2.6 EQUATIONS

In order to reflect on the validity of the procedure for solving equations as raised in A in 2.5, we have to understand the underlying ideas involved.

For any equation we have three elements:

1. A *universal set,* or replacement set (domain) of possible values. Such a set can be the real numbers, or the natural numbers, or any set of numbers, e.g. $x \in \{1; 2; 3\}$.
2. Some *condition*, or *open sentence*, e.g. $\{x: 2x + 3 = 13\}$
3. The *solution set* or *truth set*, i.e. the set of values satisfying the condition in 2. Here the truth set is {5}.

Here are some examples:

Equation	Universal set	Truth set
$2x + 3 = 5$	N	{1}
$2x + 3 = 6$	R	$\{\frac{3}{2}\}$
$2x + 3 = 6$	N	{} (empty set)
$\frac{1}{x+2} = \frac{1}{2x}$	$x \neq -2$; $x \neq 0$; $x \in$ R	{2}
$2x + 3x = 5x$	$x \in$ R	$x \in$ R

We have the same distinction we raised in the VAT-AVIS problems. Equations can be

- *always* true (the truth set is equal to the Universal set),
- *sometimes* true (the truth set is a subset of the Universal set), or
- *never* true (the truth set is empty).

Let's revisit the equation problem in A:

3. Sove for x: $\frac{2(x-3)}{5} + \frac{x+3}{3} = 2$

Mary's solution: $\frac{2(x-3)}{5} + \frac{x+3}{3} = 2$ .... P

$$\therefore\ 6(x-3) + 5(x+3) = 30$$
$$\therefore\ 6x - 18 + 5x + 15 = 30$$
$$\therefore\ 11x - 3 = 30$$
$$\therefore\ 11x = 33$$
$$\therefore x = 3 \quad \text{..................... Q}$$

{3} is the solution of Q, so {3} is the solution of P

We must understand that our ∴ (therefore) notation, is a *one-way* implication and the reasoning here is therefore exactly the Converse trap! As shown before, a false statement can through valid reasoning lead to a true statement. The fact that the conclusion is true, i.e. that {3} is the truth set of the equation $x = 3$ (Q is true), says *nothing* about the truth of the original statement P[1].

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[1] The fact that we *know* that {3} *is* the solution of P is irrelevant! As we stressed before: In an invalid argument the conclusion can be either true or false, but the *argument* (the logic of the *reasoning*) cannot help us decide! For example, given only (1) the implication *If it rains then it is wet*, and (2) the fact that *it is wet*, we cannot by the rules of logic conclude that it is raining (the converse trap). It may really be raining, but that cannot influence the logical structure!

Here is another simple example:

$x = 2$ ……………(1)

$\Rightarrow x^2 = 4$……………….(2)

$\Rightarrow x = 2$ or $x = -2$………(3)

Each step is logically valid: Equation 1 $\Rightarrow$ Equation 2 $\Rightarrow$ Equation 3. However, the fact that {-2} makes Equation 3 true, says nothing about the truth set of Equation 1. In fact, when x = -2, Equation 1 is a false statement: -2 = 2! This is, again, the *Converse trap* (page 63): -2 = 2 (P) $\Rightarrow$ $(-2)^2 = 2^2 \Rightarrow 4 = 4$ (Q). So P $\Rightarrow$ Q is valid, and Q is true, but it does not mean that P is true!

PROBLEM 97:

Analyse these solutions. All the values in (6) do not satisfy (1). *Why not?*

$x + \sqrt{x} = 2$ .... (1)	$x + 3\sqrt{x} + 2 = 0$ .... (1)
$\Rightarrow \sqrt{x} = 2 - x$ .... (2)	$\Rightarrow 3\sqrt{x} = -(2 + x)$ .... (2)
$\Rightarrow x = 4 - 4x + x^2$ .... (3)	$\Rightarrow 9x = 4 + 4x + x^2$ .... (3)
$\Rightarrow x^2 - 5x + 4 = 0$ .... (4)	$\Rightarrow x^2 - 5x + 4 = 0$ .... (4)
$\Rightarrow (x-1)(x-4) = 0$ ...... (5)	$\Rightarrow (x-1)(x-4) = 0$ .... (5)
$\Rightarrow x = 1$ or $x = 4$ ....(6)	$\Rightarrow x = 1$ or $x = 4$ .... (6)

Equivalent equations

Note that $x = 2 \Rightarrow 2x = 4$ and $2x = 4 \Rightarrow x = 2$.

