also this [birthday problem](#) and [another birthday problem](#)

PROBLEM 20: PLAYING WITH MATCHES

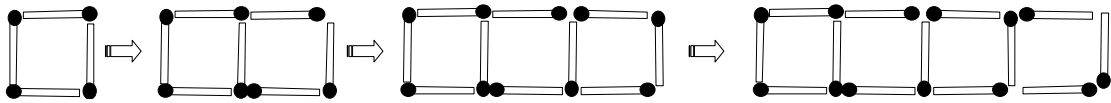
1. Sylvia forms **triangle** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of triangles	1	2	3	4	5	6	7	8	9	10	100
No of matches	3	5	7	9							

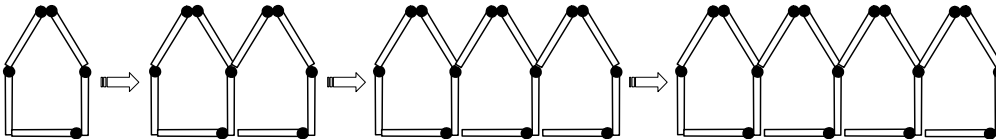
2. On another Sylvia forms **square** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of squares	1	2	3	4	5	6	7	8	9	10	100
No of matches	4	7	10	13							

3. On another day Sylvia forms **pentagon** patterns with matches as shown below:



Complete the following table. Describe any number patterns!

No of pentagons	1	2	3	4	5	6	7	8	9	10	100
No of matches	5	9									

- What is the same and what is different in the situations in 1, 2 and 3?
- How many matches will Sylvia need to build 100 **decagons** (a polygon with 10 sides) in the same way?
- How many matches will Sylvia need to build 100 **n -gons** (a polygon with n sides) in the same way?

For a discussion, see [Fireworks notes](#).

PROBLEM 22: Design a short method to calculate the square of a two-digit number ending in 5.

INDUCTIVE AND DEDUCTIVE REASONING

After having worked through some problems, we can now make our approach to problem solving more explicit. In all our work, we are basically using two different kinds of reasoning:

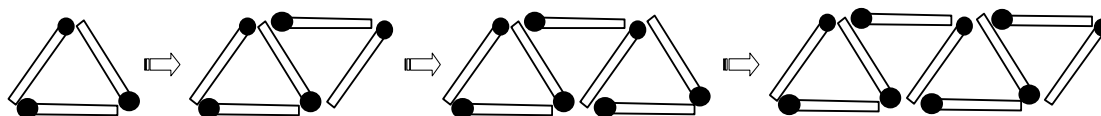
Inductive reasoning is a method of using *numerical pattern recognition* to draw general conclusions. Although inductive reasoning is of great importance in developing ideas, it cannot prove that they are correct, because it is based on a limited set of observations.

Deductive reasoning is a method of using *structural analysis* to draw conclusions from ideas we accept as true by using logic (Jacobs, 1982). We need deductive reasoning to prove that our ideas are correct.

In the examples that follow, we will further analyse the processes of inductive and deductive reasoning. Let's return to Problem 20:

PROBLEM 20: PLAYING WITH MATCHES

- Sylvia forms triangle patterns with matches as shown below:



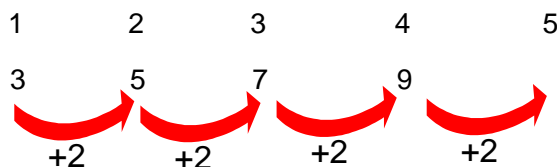
Complete the following table. Describe any number patterns!

No of triangles	1	2	3	4	5	6	7	8	9	10	100
No of matches	3	5	7	9							

Inductive reasoning consists of two sub-processes:

- pattern recognition in a finite set of data (*abstraction*)
- pattern extension to cases not in the present set (*generalisation*)

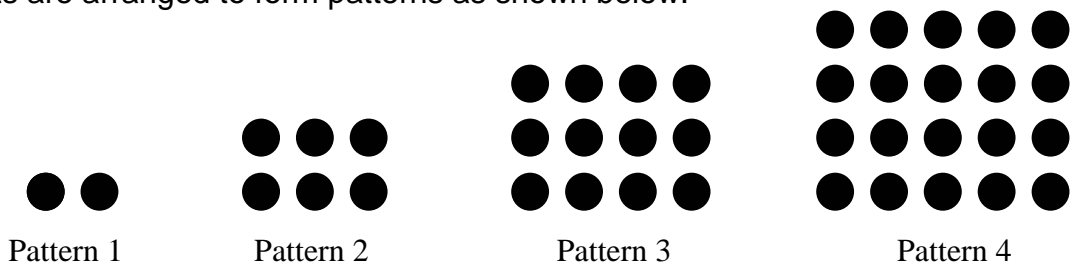
One can focus on the *numbers* given in the table (we call this the *database*) and recognise a vertical (functional) relationship $m = 2t + 1$ which easily yields all the solutions. Or one can recognise a horizontal (recursive) pattern $f(t + 1) = f(t) + 2$, which can serve as a model to generate additional information about the situation:



While the inductive approach above looked at the *numbers* and ignored the *matches* (picture), a deductive approach focuses on the process of packing the *matches* and ignore the numbers. For example, the structure of the matches can be formulated in words as "you start off with 3 matches and then add another 2 matches for every additional triangle that you build". So, for 100 triangles we will need $3 + 99 \times 2 = 201$ matches. This can be *generalised* to $3 + 2(t - 1)$ matches for t triangles.

PROBLEM 23: DOTS

Dots are arranged to form patterns as shown below:



How many dots are there in:

- Pattern 200?
- Pattern n ?

Many people prefer an *inductive approach*, and organise the information in a table and then try to identify *patterns in the numbers* in the table to solve the problem:

Pattern no	1	2	3	4	5	6	200	n
# dots (D)	2	6	12	20				

Do you see any patterns in the table? Most people recognise a recursive (horizontal) pattern +4; + 6; +8; ... but it is not so useful in this case. It is not so easy to recognise a functional (vertical) relationship in the table!

A *deductive approach* focus on the *structure* of the sketches. The figures are rectangles, and the number of dots can be seen as the area of the rectangle. So, in stead of *counting* the dots and working with the numerical *answers* 2, 6, 12, 20, ..., when we work with the *structure*, we describe the *method* for calculating the number of dots without calculating it, i.e. 1×2 , 2×3 , 3×4 , etc. Then we essentially have our functional rule!

Pattern no	1	2	3	4	5	6	200	n
# dots (D)	1×2	2×3	3×4	4×5	5×6	6×7		

Now the pattern in the numbers in the table is easy to see! This shows the inter-relationship between induction and deduction – deduction actually helped to make induction easier! We often do not use just one approach, but both.

There is a tradition of opposition between adherents of induction and of deduction. In my view it would be just as sensible for the two ends of a worm to quarrel.

Alfred North Whitehead

Let's now return to Problem 22 above:

PROBLEM 22:

Design a short method to calculate the square of a two-digit number ending in 5.

