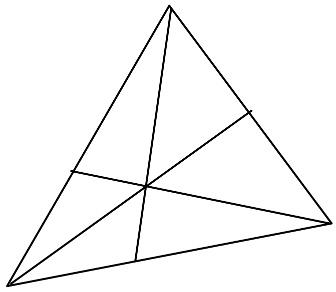
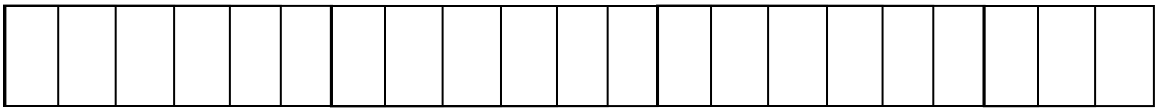
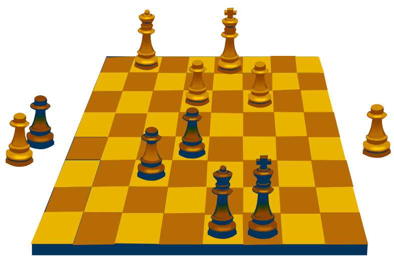
How many?



1. How many triangles (of all sizes) are there in this sketch?

2. How many rectangles of all sizes are there in this figure?

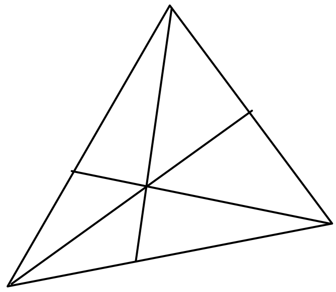




3. How many squares (of all sizes) are there on a standard 8×8 chess board?

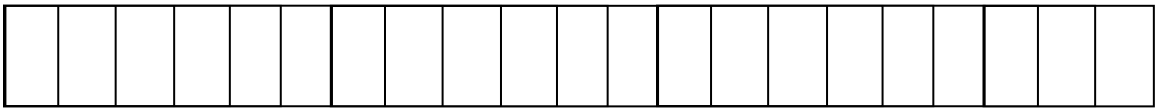
   The answer is not 64 ☺

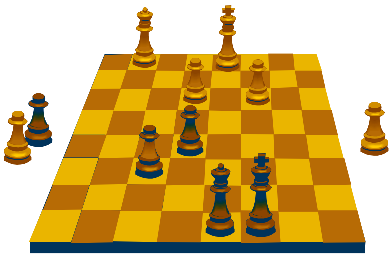
Hoeveel?



1. Hoeveel driehoeke (van alle groottes) is daar altesaam in hierdie skets?

2. Hoeveel reghoeke van alle groottes is daar in hierdie figuur?





3. Hoeveel vierkante van alle groottes is daar op 'n standaars 8×8 skaakbord?

   Die antwoord is nie 64 nie ☺

Hoeveel driehoeke?

Vooraf

Die groter konteks vir die les is verskillende soorte kennis en die bron van daardie kennis (vergelyk die artikel *On Constructivism*) en die implikasies daarvan vir *onderrig*:

- Fisiese kennis: Geleer deur fisiese aksie en empiriese abstraksie
- Sosiale kennis: Geleer deur sosiale transmissie
- Konsepsuele (logies-wiskundige) kennis: Geleer deur eie reflektiewe abstraksie
- Strategiese kennis (heuristieke): ?

In ’n probleem-gesentreerde benadering (Onderrig *via* probleemoplossing [1] wys ons nie vir kinders hoe om die probleem op te los nie (ons wys nie konsepsuele kennis nie)! Die onderwyser kan (*moet!*) egter vir leerlinge sosiale kennis (bv. notasies) wys, en heuristieke suggereer (bv. trek ’n diagram, maak ’tabel). Die vraag/probleem is: beskik leerlinge oor die nodige heuristieke vir probleemoplossing?

Ek beweer leelinge word nie gebore met geskikte heuristieke nie, en hulle kan dit nie *ontdek nie!* Sommige heuristieke soos die invoer van notasies is essensieel ‘n kreatiewe *skepping!* Maar meestal is heuristieke eintlik *konvensies* (bv. die manier hoe ons data in ’n tabel voorstel, of die manier hoe ons ’n drie-dimensionele skets maak en interpreteer). Dus leer ons heuristieke deur sosiale interaksie. *Dus is dit die onderwyser se taak om heursitieke te illustreer/onderrig!*

Die “kleiner” konteks is ’n voorbeeld van ’n spesifieke heuristiek, naamlik *sistematiese telling* (eintlik ’n groep heuristieke, insluitend *maak ’n diagram*, *ontwerp ’n notasie*, *probeer ’n spesiale geval*, *organiseer jou data*, soek *’n patroon*, ...). *Inhoudelik* kry ons ’n kykie op leerders se *visuele* intuïsies as inset-kennis (*resources)* vir belangrike wiskunde, bv. kongruensie, gelykvormigheid, trigonometrie ...

