Focusing on the structural aspects of numerical expressions

Rolene E. Liebenberg, Liora Linchevski, Marlene C. Sasman & Alwyn Olivier
Mathematics Learning and Teaching Initiative, South Africa

This paper reports on the findings of an on going research study which investigates the problems that a class of Grade 6 students experience in the process of learning about the structural aspects of numerical expressions. The aim of this study is to see to what extent a focus on the structural properties of numerical expressions can assist students' handling of algebraic structures of numerical expressions. Data collected thus far shows that students are not always able to transfer their structural knowledge of numerical expressions to new situations. This was especially evident in situations that involved the solving of an equation or judging the equivalence of expressions without doing a calculation. The findings support the need to focus on tasks other than computational tasks in developing students' structural understanding of numerical expressions.

Introduction

This research developed in the context of a number of studies (Booth 1988, Matz 1980, Lins 1990) that investigated the difficulties that students have in algebra and more specifically the difficulties in the manipulation of algebraic expressions. The research of Chaiklin and Lesgold (in Kieran, 1991) provide evidence that these difficulties are rooted in a poor understanding of the structure of *numerical* expressions. By the structure of an expression is meant its *surface structure* in the sense of Kieran 1989. Linchevski and Livneh (1996) analysed the solution procedures of students to different numerical versions of a particular algebraic structure. This study revealed that the specific number combinations often shifts the focus of attention from the structure (the rules for the order of operations) to the numerical properties of the given expression in such a way that the meaning of the expression is changed.

In most traditional curricula, as pointed out by Kieran (1989), the emphasis in arithmetic is on "finding the answer," which allows the students to get by with informal, intuitive procedures. In algebra, however, they are required to recognise the structures that they have been able to avoid in arithmetic. We take note of research that cautions against taking too simplistic a view of algebraic expressions and their analogous arithmetical transformations. Demby (1997), for example, analysed the procedures used by students in simplifying or transforming algebraic expressions. This research shows a correlation between the ability to transform algebraic expressions and the dominant feature (semantic or syntactic) of the procedure. The most successful students used semantic procedures more frequently. Demby (1997) uses semantic in the sense of "algebra as generalised arithmetic." Students who were able to use different kinds of procedures also performed well. Of interest to us in this research is the inability of students to use their understanding of the basic operations in explaining the validity of an algebraic transformation. There is, however, evidence that procedures that most often lead to the correct transformation involved a change in the surface structure of the expressions. Based on this research we believe that a focus on the structural aspects of numerical expressions is a necessary *part* in laying the foundation of algebra. Our pedagogical approach is set within a constructivist framework taking into consideration that students' construction of knowledge follows experiences in which they, for example, construct rules for the order of operations through the process of generalisation of data given. Both computational and non-computational tasks are given to consolidate these rules. These tasks are designed with the

Liebenberg, R.E., Linchevski, L., Sasman, M.C. & Olivier, A. (1999). Focusing on the structural aspects of numerical expressions. In J. Kuiper (Ed.), **Proceedings of the Seventh Annual Conference of the Southern African Association for Research in Mathematics and Science Education** (pp. 249-256). Harare, Zimbabwe.

purpose of generating discussion on the structure of the expression and its underlying meaning. We also take into consideration the diversity in ability and address the specific misconceptions and difficulties of students in the process.

In this paper we will describe the experiences of students with a particular focus on:
1. The ability of students to generalise structural features of numerical expressions.
2. The ability of the students to transfer the new structural knowledge to different contexts.

Research Setting

Background of sample.

The grade 6 class of 44 students is from one of our project schools. Students in this school are grouped heterogeneously. The four basic operations on the whole numbers are addressed in the earlier grades. Students are formally introduced to numerical expressions involving more than one operation for the first time in grade 6. They do, however, use informal methods in word problems and computations, for example, 6×12, in which they decompose 12 (as 6×10; 6×2; and then adding $60 + 12 = 72$) to obtain the total. The students do not however structure the computation as $6 \times 10 + 6 \times 2$. This research started after eight months of the project's involvement in the school. This report is based on data collected after a period of eight lessons of forty minutes. The teacher involved had established a culture of groupwork with the students but was having difficulty in handling the diversity of student ability.