To make sense of the question, i.e. to *understand the problem*, you must connect it to your previous knowledge and experiences. What is meant with a "short method"? Do you know the term? Maybe you recall these methods you learned in primary school:

- To multiply a number by 25, first multiply by 100 and then divide by 4.
- To multiply a number by 125, first multiply by 1000 and then divide by 8.

A short method therefore means a method for getting the answer by not doing the given problem, but rather some other process which you can basically do mentally without much use of paper and pencil.

"A two-digit number" of course is a general statement, not meaning any specific two-digit number, but really *any* two-digit number. So the problem really wants us to be able to *look* at a calculation like 65^2 and write down the answer without really calculating 65×65 by long multiplication.

How to get started? We emphasise again that there are two basic routes: induction or deduction. One should develop a feeling for when the one could be more accessible than the other. Let's try the inductive approach: Try some special cases, be systematic, organise your data. Use a calculator to generate the following answers:

$$\begin{aligned}15^2 &= 225 \\25^2 &= 625 \\35^2 &= 1225 \\45^2 &= 2025 \\55^2 &= 3025\end{aligned}$$

So it is all about seeing useful patterns that will enable us to write down such answers. It is easy to recognise/abstract and to conjecture that the answer will always end in 25. There is enough structural backing ($5^2 = 25$) to accept and believe that this is always true. So we already know that $85^2 = ??25$

We need more. We suggest that we ignore the 25s in the answer as *noise* which distracts us from seeing other patterns. Many times when working on this problem, students get very excited and say that they see

$$\begin{array}{r}2 \\+4 \downarrow \\6 \\+6 \downarrow \\12 \\+8 \downarrow \\20 \\+10 \downarrow \\30\end{array}$$

This is a recursive pattern: $T_n = T_{n-1} + 2n$. It is very interesting, but not very useful. If we want to use this pattern to calculate 85^2 we will first have to know the answer of 75^2 , i.e. we will have to write down the whole pattern from the beginning. Hardly a short method!

What would be more useful, would be a functional relationship. Let's remove more noise – the 5-part probably takes care of the 25-part on the right, so we expect

Input	Output
1	2
2	6
3	12
4	20
5	30
n	?

Maybe you recognise the following pattern:

Input	Output
1	$\times 2$ 2
2	$\times 3$ 6
3	$\times 4$ 12
4	$\times 5$ 20
5	$\times 6$ 30
n	?

To see the pattern, ignore the rest and concentrate only on the relevant data:

Input	
1	2
2	3
3	4
4	5
5	6
n	$n+1$

It should be clear that we multiply the ones-digit by the next whole number. So our conjecture is:

To square a two-digit number ending in 5: The number ends in 25 and the first digits are given by multiplying the tens-digit by the next whole number.

Is it true? We can easily check all 9 cases, so we know it is true. But still we want to know *why* it is true. Deduction will help:

Any two-digit number is of the form $10a + b$. We have a special case where the units-digit is 5, so $b = 5$:

$$\begin{aligned} (10a + 5)^2 &= 100a^2 + 100a + 25 \\ &= 100a(a + 1) + 25 \end{aligned}$$

That pretty much explains it, provided we can interpret the symbolism. The purpose of the 100 is to guarantee that the answer ends in 25. Understanding the function of the 100 means that we can also understand why this method works for numbers ending in 5 – we only get 100 in the middle term of the expression because of the 5!

PROBLEM 25: WIMBLEDON

You are the sports secretary of your sports club and you need to know how many games will be played in your knock-out competition. The competition works like the Wimbledon tennis championship: two players are drawn to play against each other in the first round. The loser falls out and the winner plays against a winner of another game. This continues, with losers being eliminated and winners playing against winners until the champion is crowned. If there are 60 players, how many games are played in total? And if there were 35, or 28 or 44 players, i.e. can you develop an easy method (formula) to calculate how many games are played for any number of players?

A diagram will help us to understand the situation. For example, Figure 1 is a special case with 4 players: Player P_1 plays against P_2 and the winner, W_1 , plays against W_2 , the winner of the game between P_3 and P_4 .

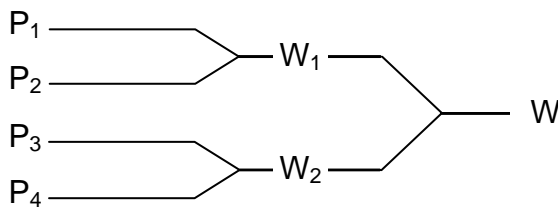


Figure 1

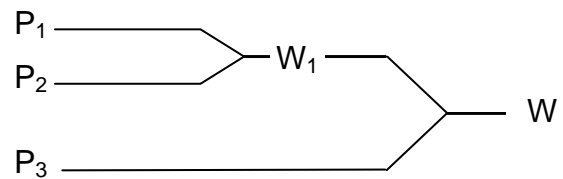


Figure 2

We also need to understand some finer details of the structure of the competition. For example, Figure 2 shows what happens when there is an odd number of players: one player gets a bye, i.e. does not play and proceeds automatically to the next round.

Thinking inductively, we can now organise our data of special cases into a table:

# players	1	2	3	4	5	6
# games	0	1	2	3	4	5

We easily recognise the pattern in the table and generalise it: The number of games is one less than the number of players, or written in symbols:

For n players there are $n - 1$ games.

Sometimes a *deductive explanation* can be deceptively simple. See if you follow this reasoning:

There can be only one champion, i.e. one player who did not lose. So, with 60 players, to get 1 winner, we need 59 losers. To get 59 losers we need 59 games. That is all!

The reasoning is completely general: For n players, to get $n - 1$ losers, we need $n - 1$ games.

PROBLEM 27: ROUND ROBIN

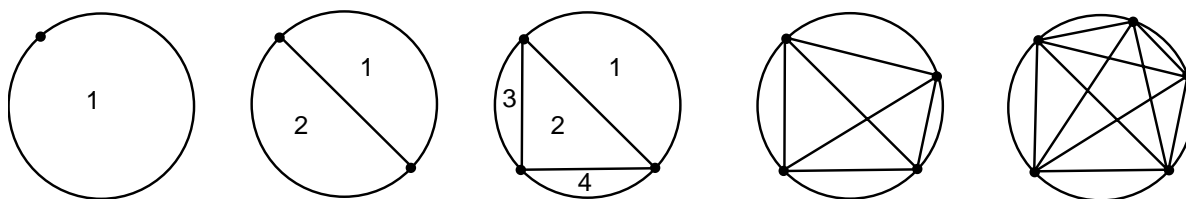
Mr Daniels is the match secretary for the Mpumalanga soccer league. He must arrange the soccer schedule for next year. Each team plays each other team twice – one match at home, and one match away.

How many league matches will be played if there are 9 soccer teams in the league?

Find a formula to work out how many matches must be played if there are n teams.

THE PITFALLS OF INDUCTION

PROBLEM 33: REGIONS



If 3 *points* on a circle are joined, 4 *regions* are formed, as shown above. Complete the table for 4 and 5 points by using the given sketches above. What is the *maximum* number of regions into which 6 points on a circle will divide the circle if the points are joined? And 20 points?

# points (p)	1	2	3	4	5	6	20
# regions (R)	1	2	4				

You should really first solve the problem before continuing reading!