This means that we can write $x = 2 \Leftrightarrow 2x = 4$, i.e. the two equations are *equivalent*.

Equivalent *equations* is a special case of equivalent *statements*. Everything we said about equivalent statements in 2.5 applies to equivalent equations, and the logic described there applies equally to equations: the truth of the one implies the truth of the other, and *we can replace the one with the other whenever we want.*

Because the truth value of an equation depends on the value of the variable, we can define equivalent equations in a different way: *two equations are equivalent if they have the same truth or solution set*. Do you agree that $x = 2$ and $2x = 4$ above have the same truth sets? Click here for a brief formal proof of the statement.

Let's look at the process of solving an equation using *equivalent equations* (make sure you agree that each step produces equivalent equations):

$$24(x+7) - 160 = 80 \text{ .... } (1)$$
$$\Leftrightarrow 3(x+7) - 20 = 10 \text{ .... } (2)$$
$$\Leftrightarrow 3(x+7) = 30 \text{ .... } (3)$$
$$\Leftrightarrow x + 7 = 10 \text{ .... } (4)$$
$$\Leftrightarrow x = 3 \text{ .... } (5)$$

Why can we now say that {3}, the truth set of Equation 5, is indeed also the truth set of Equation 1? What is happening here is that:

$E_1 \Rightarrow E_2 \Rightarrow E_3 \Rightarrow E_4 \Rightarrow E_5$ (from top to bottom)

$E_1 \Leftarrow E_2 \Leftarrow E_3 \Leftarrow E_4 \Leftarrow E_5$ (from bottom to top)

The *validity* of the logic can be seen in two ways:

- $E_5 \Rightarrow E_1$ is valid, {3} is the truth set of E_5. So {3} is the truth set of E_1, in other words classical Modus Ponens. (The usual procedure, from top to bottom, is really *analysis* – the *synthesis* (the *proof*) is from bottom to top!!)
- $E_5 \Leftrightarrow E_1$, i.e. they are equivalent, and therefore E_5 and E_1 have the same truth set.

The procedure for solving an equation is therefore simple:

Replace the given equation by a sequence of simpler equivalent equation, until you arrive at an equation whose solution set is obvious, and then you are finished.

The point is that if we work only with equivalent equations, they all have the same solution set, so we can deduce the solution set from an easy equation, and that is then immediately also the solution set of the original equation.

In order to can construct equivalent equations, we need some rules of transformation that preserves equivalence. As could be expected, the rules are exactly those operations that are *reversible!*

The following are therefore the *axioms* for constructing equivalent equations:

1. $a = b \Leftrightarrow a + c = b + c$
2. $a = b \Leftrightarrow a - c = b - c$
3. $a = b \Leftrightarrow ac = bc, \; c \neq 0$
4. $a = b \Leftrightarrow \frac{a}{c} = \frac{b}{c}, \; c \neq 0$
5. $a = b \Leftrightarrow \sqrt[n]{a} = \sqrt[n]{b}, \; a, b \geq 0$

As long as we use *only these axioms*, our equations are equivalent, and we are *guaranteed* that that the solution sets of all the equations are the same, and therefore that the solution procedure will lead to a correct solution set.

PROBLEM 98:
Of course, if we make an error in manipulation, or logically do not maintain equivalence, the solution process will not lead to a correct solution set. Can you find the error in this solution process?

$$\sqrt{x+2} = x \text{ ....(1)}$$
$$\Leftrightarrow x + 2 = x^2 \text{..................(2)}$$
$$\Leftrightarrow x^2 - x - 2 = 0 \text{.............(3)}$$
$$\Leftrightarrow (x+1)(x-2) = 0 \text{...........(4)}$$
$$\Leftrightarrow x = -1 \text{ or } x = 2 \text{.............(5)}$$

Click here for some notes on the meaning of $\sqrt{x}$.

Extraneous solutions 1

Note that $a = b \Rightarrow a^n = b^n$ is valid, but that $a = b \Leftarrow a^n = b^n$ is not a valid implication, i.e. “raising to a power” and therefore “squaring” as a special case, *do not preserve equivalence.*

PROBLEM 99:
Explain what is wrong with the statement $a^2 = b^2 \Rightarrow a = b$, and generally, $a^n = b^n \Rightarrow a = b$.