The Teaching Experiment

The research was conducted in a co-operative teaching environment between the researcher and the teacher. The researcher often interacted with the students as they worked in groups, or as individuals on tasks, asking and sometimes answering questions. The researcher also conducted some of the whole-class discussions. The researcher assisted the teacher in handling the diversity in the class which involved making decisions for written assessments, the analysis of tests and the grouping of students into either heterogeneous or homogeneous groups based on the results as well as preparing materials for the different groups (Linchevski and Kutscher, 1998). Unstructured observational field notes were collected during every lesson. The data collected also included the written work of the students (including assessment) and interviews with individual students. The interviews were mostly informal, with notes taken during the interview and expanded immediately afterwards.

The Teaching Materials

The activities designed for the rules for the order of the operations was set in the context of finding the value of numerical expressions using both the scientific and non-scientific calculator as computational tools. The students were presented with numerical expressions whose structure created conflict, for example, $4 \times 6 + 7 \times 9$, produced the answer 87 for the scientific calculator while the non-scientific calculator produced the answer 279. Numerical expressions that did not create conflict, for example, $4 \times 7 \times 9 + 5$ were also given. The first activity, for example, only included expressions containing multiplication and addition. Expressions with either only multiplication or addition were also included for students to reflect on structures that they were familiar with. The students were asked to complete the table (taken form the first activity) below:

Number Expression	Scientific Calculator	"Sequential" Calculator
4+5 × 17		
12 × 9 + 28		
14 × 1 + 11		
4 × 6 + 7 × 9		
12 + 9 + 7		
15 + 8 × 9 + 12		
4 × 7 × 9 + 5		
4 × 7 × 12 × 9		
14 + 7 + 23 × 7		
12 + 15 × 3 × 9		

Table 1

Students were told that when the two calculators give different values, the scientific calculator gives the correct answer. The students were then expected to analyse the data in the table and to formulate rules about the order of operations to produce correct answers, i.e. to "discover" the rules used by the scientific calculator.

Activities were then given in which the students focussed on the rules which were set in different contexts. Here we distinguish between two kinds of contexts, one that involves doing calculations and another that focuses on the recognition of the structure without the need to calculate.
Below is an example of tasks focusing on the structure without the need to calculate:

1. Find an order in which the operations have to be done in the following number sentences.
   In each case find two ways in which it can be done if possible.

   <u>Example:</u>

   3 + 8 × 7 + 5

$\mathbf{3 \overset{3}{+} 8 \overset{1}{\times} 7 \overset{2}{+} 5}$	$\mathbf{3 \overset{2}{+} 8 \overset{1}{\times} 7 \overset{3}{+} 5}$
1a. 4 × 5 + 6 × 2	1b. 4 × 5 + 6 × 2

2. Find numbers to put in the following structure that gives an answer of 124:

   $$\square + \square \times \square$$

The teaching sequence and class-setting

The first activity was given to all the students who worked in heterogeneous groups. Based on the nature of the observed group discussions and the whole-class discussion it was decided to give the whole class a short diagnostic test to probe the source of the difficulties that the students had in dealing with these numerical expressions. The students were asked to calculate the value of eight numerical expressions without the use of a calculator. The test items where designed to see whether the nature of the operations in the expressions as well as the nature of the numbers played any significant role as distracters. This is illustrated by the following three items taken from the test:

1. $4 \times 3 + 5 \times 2 \times 6$
2. $4 \times 3 + 5 + 2 \times 6$
3. $4 \times 5 + 5 \times 2 \times 6$

Based on the analysis of the test results the students were re-grouped according to their misconceptions or difficulties identified. Students worked in these homogenous groups on various activities designed to address their specific problem. The students were then given another assessment to see whether they could transfer their structural knowledge to situations that required determining whether numerical expressions were equivalent. The students were at this stage not familiar with the term equivalence and were simply asked in the test to identify expressions that would give the same answer. Based on the previous assessment and observations in the lessons there appeared to be definite interference of the structural properties of expressions involving only one operation. Students intuitively understood the commutativity of addition ($3 + 4 + 5 = 3 + 5 + 4$) and generalised this to structures involving more than one operation, for example, $3 + 4 + 5 \times 6$ being equal to $3 + 5 + 4 \times 6$. The test items were designed once again on the basis of the nature of the operations to check for this kind of overgeneralisation. The students were not allowed to use calculators and the numbers chosen forced them to focus on the structure rather than calculating. Here are three of the six items given in the test:

1. Which of the following has the same answer as $302 + 79 + 128 \times 29$?
   a. $302 + 128 + 79 \times 29$
   b. $128 + 79 + 302 \times 29$
   c. $302 + 79 + 29 \times 128$
   d. $79 + 302 + 128 \times 29$
   e. $79 + 302 + 29 \times 128$

2. Which of the following has the same answer as $47 + 302 \times 58 \times 894$?
   a. $47 + 58 \times 302 \times 894$
   b. $47 + 894 \times 58 \times 302$
   c. $302 + 47 \times 58 \times 894$
   d. $47 + 302 \times 894 \times 58$
   e. $47 + 302 \times 894 \times 58$

3. Which of the following has the same answer as $302 + 79 \times 128 + 29$?
   a. $302 + 128 \times 79 + 29$
   b. $79 + 302 \times 128 + 29$
   c. $302 + 79 \times 29 + 128$

Results

On Formulating A Rule

The following question by one student in a group after having completed the data in Table 1of the first activity sums up the problem for nearly all of the students:

S_1: *"I know what a rule is, but how are we going to find it?"*

In the activity the students were told that the scientific calculator produced the correct answers in situation were there was conflict. The rules formulated by the students was based on this:

S_2: *" The scientific calculator gives the correct answer"*

The student reverted back to this kind of reasoning even after being challenged to find out *how* the scientific calculator gets the correct answer:

S_2: *"It takes in the number expression and then works out the answer right"*

Only one student made a suggestion of how the scientific calculator might work in a specific numerical expression:

S_3: *"The number 17 in $4 + 5 \times 17$ is broken down to 10 and 7 first"*

The students were clearly not reflecting on how the two calculators produced the answers. We realised that if we had perhaps asked the students to work out the answers to the numerical expressions first before introducing the two calculators the students would maybe have reflected on how they obtained their answers.

The students were then given the following calculation to do without the calculator:

$$762 + 5 \times 4$$

The students arrived at three different answers as shown by their methods:

1.
$$\begin{aligned} &762 + 5 \times 4 \\ &= 5 \times 4 + 762 \\ &= 20 + 762 \\ &= 782 \end{aligned}$$

2.
$$\begin{aligned} &762 + 5 \times 4 \\ &= 767 \times 4 \\ &= 3068 \end{aligned}$$

3.
$$762 + 5 \times 4$$

$$\begin{array}{ll} 700 \times 4 & = 2800 \\ 60 \times 4 & = 240 \\ 2 \times 4 & = 8 \\ \hline \multicolumn{2}{l}{3048 + 5 = 3053} \end{array}$$

The students returned to the activity trying to find the rule by calculating the values of the numerical expressions to see if they could get the answer of the scientific calculator. The range of expressions presented to the students provided valuable discussion in trying to formulate the rule. Students who, for example, initially generalised the rule as "when the multiplication is first in the expression, the answers are the same" were immediately challenged by others who pointed out that this was not true in the case of expressions like $a \times b + c \times d$. The rule was formulated in two slightly different ways:

R_1: *"Break it up into parts, do the multiplication and then add"*
R_2: *"Put brackets around the multiplication sums first, then add"*

Both of these formulations led to misinterpretations. For example, in calculating the value of $5 \times 3 + 2 \times 4 \times 6 + 7 \times 9$, a student explained that he used brackets "to break it up into parts" and then multiplied:

$$(5 \times 3) + (2 \times 4) \times 6 + (7 \times 9)$$

$$15 + 8 + 63 = 86$$

$$86 \times 6$$

The rest of the students immediately disagreed with this strategy indicating that "he was not supposed to multiply at the end". A student went on to explain that the six should have been added to eighty-six:

S_4: $\quad 4 \times 6 + 7 \times 9$

$\quad = (4 \times 6 \times 9) + 7$

S_4 explained his method: "I did the multiplication sums, I put a bracket around it and then added." The identification of the multiplication as a unit is clearly the problem. A possible cause for this could be the emphasis on doing the multiplication first or interference of previous knowledge of the commutativity of multiplication in structures like
$a \times b \times c \times d = a \times b \times d \times c$ which is overgeneralised to the structure $a \times b + c \times d$.

This emphasis on "doing the multiplication first" also prevented students from accepting that it was possible to add first, for example in the expression $7 \times 3 + 9 + 10$.