You probably noticed the recursive pattern 1, 2, 4, 8, 16, ... in the numbers in the table and then continued the pattern to *predict* $R(6) = 32$. Or you may have inductively deduced the formula $R(p) = 2^{p-1}$, which also yields $R(6) = 32$.

Unfortunately our expectation that the pattern is continued beyond the fifth case is wrong! In fact $R(6) = 31$! You are encouraged to check for yourself by drawing a nice big circle and physically counting the number of regions.

We will return to this problem later (see page 59).

This example serves to remind us that inductive reasoning, powerful as it may be in discovering new patterns, is prone to error:

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks. Kasner & Newman, 1940

PROBLEM 34: ONES

Check these patterns:

x	x^2	Digit sum of x^2
1	1	1
11	121	4
111	12321	9
1111	1234321	16
11111	123454321	25

Now predict the digit sum of 11111111111^2 (11 ones).

Most of us will recognise the pattern in the digit sum of x^2 in the last column as squares, and then extend beyond the given five examples to predict that the digit sum of 1111111111^2 is $11^2 = 121$. However, if you actually calculated it, you will see that the predicted answer is incorrect!

Our abstraction that the numbers in the given examples are squares is correct. But this pattern is not continued, because the *structure* breaks down after 9 ones because of carrying. If you have not yet done it, you will want to check that this is correct and think about the reasons why the structure breaks down!

This example again emphasises the danger of *only* focussing on the *numbers* in a situation and to generalise through induction without considering the structure of the situation.

PROBLEM 35: PRIMES

Investigate the nature of $P(n) = n^2 - n + 11$, $n \in \mathbb{N}$

Let's approach the problem inductively by generating the following special cases:

$$P(1) = 11$$

$$P(2) = 13$$

$$P(3) = 17$$

$$P(4) = 23$$

$$P(5) = 31$$

$$P(6) = 41$$

$$P(7) = 53$$

$$P(8) = 67$$

$$P(9) = 83$$

$$P(10) = 101$$

These numbers are all prime. Would you agree that we can conclude that the answer is always a prime number?

Unfortunately, the very next case, i.e. $P(11) = 121 = 11^2$ is *not* prime! Although our observed pattern of primes is true for the given 10 cases, it is not correct to extend (generalise) the pattern beyond these 10 cases.

PROBLEM 36: ODD NUMBERS

The examples of numbers $P(n)$ in Problem 35 are all odd. Would you agree that we can conclude that the answer is always an odd number? How can you be *sure*?

PROBLEM 37: EVEN NUMBERS

The difference between consecutive numbers $P(n)$ in Problem 35 is even. Would you agree that we can conclude that the difference is always an even number? How can you be *sure*?

PROBLEM 38: MORE PRIMES

Show that $P(n) = n^2 - n + 41$ is prime for $n = 0$ to 40 but not prime for $n = 41$.

Show that $P(n) = n^2 - 79n + 1601$ is prime for $n = 0$ to 79 but not prime for $n = 80$.

Show that $n^2 - pn + q$ cannot be prime for $n = q$.

PROBLEM 39: IQ PROBLEM

What is the next number in this sequence: 2; 4; 6; 8; ?

We would probably all opt for 10. Are you sure? This is based on an implicit function rule $2n$. But as the table below shows, the next value may as well be 34, based on another valid function rule:

n	1	2	3	4	5	6
$2n$	2	4	6	8	10	12
$2n + (n-1)(n-2)(n-3)(n-4)$	2	4	6	8	34	132

PROBLEM 40: IQ TEST

What is the next number in this number pattern:

1, 4, 9, 16 ?

Surely, we will all answer 25! It is a sequence we all know very well, namely the sequence of square numbers, i.e. n^2 . But other formulae also generate these values, but then complete the pattern differently:

n	1	2	3	4	5
n^2	1	4	9	16	25
$n^2 + (n-1)(n-2)(n-3)(n-4)$	1	4	9	16	?
$2^n - \frac{n(n-2)(n-4)}{3}$	1	4	9	16	?

PROBLEM 41: WHOLE NUMBER

Check some special cases for the conjecture that for $y \in \mathbb{N}$, $x = \sqrt{1141y^2 + 1}$ is never a whole number.

The conjecture is true for all values of y up to 30 693 385 322 765 657 197 397 207, but for the next, x is a whole number!

Can you find the value of x ?

This means that the conjecture is true for 30 693 385 322 765 657 197 397 207 consecutive cases, yet the conjecture turns out to be false!

PROBLEM 42: REMAINDER

Investigate the remainder when the square of an odd number is divided by 8. Are you *sure*?

An inductive approach yields:

$$3^2 = 9 \text{ and } 9 \div 8 = 1 \text{ rem } 1$$

$$5^2 = 25 \text{ and } 25 \div 8 = 3 \text{ rem } 1$$

$$7^2 = 49 \text{ and } 49 \div 8 = 6 \text{ rem } 1$$

$$11^2 = 121 \text{ and } 121 \div 8 = 15 \text{ rem } 1$$

A conjecture that the remainder is always 1 seems reasonable. But how can we be sure that it will *always* be 1? Considering our previous examples of the pitfalls of induction, how can you be sure that suddenly after a million cases a remainder of 2 will not appear? As someone said:

Absence of evidence does not mean evidence of absence!

In order to be sure the conjecture is *always* true, we must check *all* the cases, not just a few, and not just a few million! It is impossible to check all because the natural numbers are infinite! But we can check all by using the power of algebraic symbols, representing *any* case or *all* cases.

In order to explain *why* the remainder is only 1, it is necessary that we analyse the *structure* of the situation. As we have mentioned before (see, for example page 15), we can only explain the structure of a situation when we work with the general case, i.e. use *deductive reasoning*, as follows:

We can express *any* odd number and *all* odd number as $2n + 1$, $n \in \mathbb{N}_0$.⁴ Make sure that you understand the meaning and structure of this statement! Check it for special cases. Can 731 be written in the form $2n + 1$, $n \in \mathbb{N}_0$? Now we have

$$\frac{(2n+1)^2}{8} = \frac{4n^2 + 4n + 1}{8} = \frac{4n(n+1) + 1}{8}$$

Now it is necessary to know that if n is odd, $n + 1$ is even and the other way around, so $n(n + 1)$ is even and therefore $4n(n + 1)$ is divisible by 4 and by 2, i.e. 8. From the right distributive property of division over addition, i.e.

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$$

we can now deduce that the remainder is 1.