This is why the procedure in Problem 98 above leads to a wrong solution: Equation 2 does not imply Equation 1, i.e. Equation 2 does not *lead back* to Equation 1. The set {-1, 2} is therefore the solution of all the equations in Problem 98, *except Equation 1!*

When we use squaring, we introduce *extraneous (extra) solutions*, which are not solutions of the original equation, as the following simple example shows:

$x = 2$ *solution is {2}*

$\Rightarrow x^2 = 4$ *solution is {-2; 2}*

$x = 2$ and $x^2 = 4$ are therefore not equivalent equations, because:

- They do not have the same solution set
- $x = 2 \Rightarrow x^2 = 4$ is valid, but not $x = 2 \Leftarrow x^2 = 4$ (what is valid is $x = \pm 2 \Leftarrow x^2 = 4$)

Extraneous solutions 2

In other cases we should be careful to maintain the universal set of the original equation for all the equations, otherwise equivalence is not maintained. The universal set is determined by the *domain* and *range* of the functions contained in the equation, for example:

$2\log x = 2$ *solution is {10}*

$\Rightarrow \log x^2 = 2$ *solution is {-10; 10}*

The universal set, i.e. possible replacement values for the first equation is $x > 0$ because the domain of the log function is $x > 0$. In the second equation the domain is $x^2 > 0$, so x can be negative! Or viewed differently, the "log rule" $n\log A = \log A^n$ is only true if $A > 0$, so we may not replace $n\log A$ with $\log A^n$ if $A \leq 0$!

If we do not maintain equivalence (i.e. if we use only one-way implications) as in the two examples above, the best we can say is that *the solution set of the original equation is a subset of the final equation*.

The correct way to solve equations to ensure that it leads to the correct solution set, is to make sure that equivalence is maintained at all cost by ensuring that the same universal set applies to all the equations in the procedure, as the following example shows (see problem 98):

PROBLEM 100:

Describe why *all* these equations are *equivalent*. How do we *know* that the solution of Equation 6 is also the solution of Equation 1?

$$\sqrt{x+2} = x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (1)$$

$$\Leftrightarrow x + 2 = x^2,\ x \geq -2 \text{ and } x \geq 0 \quad \ldots\ldots (2)$$

$$\Leftrightarrow x^2 - x - 2 = 0,\ x \geq 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\Leftrightarrow (x+1)(x-2) = 0,\ x \geq 0 \quad \ldots\ldots\ldots\ldots\ldots (4)$$

$$\Leftrightarrow x = -1 \text{ or } x = 2 \text{ and } x \geq 0 \quad \ldots\ldots\ldots\ldots (5)$$

$$\Leftrightarrow x = 2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Remark on teaching

In general, textbooks and teachers are not really using the principle of maintaining equivalent equations (implications going both ways), but are using "therefore", i.e. one-way implications in solving equations. We repeat that the method is logically invalid. What textbooks and teachers do to "save" the situation, is to insist that children check the solution set by substitution in the original equation and in this way identifying false solutions.

We think this is unfortunate, for at least the following reasons:

- Children forget to check and therefore get wrong answers.
- Children check mostly because they think they may have made a manipulation error somewhere, i.e. they do not check because they understand the logic of equivalent equations.
- This leads to a distorted view of the nature of mathematics, namely that a correct procedure can lead to wrong results. This is totally contrary to the very nature of an algorithm – if you correctly implement an appropriate algorithm (e.g. on a computer), the correct answer is guaranteed!
- Because children do not understand the principle of and meaning of equivalence, they make further mistakes, e.g. by checking the solution set in the second equation, not in the original equation.

Teachers are often hesitant to teach this correct procedure, fearing that children will not understand. But apart from the *meaning* of equivalence, there is nothing extra here. The knowledge of the universal set, i.e. the possible replacement values for *x*, used to maintain equivalence is exactly the knowledge learners need when they check the solution in the original equation, namely knowledge of the domain of the functions contained in the equation. Why only consider the universal set *at the end* and not in *the beginning*?