On Applying A Rule in Computational Contexts

The analysis of the first assessment confirmed that the recognition of the multiplication as a unit in the expressions was a problem for the students. Four groups of students were identified on the basis of this ability to recognise the multiplication as a unit:

Group 1.	Recognises the unit and applies the rule correctly in all expressions (23 students)
Group 2.	Recognises the unit but gives up the rule in certain expressions (3 students)
Group 3.	Does not recognise the unit in all expressions (14 students)
Group 4.	Does not recognise the unit in expressions where two or more consecutive multiplication operations (2 students)

Students in the first group often produced incorrect answers due to overgeneralisations of previous rules. In the following example, $4 \times 3 + 5 \times 2 \times 6$, many students showed their method of calculation as $12 + 10 + 12$. In an interview one student explained:

S: *"We did this in grade 5."*
The student writes down her own example, $6 \times 12 + 30$ and proceeds to explain:
S: *"Six times ten is sixty plus the six times two is twelve"*

It is evident here that the student applies the distributive law intuitively:

$$6 \times (10 + 2) = 6 \times 10 + 6 \times 2$$

and then overgeneralises incorrectly to $5 \times 2 \times 6 = 5 \times 2 + 6 \times 2$. This overgeneralisation is only induced in expressions that contained the addition and not in those that contained only multiplication.

Students in the second group often gave up the rule where the expression had more than one multiplication unit, for example, $4 + 3 + 6 \times 2 + 3 \times 5$, as shown in a student's method:

$$4 + 3 + (6 \times 2) + (3 \times 5)$$

$$12 + 3 = 15 \times 5 = 75 + 4 = 79 + 3 = 81$$

From the interviews it became evident that students misinterpreted the rule "first multiply and then add" to mean that once *a multiplication* was carried out the rule was applied. Here we see conflict between the recognition of the multiplication units and the interpretation of the rule.

In the third and fourth groups students either worked sequentially or paired off the numbers, for example:

$$\begin{aligned} & 4 + 5 + 5 \times 2 \times 6 + 4 \\ &= (4 + 5) + (5 \times 2) \times (6 + 4) \\ &= 9 \quad + 10 \times 10 \\ &= 190 \end{aligned}$$

On Applying A Rule in Non-Computational Contexts
The strategies of students working on the task where numbers had to be placed in a given expression to produce a given number indicated that they were not using the agreed structure of the expressions. This was also evident in the numbers that the students chose to put into the expressions. Students who were able to find the numbers in the structure $a + b \times c = 124$[1], had difficulty with the specific example $5 + 5 \times c = 124$. Most of the students worked sequentially, first adding to get $10 \times c$. The trial and error method was then used to find c. It is interesting to note that many of the students who resorted to the sequential method started by saying "5 times what?". This suggests that they *did recognise* the multiplication as a unit but viewed it procedurally, from left to right i.e. they did not handle it as $5 + 5 \times c = 5 + x$ where $x = 5 \times c$. It is also possible that students were distracted from the structure by the nature of the specific seductive numbers (5 + 5), as reported by Linchevski and Livneh (1996).

The students did not go back to the original structure to check their solution. When checking their solution most of the students worked sequentially, including students from Group 1 in the first assessment. This did not help in creating conflict and we had to ask the students to reflect on the rule for the order of operation (compare Herscovics and Linchevski, 1994). We encouraged these students to use the calculators to find c. Students entered various

[1] Letters were not used in the tasks. It is merely used here for reporting on the students.

computations into the calculator, for example, 124 ÷ 5 and 124 × 5. It is evident from this that they were not using the structure correctly. One of the students from Group 1 determined the value of c = 23,8 by trial and error and the correct application of the rule. In his method he used the commutative property of addition but from his explanation it is evident that this is based on the rule for the order of operation:

S: *"I know that it cannot be 24, the answer will be 125. So I started with 23,4 and I must multiply first and then add"*

Method written on the board by the student: 23,4 × 5 + 5 = 122
23,6 × 5 + 5 = 123
23,8 × 5 + 5 = 124

The students were then asked how they could determine the answer of 5 × 23,8 without multiplying. Only one student, worked with the structure, stating that it has to be 119 because 5 + 119 = 124.

In the second assessment on judging the equivalence of numerical expressions, only four students (all from Group 1 of the first assessment) were able to judge the equivalence of all the expressions given. These students determined the equivalence by checking to see whether the numbers in the multiplication unit were the same: For example, in the item

Which of the following has the same answer as 302 + (79 × 128) + 29?
a. 302 + (128 × 79) + 29
b. 79 + (302 × 128) + 29
c. 302 + (79 × 29) + 128

one student explained his choice of 302 + (128 × 79) + 29 *"because that one only had the same numbers in the brackets as the one given"*.