So, in hierdie les “konfronteer” ons leerlinge met ’n probleem, en ek vermoed dat baie nie suksesvol sal wees nie. Kyk hier vir voorbeelde van leerder response.

Ek teoretiseer dat een van die redes is dat hulle nie oor geskikte heuristieke beskik om die probleem op te los nie. So, kom ons *illustreer/onderrig* die nodige heuristieke, en kyk dan of die kennis *oordraagbaar* is, d.i. of leerlinge die nuwe kennis in nuwe situasies kan toepas ...

---

[1] Verskillende mense gebruik dikwels dieselfde woorde soos "probleemoplossing", maar gee heel verskillende betekenisse daaraan! Mense kan dan maklik dink hulle praat dieselfde taal of praat oor dieselfde ding, maar eintlik praat hulle oor heel verskillende goed!

<u>*Wat is ‘n probleem?*</u>
Vir iets om ‘n *probleem* te wees, moet daar een of ander blokkasie of struikelblok wees, d.w.s ‘n wiskundige probleem is ‘n taak wat jy nie weet hoe om op te los nie. Anders is dit mos nie ‘n probleem vir jou nie! Ek sal dit effens aanpas: ‘n Probleem is ‘n taak waarvoor jy nie ‘n geroetineerde (outomatiese) oplossingsmetode het nie.

<u>*Wat is probleemoplossing?*</u>
Wanneer ‘n onderwyser of ‘n handboek *eers* die nodige wiskundige konsepte en metodes vir ‘n probleemtipe ontwikkel en dit met 5 uitgewerkte voorbeelde illustreer, en dan leerlinge *toepassings* aan die einde van die hoofstuk laat doen, sê hulle dit is probleemoplossing. Dit kan tog nie wees nie – leerlinge weet dan reeds hoe om dit te doen! Ons noem hierdie soort take bloot *oefeninge!*

Ons sou in plaas van *probleme* vs *oefeninge*, die terminologie *nie-roetine probleme* en *roetine probleme* kon gebruik. Maar wat ek met *probleme* bedoel is dus nie-roetine probleme en *nie* oefeninge of roetine probleme nie!

Die volgende is ‘n nuttige raamwerk om tussen verskillende benaderings te onderskei:

- *Onderrig vir probleemoplossing:* Die onderwyser onderrig *eers* vooraf en apart die "tools" – die nodige wiskundige konsepte en metodes in die abstrak, en dan word dit *agterna* toegepas.
- *Onderrig via (deur) probleemoplossing:* Die onderrig *begin* met relevante probleme en die wiskundige konsepte en metodes word ontwikkel *terwyl* leerlinge die probleme oplos.
- *Onderrig omtrent/van probleemoplossing:* Die onderwyser illustreer in die algemeen probleemoplossingsmetodes.

Met probleemoplossing in die klaskamer bedoel ek dus *onderrig via (deur) probleemoplossing.*

<u>*Die onderliggende perspektiewe*</u>
Die twee benaderings *onderrig vir probleemoplossing* en *onderrig via (deur) probleemoplossing*

- Is gebaseer op heel verskillende onderliggende teorieë oor die aard van kennis (epistemologieë), in besonder die aard van Wiskunde, verskillende leerteorieë (insluitend die aard van geheue) en verskillende perspektiewe oor die betekenis van "verstaan", en
- dit lei tot totaal verskillende klaskamerkulture die onderlinge verwagtings en verantwoordelikhede van onderwysers en leerlinge), en
- dit bepaal op sy beurt wat (en of) leerlinge leer.