PROBLEM 101:

Look at this example and contrast the two procedures to solve for *x* if $\sqrt{x-2}=1-x$:

The "usual procedure":

$$\sqrt{x-2}=1-x$$
$$\Rightarrow x-2=1-2x+x^2$$
$$\Rightarrow x^2-3x+3=0$$
$$\Rightarrow x=\frac{3\pm\sqrt{-3}}{2}$$
$$\Rightarrow x\in\varnothing$$

Considering the Universal set:

In $\sqrt{x-2}$, $x-2\geq 0\Rightarrow x\geq 2$

and

$\sqrt{x-2}\geq 0$, so $1-x\geq 0\Rightarrow x\leq 1$

So $x\leq 1$ and $x\geq 2\ \Rightarrow x\in\varnothing$

PROBLEM 102:

Discuss the *logic* of the following procedures to solve the equations.
The solutions are wrong! How do we *know* they are wrong?
How can a *valid* procedure lead to *wrong* results? Exactly where is the logical error here?
Now correct the procedure for each solution by strictly using *equivalent equations*.

$$\frac{1}{x-2}=\frac{4}{x^2-4}+1$$
$$\Rightarrow x+2=4+x^2-4$$
$$\Rightarrow x^2-x-2=0$$
$$\Rightarrow (x+1)(x-2)=0$$
$$\Rightarrow x=-1 \text{ or } x=2$$

$$2\log x=\log(x+2)$$
$$\Rightarrow \log x^2=\log(x+2)$$
$$\Rightarrow x^2-x-2=0$$
$$\Rightarrow (x+1)(x-2)=0$$
$$\Rightarrow x=-1 \text{ or } x=2$$

$$\sin\theta+\cos\theta=1 \text{ and } \theta\in[0°;360°]$$
$$\Rightarrow (\sin\theta+\cos\theta)^2=1$$
$$\Rightarrow \sin^2\theta+2\sin\theta\cos\theta+\cos^2\theta=1$$
$$\Rightarrow 2\sin\theta\cos\theta=0$$
$$\Rightarrow \sin 2\theta=0$$
$$\Rightarrow 2\theta=0^\circ+180^\circ k$$
$$\Rightarrow \theta=0^\circ+90^\circ k$$
$$\Rightarrow \theta=0^\circ;90^\circ;180^\circ;360^\circ$$

Click here to see graphs of the relationships above.

PROBLEM 103: WHAT WENT WRONG?

These are some children's solutions of some equations. Your comments?

$$\frac{x+5}{x-7}-5=\frac{4x-40}{13-x}$$
$$\Rightarrow \frac{-4x+40}{x-7}=\frac{4x-40}{13-x}$$
$$\Rightarrow \frac{4x-40}{7-x}=\frac{4x-40}{13-x}$$
$$\Rightarrow 7-x=13-x$$
$$\Rightarrow 7=13$$

$$\frac{x}{x-2}-7=\frac{2}{x-2}$$
$$\therefore \frac{x}{x-2}-\frac{2}{x-2}=7$$
$$\therefore \frac{x-2}{x-2}=7$$
$$\therefore 1=7$$

I must have made a mistake!

$$\frac{x}{x-2}-7=\frac{2}{x-2}$$
$$\therefore \frac{x}{x-2}-\frac{2}{x-2}=7$$
$$\therefore \frac{x-2}{x-2}=7$$
$$\therefore 1=7$$

which is false, and because the reasoning is reversible, the original equation is always false

$$\frac{x}{x-2}-7=\frac{2}{x-2}$$
$$\Leftrightarrow (x-2)\left[\frac{x}{x-2}-7\right]=(x-2)\left[\frac{2}{x-2}\right],\ x-2\neq 0$$
$$\Leftrightarrow x-7(x-2)=2,\ x\neq 2$$
$$\Leftrightarrow 12=6x,\ x\neq 2$$
$$\Leftrightarrow x=2,\ x\neq 2$$

This leads to a contradiction, so the original equation is never true

2.7 IDENTITIES

We have previously defined an identity as an algebraic statement like $2x + 3x = 5x$ which is true for *all* values of the variables, while an equation is a statement that is only true for *some* values of the variable (see page 13).

- An *identity* is therefore simply an *equation* which is true for all values of the replacement set, i.e. the truth set is equal to the replacement set. Mathematicians write $\forall\, x \in U, 2x + 3x = 5x$.
- An *equation*, on the other hand, is true for a subset of the replacement set. Mathematicians write $\exists\, x \in U, 2x + 3 = 5$.