You should note the power of deductive reasoning: While in Problem 41 we may have felt sure that for $n \in \mathbb{N}$, $\sqrt{1141n^2 + 1}$ is never a whole number, it turned out to be false for n equal to 30 693 385 322 765 657 197 397 208. But in problem 45, after our deductive analysis, we are *absolutely certain* that no matter what odd number we take – take 30 693 385 322 765 657 197 397 209 if you like! – if we square it and we

⁴ Note that we are using the notation \mathbb{N} for the natural numbers, i.e. $\{1, 2, 3, 4, \dots\}$ and \mathbb{N}_0 for the whole numbers. i.e. $\{0, 1, 2, 3, \dots\}$. If we wrote any odd number as $2n + 1$, $n \in \mathbb{N}$, i.e. started with $n = 1$, the first odd number would be 3, but starting with $n = 0$, the first odd number is 1. Of course we can also write any odd number as $2n - 1$, $n \in \mathbb{N}$, but the form $2n + 1$ is often more convenient.

then divide by 8, the remainder will *definitely* be 1. In fact, we can predict it with certainty, without having to actually do the calculations. That is the power of mathematics, and *that power lies in the generality of algebraic symbols and in deductive thinking!*

Looking back

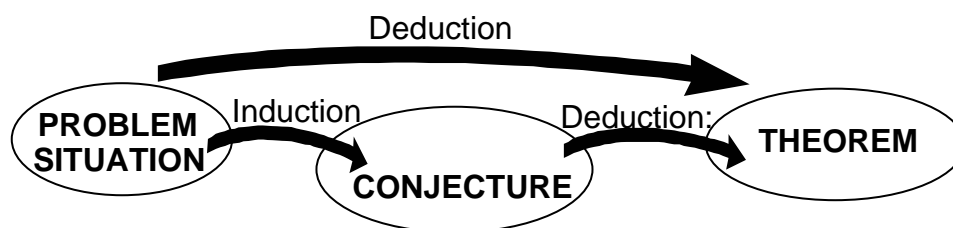
Let us repeat: inductive reasoning is a powerful method to discover new relationships. But we can never be sure that an inductive pattern will not somewhere break down, even after millions of cases. To be *sure that* it is always true (validity), and to *explain why* it is true – why the pattern has this form and not another, we must use *deductive reasoning*, i.e. reason using the structure of the situation.

This means that in mathematical activity there are two approaches:

- One can work deductively, or
- One can work inductively, but to make sure, it should be *followed by deduction*.

But just induction on its own is not adequate! When we use induction we cannot be sure our conclusion is correct, no matter how many cases we check.

The following diagram depicts the relationship between induction and deduction and the status of knowledge (a conjecture is not yet proved; a theorem is a proved conjecture):



MORE INDUCTION AND DEDUCTION

After seeing the pitfalls of inductive reasoning, our attitude towards problem solving should now be different. We should be *sceptical* about each result! It does not mean that we should not use inductive reasoning. To the contrary! Inductive reasoning is very powerful – most discoveries in mathematics are probably made through inductive reasoning. But we should be sceptical, doubt every result and insist on proof before we are convinced. But we must realise that certainty, proof, can only be reached through deductive reasoning, and it should become part and parcel of all our work.

Let's return to Problems 35-37. We again give the first 10 cases of $P(n) = n^2 - n + 11$, $n \in \mathbb{N}$:

$$P(1) = 11$$

$$P(2) = 13$$

$$P(3) = 17$$

$$P(4) = 23$$

$$P(5) = 31$$

$$P(6) = 41$$

$$P(7) = 53$$

$$P(8) = 67$$

$$P(9) = 83$$

$$P(10) = 101$$

Thabo says that he thinks the answer is always an odd number. Is this correct? Surely, it is correct for all the cases above. If we try more cases, we find that they are all also odd. No matter what number we choose for n , we get an odd number every time. Can we conclude that the answer is always odd for all $n \in \mathbb{N}$?

No ways! Not until we have checked *all* cases, and that we can only do through deduction. Look at this:

$$\begin{aligned} P(n) &= n^2 - n + 11 \\ &= n(n-1) + 11 \end{aligned}$$

$(n-1)$ and n are consecutive natural numbers. So if one is even, the other one is odd. So $n(n-1)$ is the product of an even and an odd number. Are you happy that it always is an even number? How do you know? So $P(n)$ is an even number plus 11, i.e. an even number plus an odd number, which always is an odd number. Are you happy with that?

Note that we have in this deduction assumed several statements without trying to prove them (e.g. any even number plus any odd number is always an odd), assuming that it is accepted background knowledge. But if someone does not agree, one should prove it. So, is an even number plus an odd number always odd? How do we know? We probably developed this knowledge through generalisation from special cases:

$$2 + 5 = 7$$

$$4 + 7 = 11$$

$$8 + 9 = 17$$

A deductive proof and explanation would be something like:

If $n \in \mathbb{N}_0$, $2n$ is any even number; if $k \in \mathbb{N}_0$, $2k + 1$ is an odd number

$2n + 2k + 1 = 2(n + k) + 1$, which is an even number plus one, which is odd.

Now we have *proved* it for *any* even number and *any* odd number!

In Problem 37 we noted that the difference between consecutive numbers is even. How can we be sure this is correct? The following argument works generally with *any* number of the form $n^2 - n + 11$, i.e. $P(n)$ and the next number, i.e. $P(n+1)$ and therefore leads to a general result for all such pairs:

$$\begin{aligned} & P(n+1) - P(n) \\ &= (n+1)^2 - (n+1) + 11 - (n^2 - n + 11) \\ &= n^2 + 2n + 1 - n - 1 + 11 - n^2 + n - 11 \\ &= 2n \end{aligned}$$

$2n$, $n \in \mathbb{N}$ is a general description for any even number ...

PROBLEM 43:

Take any two-digit number, then subtract the sum of the digits. What do you find?

First *specialise*, for two reasons.

First, to understand what is going on.

So take 34. What we must do is $34 - (3 + 4) = 27$

Second, to get a sense of pattern – there seems nothing special in this number, so if there is an underlying pattern we need to list several cases:

$$\begin{aligned} 34 - (3 + 4) &= 27 \\ 35 - (3 + 5) &= 27 \\ 36 - (3 + 6) &= 27 \\ 37 - (3 + 7) &= 27 \end{aligned}$$

Can it be that the answer is always 27? Hardly. Maybe it is true that for numbers in the *thirties* the answer is always 27. As a matter of fact the statement is definitely true, because we can easily test it for *all* the cases – if we also 30, 31, 32, 33, 38 and 39 we know for sure that the answer is always 27. But what about other decades? Let's try the sixties:

$$\begin{aligned} 64 - (6 + 4) &= 54 \\ 65 - (6 + 5) &= 54 \\ 66 - (6 + 6) &= 54 \end{aligned}$$

So, for the sixties, the answer seems to be always 54. We make up conjectures about the structure as we go along – we *guess* an underlying pattern and then we check it. . Different people will guess different patterns. For example, I notice that 54 is double 27, and then I get excited as I notice that the sixties is double the thirties. So I can now *predict* that , for example, the answer in the forties will be double the answer in the twenties. I check:

$$\begin{aligned} 21 - (2 + 1) &= 18 & 44 - (4 + 4) &= 36 \\ 25 - (2 + 5) &= 18 & 42 - (4 + 2) &= 36 \\ 28 - (2 + 8) &= 18 & 47 - (4 + 7) &= 36 \end{aligned}$$

So I was correct! I even see a "bigger" pattern: the sixties is three times the twenties, and $3 \times 18 = 54$.

I also notice that the sum of the digits of the answer is *always* 9! See what I mean? So far we have the following answers:

- 18 and $1 + 8 = 9$
- 27 and $2 + 7 = 9$
- 36 and $3 + 6 = 9$
- 54 and $5 + 4 = 9$

But *why* will this be?