Dus is die lesplan (ge-antisipeerde verloop):

1. Gee leerlinge 'n geskikte probleem – ek stel voor die vraag oor hoeveel driehoeke op die werkvel. Maak seker dat hulle die probleem (die konteks, die stuktuur, die vraag) verstaan, bv. wat bedoel ons met driehoeke van verskillende groottes? Laat hulle aan die probleem werk, daarmee worstel, sonder dat ons hulle vooraf wys hoe om dit te doen en sonder om eers vooraf die nodige voorkennis te hersien, behandel of op te som.
2. As leerlinge (*sommige* leerlinge) dit kan doen is dit wonderlik! Ons taak is dan om hulle so te laat verduidelik dat almal dit verstaan. Ons kan help deur bv. terwyl hulle verduidelik geskikte notasies of diagramme op die bord te teken.
3. As leerlinge dit nie kan doen nie, en besef hulle kan nie of besef hulle is verkeerd (hoe gaan hulle dit besef??) is dit ook in orde. Ons het dan 'n *behoefte* vir nuwe kennis geskep. Ideaalgewys is leerlinge nou *gemotiveerd* om die nuwe kennis te bemeester wat hulle benodig om die probleem op te los.
4. Onderrig/illustreer die nuwe kennis (bou egter op enige kennis van sommige leerders!)
5. Laat leerlinge nou beleef dat die kennis *nuttig* is, d.w.s dat dit hulle bemagtig om die oorspronklike probleem "maklik" of "makliker" op te los.
6. Voorsien vir die *vaslegging* van kennis deur verdere *toepassing* in nuwe probleme – ek stel voor vraag 2, reghoeke.

As die les nie verloop soos geantisipeer nie, moet ons aanpassings maak ... (Dit beteken ons aannames of ons teorie is onvoldoende! Ons moet dus ons aannames of teorie aanpas en dit vir nuwe voorspellings/antisipasies gebruik, dit weer toets, ...)

Lees ook hierdie agtergrond artikel oor sistematiese telling.

Agterna: Waarneming/refleksie

How many triangles?[2]

Alwyn Olivier

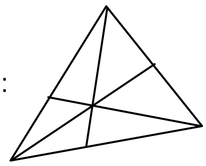
Current efforts to improve mathematics education are based on a different view of what it means to “do” mathematics. We are moving away from a view of mathematics as a static, finished, structured system of facts, procedures, and concepts, towards a dynamic and exploratory view, where the focus is on the active, generative processes one engages in when doing mathematics.

This more dynamic view of mathematical activity has implications for what learners need to learn, how they learn it, and the kinds of activities in which learners and teachers should engage during mathematics lessons. Much of learners' learning is seen as the process of acquiring a "mathematical point of view", which is characterized by such activities as looking for and exploring *patterns* to understand mathematical *structures* and underlying *relationships*; to formulate and solve problems; making sense of mathematical ideas, thinking and reasoning in flexible ways: conjecturing, generalising, justifying, and communicating one's mathematical ideas (Schoenfeld, 1992; Department of Education, 2002).

Schoenfeld (1992) distinguishes the following variables influencing learners’ problem solving:

- Resources: their knowledge base
- Problem solving strategies (heuristics)
- Monitoring and control
- Beliefs and affects
- The classroom environment

Let’s now briefly discuss some of these issues with reference to a well-known problem:



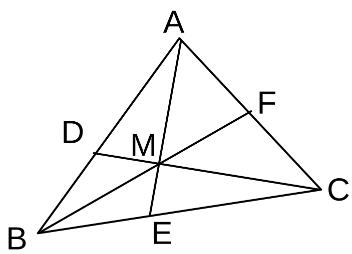
How many triangles (of all sizes) are there in this figure?

We remark that learners seem to have much trouble with it. We also remark that teachers too often seem to accept that learners with excellent *visual sense* simply “see” it, and others do not. We argue that it is not merely a case of “seeing” the triangles – we should help learners *develop* the necessary underlying skills!

We can solve the problem by implementing one of Schoenfeld’s problem solving strategies (heuristics), namely *to systematically list all cases*. To systematically list all cases means that we should employ a *system* and not work haphazardly. In order to effectively implement the strategy, we need to introduce some notation to record and communicate our results, and especially to monitor and control our progress (see Schoenfeld’s categories above).