The following examples should make the distinction clear:

Condition	**Universal set**	**Truth set**
$2x + 3 = 5$ (an *equation*)	N	{1}
$2x + 3x = 5x$ (an *identity*)	R	R
$\frac{1}{x+2} = \frac{1}{2x}$ (an *equation*)	$x \neq -2$; $x \neq 0$; $x \in$ R	{2}
$\frac{1}{x} + \frac{1}{x} = \frac{2}{x}$ (an *identity*)	$x \neq 0$; $x \in$ R	$x \neq 0$; $x \in$ R
$\sin\theta + \cos\theta = 1$ (an *equation*)	$-360^0 \leq \theta \leq 360^0$	$\theta = 0^0, 90^0, -270^0, -360^0$
$\sin^2\theta + \cos^2\theta = 1$ (an *identity*)	$-360^0 \leq \theta \leq 360^0$	$--360^0 \leq \theta \leq 360^0$

Sometimes we refer to an equation as a *conditional equation* (meaning it is only true under certain conditions, meaning it is only true for *some* values of the variable), and refer to an identity as a *universal equation* (meaning it is universally true, or always true, meaning it is true for *all permissible values* of the variable).

Permissible values, "not defined", replacement set

While an identity like $2x + 3x = 5x$ is true for *all* values of the variable, a statement like $\frac{1}{x} + \frac{1}{x} = \frac{2}{x}$ is true for all values of the variable except $x = 0$. We can handle this in many different ways. We could say that $x = 0$ is a counter-example which shows that the statement is not always true. We can say it is *sometimes true*, i.e. it is true for all the values of x except $x = 0$. Mathematicians prefer to say it is true for all *permissible* values of the variable, i.e. it is true for *all* values of the universal set. We also say that the statement is *not defined* for $x = 0$.

We too easily write down "simplifications" like $\frac{x^2 - 4}{x - 2} = x + 2$ as if it is universally true. It is very important that we take into consideration the exceptions when we use identities to make replacements like in the following example:

PROBLEM 104:

Analyse the logic and where the mistake is in the following solution:

Solve for x: $\frac{x^2 - 4}{x - 2} = 4$

Simplify the LHS: $\frac{x^2 - 4}{x - 2} = \frac{(x + 2)(x - 2)}{x - 2} = x + 2$

$\Rightarrow \quad x + 2 = 4$

$\Rightarrow \quad x = 2$

But x is *not* a solution of the original equation – we cannot divide by 0!

PROBLEM 105:

Criticise the statement that $\frac{x-2}{x-2} = 1$

We can now conclude that *we can prove any identity by handling it as an equation and showing that the truth set is equal to the universal set*. Make sure that you understand the logic in the following example:

Prove that: $(x - 2)^2 + 8x = (x + 2)^2$

Proof:
$$(x - 2)^2 + 8x = (x + 2)^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{P}$$
$$\Leftrightarrow x^2 - 4x + 4 + 8x = x^2 + 4x + 4$$
$$\Leftrightarrow x^2 + 4x + 4 = x^2 + 4x + 4 \quad (\text{or } 0 = 0) \quad \ldots\ldots\text{Q}$$

Q is true for all values of x, so P is true for all values of x, i.e. P is an identity.

To repeat: we can prove an identity by handling it as an equation and showing that the solution set is equal to the replacement set (i.e. it is true for all permissible values of the variable).

We can therefore prove the identities in our original problems in A similarly:

E

EQUIVALENT

EQUATIONS

1. Prove that $\sin^2\theta + \cos^2\theta = 1$

Handle it as an equation and solve for θ:

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{----------- P}$$
$$\Leftrightarrow \quad (\frac{y}{r})^2 + (\frac{x}{r})^2 = 1 \text{ for any } \theta$$
$$\Leftrightarrow \quad y^2 + x^2 = r^2 \text{ and } r \neq 0 \quad \text{-------- Q}$$

The solution set for Q and therefore for P is R. So they are identities.

2. Prove that $$\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta}$$

Handle it as an equation and solve for θ:

$$\frac{\sin\theta}{1-\cos\theta} = \frac{1+\cos\theta}{\sin\theta} \quad \text{------------ P}$$
$$\Leftrightarrow \sin^2\theta = (1-\cos\theta)(1+\cos\theta) \quad \sin\theta \neq 0,\ \cos\theta \neq 1$$
$$\Leftrightarrow \sin^2\theta = 1 - \cos^2\theta$$
$$\Leftrightarrow \sin^2\theta + \cos^2\theta = 1 \quad \text{--- ------- Q}$$

The solution set for Q and therefore for P is R. So they are identities.

PROBLEM 106: EQUATION OR IDENTITY?

1. Solve for x: $\dfrac{x(x-1)^2-5x+8}{x+2}=(x-2)^2, \quad x \neq -2$
2. Prove that $\dfrac{x(x-1)^2-5x+8}{x+2}=(x-2)^2, \quad x \neq -2$

What is the difference between the two situations?