I also begin to notice that the answer is always a multiple of 9. We have already tested nearly all 99 cases and we can if we want to. There can be no doubt – I am confident that the answer always is a multiple of 9. But *why* is it true, meaning why a multiple of 9, why not 7 or 8 or something else?

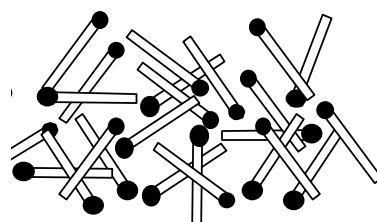
Can you explain *why* by looking at all our special cases above? I cannot. So let's go general.

Any two-digit number can be written as $10a + b$, where $a, b \in \mathbb{N}_0$ and $a, b < 10$.

So the number subtract the sum of its digits is $10a + b - (a + b)$

Again we have no clear idea where we are going. We want to prove that it is a multiple of 9. What is the structure of a multiple of 9? How will we recognise it? Let's for the moment simply simplify:

$$\begin{aligned} & 10a + b - (a + b) \\ &= 10a + b - a - b \\ &= 9a \end{aligned}$$



This was my matchbox problem!

Surely, $9a$ is a multiple of 9! Do you agree? If not, you can understand it by *specialising*:

a	1	2	3	4	5	6	7	8	9
9a	9	18	27	36	45	54	63	72	81

We get more out of the deduction. The answer is not only a multiple of 9, it also tells us which multiple of 9! How did I miss it in recognising the inductive pattern?

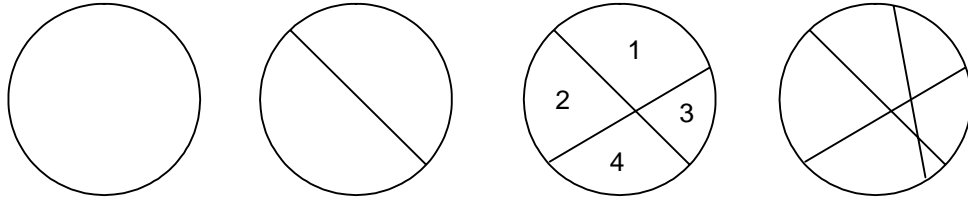
We must *interpret* the *meaning* of the expression and of the variable. There is the syntactical meaning: $9a$ means $9 \times a$. a is not just any number. It is the tens-digit in $10a + b$. Therefore, if I have 57, I can predict that the answer is $9 \times 5 = 45$ without doing the subtraction calculation!

PROBLEM 44:

- Try to generalise the result in the previous activity:
- Take any *three*-digit number, then subtract the sum of the digits ...
- What about a four-digit number?
- What about an *n*-digit number?

PROBLEM #: CIRCLES, REGIONS AND CORDS

1.

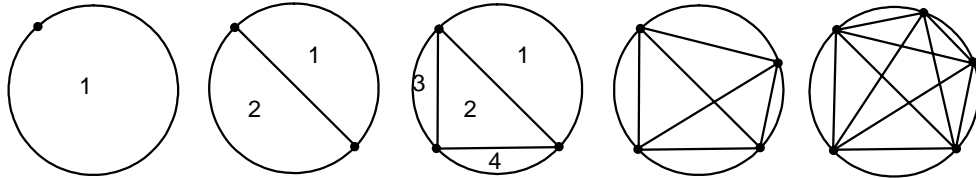


2 chords divide a circle into 4 regions.

What is the *maximum* number of regions into which 6 chords will divide a circle?
And 20 chords?

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4					

2.



If 3 points on a circle are joined 4 regions are formed.

What is the *maximum* number of regions into which 6 points on a circle will divide the circle if the points are joined?

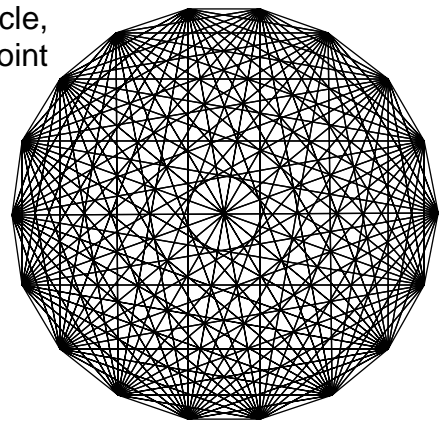
And 20 points?

# points (p)	1	2	3	4	5	6	20
# regions (R)	1	2	4				

3. (a) In this figure, there are 18 points on the circle, and every point is connected to every other point on the circle.

How many connecting lines (chords) are there all together?

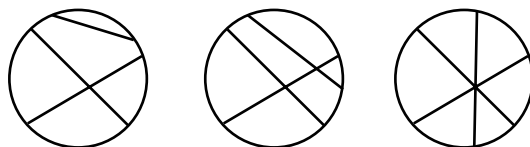
(b) In another circle there are 465 connecting lines. How many points are there on the circle?



We discuss these three problems on the next pages ...

1. Chords and regions

First *understand* the situation! *Maximum* number of regions implies that the chords cannot be drawn like these below; each new chord must cut every other chord and no three chords should be concurrent (cut in one point).

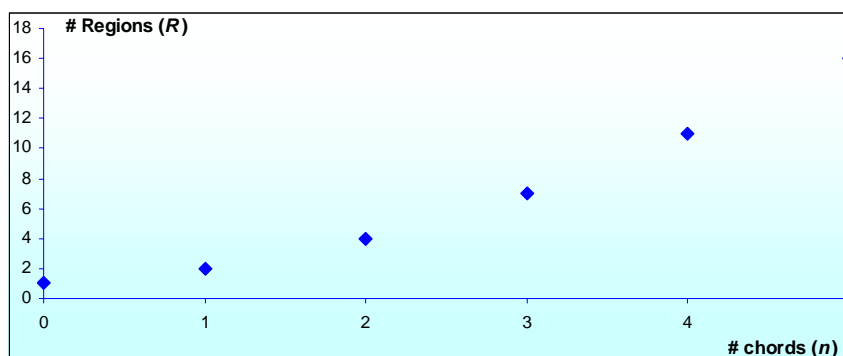


An *inductive* attack tries to find a pattern in the *numbers*.

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4	7	11	16		

If you want to find a *functional* (here *vertical*) relationship by *inspection* (i.e. by just “looking”) you will have to systematically *ask* – is it maybe $+1?$, is it $\times 2 - 2?$, is it $\dots??$, i.e. the only way of “seeing” a pattern is to first *beforehand* formulate a *conjecture* (an unproven theorem) and use it as a *lense* to look at the data and to *test/check* each conjecture and do not stop trying different alternatives until you find one that fits!!

It may help if you represent the data *graphically* and then use your knowledge of the relationship between the shape of a graph and its formula! Here is an Excel *scatterplot* of the data. What kind of a formula do you think may *fit* the points?



If you are using a technology tool like Excel, you may as well go all the way and let it find the *regression formula*⁵ (the “curve of best fit”) for you!

⁵ Click here to [see Excel's Trendline](#). There is also a [How to ... file](#).