So let’s name the vertices of the triangle in the conventional way as shown below. I am going to start at corner B and systematically work my way anti-clockwise around the triangle, successively considering BE, EC, BC, CF, … as one side of possible triangles. Starting with BE, I systematically consider C, F, A, D, M as third corner of the triangle and decide which ones form a triangle and which not, depending on which lines are drawn (this is part of the *knowledge* I bring to the situation, one of Schoenfeld’s categories). So BFE does not form a triangle, but BAE does. Also, it is necessary to record and *organise* our results, e.g. to record it in a table to keep track of our progress, otherwise we may miss out some possibilities or lose our place. Here is the result:

Base	**Triangles**	**# Triangles**
BE	BAE, BME	2
EC	EAC, EMC	2
BC	BFC, BAC, BDC, BMC	4
CF	~~CBF~~, CMF	1
FA	FBA, FMA	2
CA	CDA, CMA, ~~CEA~~, ~~CBA~~	2
AD	AMD	1
DB	~~DCB~~, DMB	1
AB	AMB, ~~AEB~~, ~~AFB~~, ~~ACB~~	1
	Total	**16**



As is often the case, our recorded *product* does not show much of our underlying thinking *processes*. For example, in the table above, it was necessary that we carefully *keep track* so that we do not count a triangle twice, e.g. in the fourth row we must recognise that ∆CBF is the same as ∆BFC that we have already counted. Also, I actually checked that the inner bases BM, MF, BF, ... do not give new triangles, and I am wondering if such inner bases will ever give new triangles, so is it necessary to test them?

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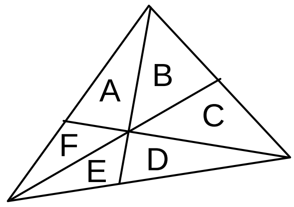
[2] Parts of this paper were previously published in Western Cape AMESA NEWS

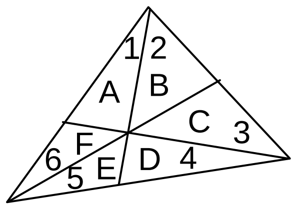
A second approach would be to use *regions* in the triangle as our organizing principle instead of vertices and sides. Name the regions as shown below. Note that the symbols (A, B, …) and notation (e.g. ABC) on the *surface* may seem the same as above, but they now have different *meanings*. Our system in this case is to systematically combine 1 region, 2 regions, 3 regions, … and to decide and record which of them form triangles (e.g. AB does not form a triangle, but BC does):

# Regions	Regions	# Triangles
1	A, B, C, D, E, F	6
2	BC, DE, FA	3
3	ABC, BCD, CDE, DEF, EFA, FAB	6
4	-	0
5	-	0
6	ABCDEF	1
	Total:	**16**



A third approach is to use the *angles* in the sketch as organiser. Below I have numbered the angles and named the regions as before and systematically listed the regions forming triangles containing the indicated angles, without repeating any. The table again shows much less than our actual thinking process, e.g. I also checked regions containing the angles in the middle.

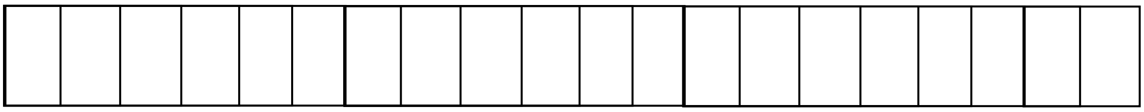
Angle	Regions	# Triangles
1	A, AF, AFE	3
2	B, BC, BCD	3
1 & 2	ABCDEF	1
3	C, ~~CB~~, CBA	2
4	D, DE, DEF	3
3 & 4	~~BCD~~, CDE, ~~ABCDEF~~	1
5	E, ~~ED~~, ~~EDC~~	1
6	F, ~~FA~~, FAB	2
	Total:	**16**



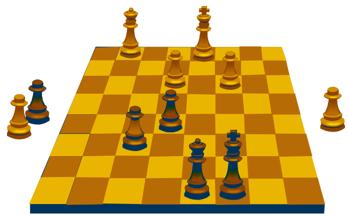
Here are some problems to solve:

How many?

1. How many rectangles of all sizes are there in this figure (there are 20 small rectangles)?



   Generalise to any such figure …



2. How many squares (of all sizes) are there on a standard 8×8 chess board?

   Generalise to an *n*×*n* chess board …

3. *Looking back …* How will you generalise 1 and 2 …?

PTO

Rectangles

Mathematicians are lazy, so we are definitely not going to *count* the rectangles! So let's rather use one of Schoenfeld's heuristics: *analyse a simpler case*. Let's take the simpler case of six small rectangles and let's systematically list the triangles using the number of regions as organiser:

# Regions	Regions	# Rectangles
1	A, B, C, D, E, F	6
2	AB, BC, CD, DE, EF	5
3	ABC, BCD, CDE, DEF	4
4	ABCD, BCDE, CDEF	3
5	ABCDE, BCDEF	2
6	ABCDEF	1
	Total	**21**

A	B	C	D	E	F

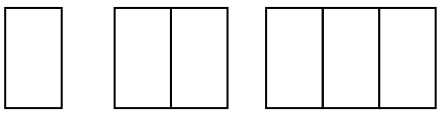
In this case it is not the *answer* (21) that is important, but the *structure!* In such an approach, we do not use systematic counting to directly solve the original problem, but we use systematic counting to uncover the *underlying structure or pattern*, and then use the structure or pattern as a *model* to solve the original problem:

For 6 small triangles, the number of triangles R(6) = 6 + 5 + 4 + 3 + 2 + 1.