PROBLEM 107: 1 = 2?

Study the following argument. You have two choices:

- If you find the reasoning is valid, you must accept that the conclusion 1 = 2 is true.
- If you find the reasoning is not valid, you can make no assertion about the truth of the conclusion 1 = 2. If so, explain!

Specific:	***General:***	
Let $x = 2$	Let $a = b$	(1)
$\Leftrightarrow$ $-2x = -4$	$\Leftrightarrow -ab = -b^2$	(2)
$\Leftrightarrow x^2 - 2x = x^2 - 4$	$\Leftrightarrow a^2 - ab = a^2 - b^2$	(3)
$\Leftrightarrow x(x-2) = (x+2)(x-2)$	$\Leftrightarrow a(a-b) = (a+b)(a-b)$	(4)
$\Leftrightarrow x = x + 2$	$\Leftrightarrow a = a + b$	(5)
$\Leftrightarrow 2 = 2 + 2$	$\Leftrightarrow b = b + b$	(6)
$\Leftrightarrow 2 = 4$	$\Leftrightarrow b = 2b$	(7)
$\Leftrightarrow 1 = 2$	$\Leftrightarrow 1 = 2$	(8)

PROBLEM 108: PROOF

Prove that $\dfrac{\tan x-\sin x}{\sin^3 x}=\dfrac{\sec x}{1+\cos x}$

Explain in detail why this is a valid proof.
Comment on the objection “the proof is invalid, because you cannot assume what you must prove.”

$$\frac{\tan x-\sin x}{\sin^3 x}=\frac{\sec x}{1+\cos x} \quad \ldots \mathrm{U}=\{x: \mathrm{x}\in \mathrm{R}, \cos x\neq 0, \sin x \neq 0,\ \cos x \neq -1\} \quad \ldots \mathrm{P}$$

$$\times \sin^3 x(1+\cos x): \Leftrightarrow (\tan x-\sin x)(1+\cos x)=\sin^3 x \sec x$$

$$\Leftrightarrow \tan x+\sin x-\sin x-\sin x\cos x=\sin^3 x\sec x$$

$$\times \cos x: \quad \Leftrightarrow \sin x-\sin x\cos^2 x=\sin^3 x$$

$$\Leftrightarrow \sin x(1-\cos^2 x)=\sin^3 x$$

$$\Leftrightarrow \sin x\sin^2 x=\sin^3 x$$

$$\Leftrightarrow \sin^3 x=\sin^3 x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \mathrm{Q}$$

$$\mathrm{Q} \text{ is true for all } x\in \mathrm{U}, \text{so P is true for all } x\in \mathrm{U}$$

2.8 UNFINISHED BUSINESS

Draft – work in progress!

2.8.1 *Equivalent equations*

Theorem:
Two equations are logically equivalent if and only if they have the same solution set.

Proof:
Let $E_1$and E_2 be two equations, and let their solution sets be $\{E_1\}$ and $\{E_1\}$ respectively.
What we must prove is that $(E_1 \Leftrightarrow E_2) \Leftrightarrow \{E_1\} = \{E_1\}$, i.e. that the two definitions of equivalent equations are equivalent.

First prove $[E_1 \Leftrightarrow E_2] \Rightarrow [\{E_1\} = \{E_2\}]$

$E_1 \Rightarrow E_2$ means that $x \in \{E_1\} \Rightarrow x \in \{E_2\}$. So $\{E_1\} \subseteq \{E_2\}$
$E_1 \Leftarrow E_2$ means that $x \in \{E_2\} \Rightarrow x \in \{E_1\}$. So $\{E_1\} \supseteq \{E_2\}$
$\{E_1\} \subseteq \{E_2\}$ and $\{E_1\} \supseteq \{E_2\}$ means $\{E_1\} = \{E_2\}$
So we have proved that if E_1 and E_2 are equivalent, then they have equal solution sets.

Now prove $[E_1 \Leftrightarrow E_2] \Leftarrow [\{E_1\} = \{E_2\}]$

$\{E_1\} = \{E_2\}$ means that $\{E_1\} \subseteq \{E_2\}$ and $\{E_1\} \supseteq \{E_2\}$:
$\{E_1\} \subseteq \{E_2\}$. So $x \in \{E_1\} \Rightarrow x \in \{E_2\}$, which means that $E_1 \Rightarrow E_2$
$\{E_1\} \supseteq \{E_2\}$. So $x \in \{E_2\} \Rightarrow x \in \{E_1\}$, which means that $E_1 \Leftarrow E_2$
$E_1 \Rightarrow E_2$ and $E_1 \Leftarrow E_2$ means $E_1 \Leftrightarrow E_2$
So we have proved that if E_1 and E_2 have equal solutions sets, then they are equivalent.