If you want to find a formula *analytically* (i.e. algebraically) you have to bring certain *resources* (knowledge) to the situation. For example, knowledge of *recursive* (horizontal) differences may be helpful here:

# chords (n)	0	1	2	3	4	5	6	20
# regions (R)	1	2	4	7	11	16		

Differences: +1 +2 +3 +4 +5

Second differences: +1 +1 +1 +1

Maybe you know: *if the second differences are constant, then the formula is quadratic.*

Do you know this theorem? If not, [click here for a brief discussion in the appendix.](#)

Now that we know the formula is quadratic, we merely have to solve for the parameters a , b and c in $R(n) = an^2 + bn + c$. The values in the *database* satisfy the equation, so substituting them makes the equation true, and this leads to three equations in three unknowns:

$$R(0) = 0a + 0b + c = 1 \dots (1)$$

$$R(1) = 1a + 1b + c = 2 \dots (2)$$

$$R(2) = 2a + 4b + c = 4 \dots (3)$$

Here is yet another *inductive, recursive* method: Look at R and the differences *with different eyes*:

$$R(0) = 1$$

$$R(1) = 1 + 1$$

$$R(2) = 1 + 1 + 2$$

$$R(3) = 1 + 1 + 2 + 3$$

$$R(n) = 1 + (1 + 2 + 3 + \dots + n)$$

If you *know* that $1 + 2 + 3 + \dots + n = \sum_{i=1}^n i = \frac{n(n+1)}{2} \dots$

or if you do not, maybe you can *deduce* it from your knowledge of the sum of an *arithmetic series*: $S_n = \frac{n}{2}\{2a + (n-1)d\} \dots$

$$\text{Anyway, } R(n) = 1 + \sum_{i=1}^n i = 1 + \frac{n(n+1)}{2} = \frac{n(n+1)+2}{2}, \quad n \in N_0 = 0, 1, 2, 3, \dots$$

You should *check it against the known database* so that at least you are sure the formula is valid for $n = 1$ to 5!!

But, of course, although we here used *algebraic reasoning* to deduce the formula, it is nevertheless based on an *inductive analysis* of the *numbers* in the table, *not on the structure of the situation!*

So you can only pray that the generalisation is *valid* after $n = 5$!!! But how can you be *sure??*

2. Points and regions

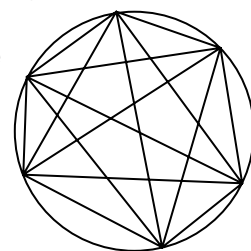
You probably noticed the *recursive pattern* 1, 2, 4, 8, 16, ... in the numbers in the table and then continued the *recursive doubling pattern* to *predict* $R(6) = 32$. Or you may have inductively deduced the *functional formula* $R(p) = 2^{p-1}$, which also yields $R(6) = 32$.

A reminder: *Induction* consists of two processes:

1. *Abstraction* (finding the pattern in the *known set of numbers* or *database*) – here the pattern in the 6 *known pairs* (1, 1), (2, 2), (3, 4), (4, 8), (5, 16) is $R(p) = 2^{p-1}$, $1 \leq p \leq 5$, $p \in N$.
2. *Generalisation* (extending the pattern *beyond* the known database) – here assuming that the next pair is (6, 32) and $R(p) = 2^{p-1}$, $p \in N$.

In this case the abstraction (1) is correct but the generalisation (2) is not! Unfortunately our expectation that the pattern is continued beyond the fifth case is wrong!

In fact $R(6) = 31$! Check for yourself by physically counting the number of regions in this sketch. *Understand the problem!* The prerequisite of *maximum* number of regions implies that no three chords should be *concurrent* (cut in one point).



This example serves to remind us that inductive reasoning, powerful as it may be in discovering new patterns, is prone to error:

When the mathematician says that such and such a proposition is true of one thing, it may be interesting, and it is surely safe. But when he tries to extend his proposition to everything, though it is much more interesting, it is also much more dangerous. In the transition from one to all, from the specific to the general, mathematics has made its greatest progress, and suffered its most serious setbacks. Kasner & Newman, 1940

But can we find the *correct* formula? Let's try differences again:

# points (p)	1	2	3	4	5	6		20
# regions (R)	1	2	4	8	16	31		

First differences: 1 2 4 8 15

Second differences: 1 2 4 7

Third differences: 1 2 3

Fourth differences: 1 1

The fourth differences are equal, so the generating formula is a fourth degree polynomial. So the formula is of the form $R(p) = ap^4 + bp^3 + cp^2 + dp + e$. So we must find a, b, c, d and e. But $R(0) = 1$, so $e = 1$. So we must find a, b, c, d, i.e. solve four unknowns and so we need four equations:

$$R(1) = 1a + 1b + 1c + 1d = 0 \dots\dots\dots (1)$$

$$R(2) = 16a + 8b + 4c + 2d = 1 \dots\dots\dots (2)$$

$$R(3) = 81a + 27b + 9c + 3d = 3 \dots\dots\dots (3)$$

$$R(4) = 256a + 64b + 16c + 4d = 7 \dots\dots\dots (4)$$

Solving these four simultaneous equations, we get:

$$R(p) = \frac{1}{24}p^4 - \frac{1}{8}p^3 + \frac{23}{24}p^2 - \frac{2}{3}p + 1$$

$$= \frac{p^4 - 6p^3 + 23p^2 - 18p + 24}{24}$$

Or you could use the known database in a technology software package to easily find the regression formula for you⁶. Do you agree that the technology formula is the same as above?

You may want to check that this formula generates the correct values, i.e. 1, 2, 4, 8, 16, 31.

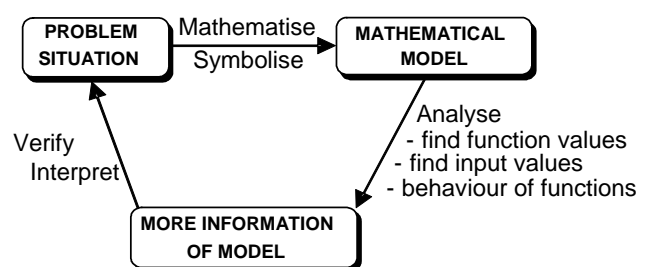
This formula produces the next value $R(7) = 57$. But is it correct? Remember, this formula is generalised from the number pattern, i.e. through *induction*, so we must wonder if induction will get us into trouble yet again! To check $R(7) = 57$, i.e. to check if 7 points on a circle yield 57 regions, you have no choice but to draw and count! You cannot trust this formula as a model of the situation, you cannot be sure!

You can only be sure if you reasoned with the *structure of the situation!*

3. Mystic rose

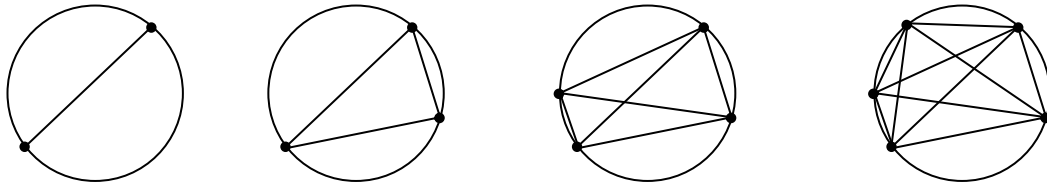
Students often simply formulate a guess like "It's 18×18 ", or "it's 18×17 " and then cannot *justify* it, expecting the lecturer to tell them if they are right or wrong. This is *not* what mathematical thinking is about! We must learn to see our efforts not as *answers*, but as *conjectures*, as *public statements* that should be discussed, explained, verified and justified through *logical argument*. And we should learn to value logical arguments, and not accept mere *authority* of someone like the lecturer as support or justification for our solutions!