The most common strategy used by students to judge the equivalence of the expressions was that of pairing off the numbers from left to right. This strategy enabled the students to establish the equivalence without an understanding of the structures in certain of the test items, for example:

Which of the following has the same answer as 302 + 79 + 128 × 29?
a. 302 + 128 + 79 × 29
b. 128 + 79 + 302 × 29
c. 302 + 79 + 29 × 128
d. 79 + 302 + 128 × 29
e. 79 + 302 + 29 × 128

This following test item helped to expose those students who used this strategy:

Which of the following has the same answer as 302 + 79 × 128 + 29?
a. 302 + 128 × 79 + 29
b. 79 + 302 × 128 + 29
c. 302 + 79 × 29 + 128

Another strategy was the incorrect application of the commutative property in parts of the expression, for example 302 + 128 + 79 × 29 was considered equivalent to 302 + 79 + 128 × 29, since 128 + 79 is equivalent to 79 + 128.

Conclusions

The initial responses of the students in their attempts in trying to formulate a rule from the data presented in Table 1 suggests that they are not familiar with the process of analysing data in tables. None of the students considered looking at the expressions in the table, for example, in which both calculators produced the same answer and then trying to see what their structures had in common. In fact, this strategy had to be suggested to them to initiate progress. In the whole-class discussion it became evident too that the students did not see the need for a rule. Finding the general rule for the structure of the expressions was very difficult for the students. It is significant to note that the students were not able to generalise the information from a structural perspective. By this we mean that the students were not able to see the relationship between the nature of the operations in the structures and the outputs of the calculators. The students needed to calculate the value of the expressions first and only then on reflecting how they calculated were they able to formulate the rule. The inability of the students to transfer their structural knowledge to non-computational contexts provides evidence of how deeply ingrained and powerful the sequential processing of information is. It has also confirmed various research findings on how difficult it is for students to develop a structural view. In the context of solving equations the students relied on procedural conceptions. In those contexts where the students had to establish numerical equivalence it is evident that they are not aware of how to judge the equivalence of two expressions. Their previous experiences are based only on doing a calculation. It is also clear that the students did not relate their knowledge acquired to the new task. As part of our ongoing research we will monitor whether the ability to handle numerical expressions structurally improves after experiences in which they focus on the structural aspects.

References

Booth, L. R. 1988. Children's difficulties in beginning algebra. In A.F. Coxford (Ed.), **The Ideas of Algebra, K-12,** 20-32. Reston, V.A: National Council of Teachers of Mathematics.

Demby, A. 1997. Algebraic procedures used by 13- to-15 – year–olds. **Educational studies in mathematics, 33,** 45 – 70.

Herscovics, N. & Linchevski, L. 1994. A cognitive gap between arithmetic and algebra. **Educational studies in mathematics, 27**(1), 59-78.

Kieran, C. 1988. Two different approaches among algebra learners. In A.F. Coxford (Ed.), **The Ideas of Algebra, K-12**, 91-96. Reston, VA: National Council of Teachers of Mathematics.

Kieran, C. 1991. A procedural-structural perspective on algebra research. **Proceedings of the 15th International Conference of the International Group for the Psychology of Mathematics Education, 2,** 245-253. Assisi, Italy.

Linchevski, L. & Kutscher, B. 1998. Tell me with whom you're learning, and I'll tell you how much you've learned: Mixed-ability versus same-ability grouping in mathematics. **Journal for Research in Mathematics Education, 29**, 533-554.

Linchevski, L. & Livneh, D. 1996. The competition between numbers and structure. In L. Puig and A. Gutierrez (Eds.), **Proceedings of the 20th International Conference of the International Group for the Psychology of Mathematics Education, 3,** 257-264. Valencia, Spain.

Lins, R. L. 1990. A framework of understanding what algebraic thinking is. In G. Booker, P. Cobb, and T.N. Mendicuti (Eds.), **Proceedings of the Fourteenth International Conference of the International Group for the Psychology of Mathematics Education, 2**, 93-101. Mexico City, Mexico.

Matz, M. 1980. Building a metaphoric theory of mathematical thought. **Journal of Mathematical Behavior, 3** (1), 93-166.