We have proven it both ways, so $[E_1 \Leftrightarrow E_2] \Leftrightarrow [\{E_1\} = \{E_2\}]$

2.8.2 *Incomplete induction ("induction"), Complete induction ("Mathematical induction") and Pseudo Induction*

Didactical dilemma?
In teaching, we often use activities like this to develop *meaning* for equivalent expressions:

Complete the following table.

x	**1**	**2**	**5**	**12**	**19**	**37**	**45**
$5x + x$							
$6x$							
$4x + 2x$							

It will be mathematically foolhardy to allow pupils to generalise after *incomplete induction* of only 7 cases that the expressions *always* have the same values!
We should not rely only on induction, but should try to also develop some *structural*

explanation (deduction), for example:

We say that expressions in the tables are *equivalent* if they produce the same output numbers for the same input numbers. For example, $4x+2x$ and $6x$ are equivalent expressions, because they produce the same output values for the same input values. We can explain the equivalence in the following way: $4x+2x$ means $4\times x+2\times x$ = $\left(x+x+x+x\right)+\left(x+x\right)$ = $6\times x=6x$

Actually, we are mathematically absolutely safe in such table examples. In fact, despite the pitfalls of incomplete induction, we can be absolutely sure that $4x+2x=6x$ for *all* values of x after testing that it is true *for only two values of x!!*

Make sure you understand what this assertion says! Do you agree with it?

Let's now investigate …

We know from the **Fundamental Theorem of Algebra** that an *n*th degree polynomial equation has n solutions.

The **Principle of Pseudo Induction** says that if an *n*th degree sentence is true for *more* than *n* values, it is *always* true!

As a simple example: While you (and Ame) may find two values that make $x + x = x^2$ true, those are the *only* values you can find, because it is a quadratic *equation*. On the other hand, showing that $x + x = 2x$ is true *for two* values, *proves* that it is *generally* true (if it was an equation, it would be true *only* for the one root of the linear equation. So if the equation has more than one root, it must be an identity, therefore true for any value if x!)

Proof of the Principle of Pseudo Induction.

Definition: An equation with all coefficients zero is called the *Zero equation*, e.g.
$0x^2 + 0x + 0 = 0$

Corollary: It is clear that a Zero equation is an *identity*, i.e. true for all values of the unknown.

Theorem of Pseudo Induction, to be proved:
If an *n*th degree equation has *more* than *n* roots, it is the Zero equation and therefore is an identity which is *always* true!

We do not prove it in general, but here is an illustration for a quadratic equation:

Take the equation $ax^2 + bx + c = 0$ ……………………………………(1)
and let x_1, x_2 and x_3 be THREE DIFFERENT roots of the equation.

Then we can write ax^2 + bx + c = 0 in the equivalent form

$$a(x - x_1)(x - x_2) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Now if x_3 is a root of the equation, if we substitute it into equation (2), it makes the equation true, so:

$$a(x_3 - x_1)(x_3 - x_2) = 0$$

From this, because $x_1 \neq x_2 \neq x_3$, it follows that a = 0.

Therefore, equation (1) becomes bx + c = 0
i.e. a *linear equation*, which, by the Fundamental Theorem of Algebra has only one root.
But suppose x_1, x_2 are TWO DIFFERENT roots of the equation, then they both satisfy the equation, i.e.

$$bx_1 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

$$bx_2 + c = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

(4) – (3): $\quad b(x_2 - x_1) = 0$

and because $x_1 \neq x_2$, it follows that b = 0.
Now if a = 0 and b = 0, then c = 0 (see equation 1).

That proves that equation (1) is the Zero Equation, from which it follows that it is an identity and it is true for all values of x.

Here is an alternative proof:

If polynomials P(x) and Q(x) name the same number for k different replacements, where k is greater than the degree of either P(x) or Q(x) then P(x) and Q(x) are the same polynomial.