Now let's try again! Of course you can count, but that will be a rather daunting task and it will be prone to error! The essence of mathematics is to construct *mathematical models* that mimic the real situation, and then we manipulate mathematical objects in stead of real-life objects to *predict* unknown information.



Let's begin with an *inductive approach* and let's use some *heuristics*: Let's investigate some special cases, let's do it systematically, let's organise our resulting data in a table, try to find a pattern in the data and then use the pattern as a model to solve the original problems. Here are some special cases, where it is very easy to count the cords:

⁶ Use a curve-fitting programme like CurveExpert (on our software page). Or click here to see the [use of Excel's Trendline ...](#)



# points (n)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10				

What we want is a *functional (vertical) formula* expressing C in terms of n . But it is not always so easy to find a formula through *inspection (just by "looking")*. In the table below I identify some easily observed patterns, suggesting a relationship:

# points (n)	2	3	4	5	6	7	18	n
		↓ $\times 1$		↓ $\times 2$		↓ $\times 3$		
# chords (C)	1	3	6	10	15	21		

I now "fill in the gaps" using the pattern:

# points (n)	2	3	4	5	6	7	18	n
	↓ $\times \frac{1}{2}$	↓ $\times 1$	↓ $\times \frac{1}{2}$	↓ $\times 2$	↓ $\times \frac{1}{2}$	↓ $\times 3$		
# chords (C)	1	3	6	10	15	21		

I now write the numbers in an *equivalent, but more useful* form:

# points (n)	2	3	4	5	6	7	18	n
	↓ $\times \frac{1}{2}$	↓ $\times \frac{2}{2}$	↓ $\times \frac{3}{2}$	↓ $\times \frac{4}{2}$	↓ $\times \frac{5}{2}$	↓ $\times \frac{6}{2}$		
# chords (C)	1	3	6	10	15	21		

Please notice that for the purpose of "seeing" the structure in this context, $\frac{3}{2}$ is "simpler" than $1\frac{1}{2}$ and $\frac{6}{2}$ is simpler than 3!!! The conventions you learned in primary school about "always writing in simplest form" are totally irrelevant in context!

To generalise our pattern, we must observe what is unchanging (*invariant*) and what changes. The invariant part is clear: every value is multiplied by something and divided by 2. It is this *invariant structure* that must be continued and *generalised* to $C(18)$ and $C(n)$:

# points (n)	2	3	4	5	6	7	18	n
	↓ $\times \frac{1}{2}$	↓ $\times \frac{2}{2}$	↓ $\times \frac{3}{2}$	↓ $\times \frac{4}{2}$	↓ $\times \frac{5}{2}$	↓ $\times \frac{6}{2}$	↓ $\times \frac{?}{2}$	↓ $\times \frac{?}{2}$
# chords (C)	1	3	6	10	15	21		

We must now remove the *noise* and concentrate only on the variable part, so that we can more easily *identify the structure*:

# points (n)	2	3	4	5	6	7	18	n
	↓	↓	↓	↓	↓	↓	↓	↓
numerator	1	2	3	4	5	6		

The functional (vertical) relationship is easily seen as -1 and extended to 18 and n :

# points (n)	2	3	4	5	6	7	18	n
	↓ -1	↓ -1	↓ -1	↓ -1	↓ -1	↓ -1	↓ -1	↓ -1
numerator	1	2	3	4	5	6	17	$n-1$

We can now answer our original question: $C(18) = \frac{18 \times 17}{2}$ and $C(n) = \frac{n \times (n-1)}{2}$.

We have our solution, but it was not so easy to find the functional formula! To emphasize that different people see the same situation differently *because they bring different background knowledge (resources) as lenses to the situation*, let's investigate the *recursive* (horizontal) pattern of differences $+2, +3, +4, \dots$

# points (n)	2	3	4	5	6	7	18	n
# chords (C)	1	3	6	10	15	21		
Differences:		2	3	4	5	6		
Second differences:			1	1	1	1		

So the second difference is constant, so the formula is quadratic of the form $C(n) = an^2 + bn + c$ and we merely have to solve for the parameters a, b and c :

$$C(1) = 1a + 1b + c = 0 \quad \dots (1)$$

$$C(2) = 4a + 2b + c = 1 \quad \dots (2)$$

$$C(3) = 9a + 3b + c = 3 \quad \dots (3)$$

$$(2) - (1): 3a + b = 1 \quad \dots (4)$$

$$(3) - (2): 5a + b = 2 \quad \dots (5)$$

$$(5) - (4): 2a = 1 \Rightarrow a = \frac{1}{2}, b = -\frac{1}{2}, c = 0$$

$$\text{So } C(n) = \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2}, \quad n \in N$$

This formula generates $C(18) = 153$. But is it correct? You cannot trust this inductively generated formula as a model of the situation, *you cannot be sure!* Do you really want to check by actually physically counting $C(18)$??

Solving the question $C(n) = 465$ is very difficult without the formula, but with formula it becomes very easy, illustrating the power of Algebra:

$$C(n) = 465$$

$$\Rightarrow \frac{n(n-1)}{2} = 465$$

$$\Rightarrow n^2 - n - 930 = 0$$

$$\Rightarrow (n-31)(n+30) = 0$$

$$\Rightarrow n = 31$$

Deduction

All the previous work was *induction*, i.e. deducing patterns from *numbers*, which are *special cases*. But inductive conclusions may be wrong! And induction does not explain *the form* of the result.

To *prove* a statement (to show *that* it is true) and to explain *why* it is true, we need to reason *deductively*, i.e. *generally*, using the *structure of the situation*.

We can prove *that* a statement is true using **complete mathematical induction**⁷. For example:

*Chords and regions*⁸:

To prove that $R(n) = 1 + \frac{n(n+1)}{2}$, $n \in N_0$ by mathematical induction, we have to prove:

$$1. R(k) = 1 + \frac{k(k+1)}{2} \Rightarrow R(k+1) = 1 + \frac{(k+1)(k+2)}{2}$$

$$2. R(0) \text{ is true, i.e. } R(0) = 1 + \frac{0(0+1)}{2} = 1$$

Suppose there are already k chords drawn in the circle. Then the next, i.e. the $(k+1)^{\text{st}}$ chord will cut each of the previous k chords and therefore pass through $(k+1)$ regions, therefore adding an additional $(k+1)$ regions.

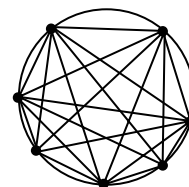
$$\text{So, if } R(k) = 1 + \frac{k(k+1)}{2}$$

$$\text{then } R(k+1) = 1 + \frac{k(k+1)}{2} + (k+1) = 1 + \frac{(k+1)[(k+1)+1]}{2}$$

which proves the implication in (1) and the rest is obvious ...