Proof:
P(x) and Q(x) can be represented respectively by

$$a_n x^n + a_{n-1}x^{n-1} + \ldots. + a_1 x + a_0$$

and

$$b_n x^n + b_{n-1}x^{n-1} + \ldots. + b_n x + b_0$$

If they should not have the same degree, then the first few coefficients of one of them would be zero. Thus n is the degree of the polynomial having the larger degree.

For any replacement for which these polynomials name the same number,

$$a_n x^n + a_{n-1}x^{n-1} + \ldots. + a_1 x + a_0 = b_n x^n + b_{n-1}x^{n-1} + \ldots. + b_n x + b_0$$

or

$$(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \ldots. + (a_1 - b_1)x + (a_0 - b_0) = 0$$

Since by hypothesis this happens for k different replacements, where $k>n$, the polynomial equation

$$(a_n - b_n)x^n + (a_{n-1} - b_{n-1})x^{n-1} + \ldots. + (a_1 - b_1)x + (a_0 - b_0) = 0$$

has more than n roots. This violates the Fundamental Theorem of Algebra, unless the polynomial is the zero polynomial. Therefore, every coefficient is zero. It follows that

$$a_n = b_n,\ a_{n-1} = b_{n-1}, \ldots.,\ a_1 = b_1,\ a_0 = b_0$$

which means that P(x) and Q(x) must be the same polynomial.

Always true, sometimes true, never true
A first degree *equation* can always be transformed to ax + b = px +q or ax + b = c
A first degree *identity* can always be transformed to 0x + 0 = 0, or to 0 = 0 or a = a
A first degree *impossibility* can always be transformed to ax + b = ax + c or b = c, where b $\neq$ c.

A. Equation true for some values	**B ….Equation true for all (permissible) values**	**C. Equation true for no values**
$3(x + 3) + 1 = x + 2$ … (1)	$3(x + 3) + 1 = x + 2(x + 5)$ (1)	$3(x + 3) + 1 = x + 2(x + 4)$ (1)
$\Leftrightarrow 3x + 9 + 1 = x + 2$	$\Leftrightarrow 3x + 9 + 1 = x + 2x + 10$	$\Leftrightarrow 3x + 9 + 1 = x + 2x + 8$
$\Leftrightarrow 3x + 10 = x + 2$	$\Leftrightarrow 3x + 10 = 3x + 10$	$\Leftrightarrow 3x + 10 = 3x + 8$

From which the solution of A is still not obvious, but it is obvious for B and C!

You will continue with A, why not with B and C?

$\Leftrightarrow 2x + 10 = 2$	$\Leftrightarrow 0x + 10 = 10$	$\Leftrightarrow 0x + 10 = 8$
$\Leftrightarrow 2x = - 8$	$\Leftrightarrow 0x + 0 = 0$ or $10 = 10$ or $0 = 0$	$\Leftrightarrow 0.x + 2 = 0$ or $10 = 8$
$\Leftrightarrow x = - 4$	$\Leftrightarrow$ (1) is true for all values of x	$\Leftrightarrow$ (1) is true for no value of x

In the case of a quadratic:

$2x(x + 1) + 2x - 3 = x(x + 2)$	$2x(x + 1) + 2x - 3 = 2x(x + 2) - 3$	$2x(x + 1) + 2x - 3 = 2x(x + 2) + 1$
$\Leftrightarrow 2x^2 + 4x - 3 = x^2 + 2x$	$\Leftrightarrow 2x^2 + 4x - 3 = 2x^2 + 4x - 3$	$\Leftrightarrow 2x^2 + 4x - 3 = 2x^2 + 4x + 1$
$\Leftrightarrow x^2 + 2x - 3 = 0$	$\Leftrightarrow 0x^2 + 0x + 0 = 0, 0 = 0$	$\Leftrightarrow 0x^2 + 0x - 4 = 0, -4 = 0, -3 = 1$

PROBLEM 109:
Prove that $\frac{(x-a)(x-b)}{(c-a)(c-b)} + \frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} = 1$

Instead of working with complicated fractions, look at it as a quadratic *equation*. Clearly one can see through inspection that it is true for x = a, x = b and x = c. So, by the Principle of Pseudo Induction, because the quadratic equation has THREE (more than *two!*) roots, it follows that it is true for *all* permissible values of *x!* Finish!!

Incomplete – will, maybe, be updated later ☺
2.8.3 *Proof by Modus Tollens*
2.8.4 *Proof by the contra-positive*
2.8.5 *Proof by Mathematical induction*

JUMP TO 2.9 SQUARE ROOTS