Mystic rose

A *deductive approach* will use the *structure of the situation*, *not the number answers for specific cases!* Analyse the *special case* when we have 7 points on the circle, and try to develop some clever way of counting the chords that will bring out the *structure* of the situation.



It should be clear that there should be 6 chords at every point on the circle, *because we are connecting every point with every other point, except itself*. So for 7 points there are 7×6 chords altogether. However, this is still not correct – *we have counted every chord twice!* Therefore, the number of chords in a 7-point Mystic Rose is $\frac{7 \times 6}{2}$, which of course confirms the answer we previously obtained for $C(7)$ through recursion.

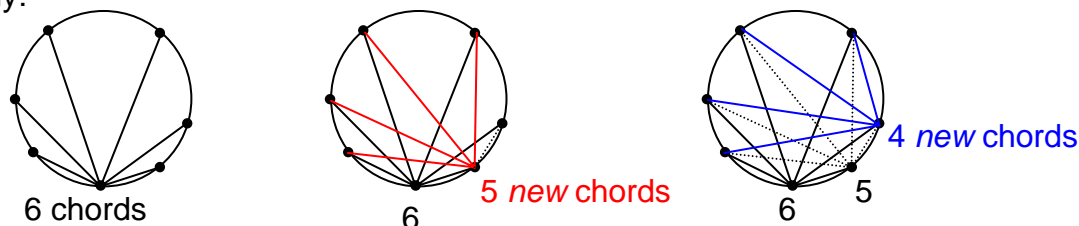
In deduction, and generally in mathematical thinking, we are not concerned with the numerical answer, but with the *method* or *structure*. So if we understand the structure in the one example $C(7) = \frac{7 \times 6}{2}$, we can *generalise* the structure – we must be able to

⁷ What we are calling “induction”, i.e. generalising from *a few special cases*, is really better named “incomplete induction”, in contrast to the method of *Mathematical induction*, which is “complete induction”, because it considers *all cases!*

⁸ Compare [Michael de Villiers](#)

see *the general in the particular*. So it should be clear that for 18 points, there will be 17 chords at each point, so 18×17 in total, except that we counted each chord twice, so $C(18) = \frac{18 \times 17}{2}$. In *general*, if we have n points, each point is connected to $n - 1$ points, so $C(n) = \frac{n \times (n-1)}{2}$. So deduction confirms and proves our previous result.

Looking back, or discussing the problem with others, we may realise that there are other ways of looking at the structure. Our approach was to count the chords at each point. This counted each chord twice, that is why we divided by 2. Now look at it differently:



It is clear that at the first point there are 6 chords. At the second point there are 5 *new* chords, because the chord to the first point has already been counted at the first point. Similarly, at the third point there are 4 new chords, etc.

So we have $C(7) = 6 + 5 + 4 + 3 + 2 + 1$

Again we emphasise that we do not want the numerical *answer*, but want to understand the *structure* of the situation. The result is not simply the sum of arbitrary numbers – the structure is clear: it is a decreasing sequence because we are not double-counting chords; 6 is not just any number, but is one less than 7 because we are drawing chords to every other point except itself. *This means that we can see the structure of the situation in this one example. We say we see the general in the particular.*

We can now without further ado say that $C(18) = 17 + 16 + 15 + 14 + \dots + 3 + 2 + 1$. Please notice that you can find $C(18)$ by adding this sequence manually, but it is nearly impossible to solve (b), i.e. solve *the equation* $C(n) = 465$ *without a formula!!*

When you *do* Mathematics, problems always lead to new problems!

PROBLEM ##: SHORT CUT

Develop a short method to calculate $17 + 16 + 15 + 14 + \dots + 3 + 2 + 1$.

Use your method to calculate $1 + 2 + 3 + 4 + 5 + \dots + 99 + 100$

Generalise!

Can you use your method to calculate $1 + 3 + 5 + 7 + 9 + \dots + 97 + 99$?

APPENDIX: VERSKILLE

Voltooi die tabelle⁹ in die volgende *spesiale gevalle* om jou *intuïtief* te oortuig dat:

As die n^{de} verskil konstant is, is die model 'n polinoom van die n^{de} graad ... (1)

n	1	2	3	4	5	6		
$y = 2n + 3$	5	7	9	11	13	15		

Eerste verskille: 2 2 2 2 2
Tweede verskille:

n	1	2	3	4	5	6		
$y = n^2$	1	4	9	16	25	36		

Eerste verskille: 3 5 7 9 11
Tweede verskille:

n	1	2	3	4	5	6		
$y = n^3$	1	8	27	64	125	216		

Eerste verskille: 7 19 37 61 91
Tweede verskille:
Derde verskille:

x	1	2	3	4	5	6		
$y = 2^n$	2	4	8	16	32	64		

Eerste verskille: 2 4 8 16 32
Tweede verskille:
Derde verskille:

Ons kan byvoorbeeld vir enige kwadratiese funksie $f(n) = an^2 + bn + c$ sê:

$$\begin{aligned} f(k+1) - f(k) &= a(k+1)^2 + b(k+1) + c - (ak^2 + bk + c) \\ &= 2ak + (a+b) \end{aligned}$$

dit wil sê die verskille tussen opeenvolgende terme is 'n eerste-graadse funksie.

Netso is die verskille tussen opeenvolgende terme van 'n 3^{de} graadse funksie 'n 2^{de} graadse funksie, die tweede verskille 'n 1^{ste} graadse funksie en die derde verskille dus konstant.

Wat bostaande toon is dat :

As die model 'n n^{de} graadse polinoom is, is die n^{de} verskil konstant ... (2)

Dit is nie wat ons moes bewys nie - (1) is in werklikheid die *omgekeerde* van hierdie stelling! Maar omdat bostaande redenasie omkeerbaar is, het ons dus albei stellings bewys!

Let op:

- Hierdie is 'n eienskap *slegs* van polinoomfunksies. 'n Funksie soos 2^n , d.w.s 1, 2, 4, 8, ... kan nooit 'n konstante verskil lewer nie. Dus weet ons dat is die verskil *nie* konstant is nie, die model *nie* 'n polinoom kan wees nie!
- As die eerste verskil konstant is, is die formule 1^{ste} -graads, d.i. $y = mx + c$. Algebraïes is hierdie konstante verskil die *gradient* m van die funksie en dit is die rede waarom die grafiek 'n reguit lyn is!
Natuurlik is 'n konstante verskil vir diskrete waardes die *definisie* van 'n Rekenkundige ry!
Maak ons die konneksies tussen rye en funksies, tussen $T_n = a + (n - 1)d$ en $f(x) = mx + c$?
- Die verskille verteenwoordig natuurlik *die afgeleide*, ons weet dat $f'(x^n) = nx^{n-1}$, dus is dit nie verrassend dat die verskille-polinoomfunksies van afnemende graad is nie!!
- Hierdie stelling is belangrik in die kurrikulum bv. in modellering en om die vergelyking van die reguit lyn deur twee punte en die vergelyking van die parabool deur drie punte te bepaal!

⁹ Kyk hierdie Excel werkvel