Can you see the *general* in the *particular*?

Use the *structure* to *predict* the result of R(20), R(100) and R(*n*)[3].

Another approach would be to investigate simpler cases and being systematic … Investigating special cases for 1, 2, 3, …small rectangles and organise our results into a table we see that the pattern giving the number of rectangles is the famous *triangular sequence*:



R(1) = 1 R(2) = 3 R(3) = 6

# small rectangles: *n*	1	2	3	4	5	6	…	20	100
# all rectangles: R(*n*)	1	3	6	10	15	21	…		

Now we will find it quite difficult to find a formula from this data!

If, however, we learn *not to calculate*, *but to analyse structure*, the relationships will be clearer:

# small rectangles: *n*	1	2	3	4	5	6	…	20	100
# all rectangles: R(*n*)	1	1+2	1+2+3	1+2+3+4	1+2+3+4+5		…	?	?

Squares

Let's look at the *simpler case* of a 4 × 4 board. I am going to "invent" a convenient notation to indicate my squares: Instead of writing ABEF for the top-left 2×2 square, I will just write AF, indicating the top-left and bottom-right corners. Here is the result:

Size	Regions	# Squares
1×1	A, B, C, …., P	16
2×2	AF, BG, CH EJ, FK, GL IN, JO, KP	9
3×3	AK, BL EO, FP	4
4×4	AP	1

A	B	C	D
E	F	G	H
I	J	K	L
M	N	O	P

The completed table does not reflect my thinking process! I soon recognised 16 as not just any number, but 4^2, then I *predicted* and *confirmed* $3^2 = 9$ for the next row, and that gave me confidence that I have counted the other cases correctly. Therefore, we use the *structure as tool* to direct and *monitor* our progress.

The point of using such a simpler case (a 4×4 instead of an 8×8 square) is to use either the *answer*, or the *structure* to solve the more complex original problem:

For a 4×4 board there are $S(4) = 4^2 + 3^2 + 2^2 + 1^2$ squares.

Can you *generalise* this *structure* to find the number of squares on a 8×8 board?[4]

---

[3] This is of course Gauss! R(20) = 1+2+3+4+ … +18+19+20 –Make a Gauss plan? $R(n) = \frac{n(n+1)}{2}$

[4] $S(8) = 8^2 + 7^2 + 6^2 + 5^2 + 4^2 + 3^2 + 2^2 + 1^2$

Of course, the mathematical viewpoint is to wonder what is a *general* formula[5] to calculate

$$S(n) = 1^2 + 2^2 + 3^2 + \ldots + n^2$$

Another approach would be to investigate simpler cases and being systematic …
So, starting with a 1×1 square there is only 1 square.
Then, for a 2×2 square there are 5 different squares, etc.
We could work *inductively, only with the numbers*, recording and organising our results in a table:

Size of square: *n*	1	2	3	4	5	6	7	8	*n*
# Squares	1	5	14	30					

Now we will find it quite difficult to find a formula from this data!
If, however, we learn *not to calculate*, but *to analyse structure*, the relationships will be clearer:

Size of square: *n*	1	2	3	4	…	8
# Squares	1 $=1^2$	1+4 $=1^2+2^2$	1+4+9 $=1^2+2^2+3^2$	1+4+9+16 $=1^2+2^2+3^2+4^2$		$1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2$

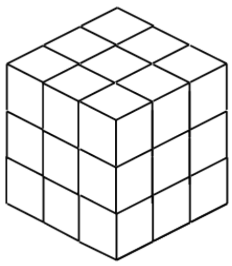
Generalising
So we have:

How many rectangles? $1 + 2 + 3 + 4$
How many squares? $1^2 + 2^2 + 3^2 + 4^2$
Furthermore, the rectangles are in one dimension (only one variable, "length", changing only horizontally), while the squares are in two dimensions (two equal variables, "area", changing horizontally and vertically). A generalization is clear: three-dimensional, working with volume, or the number of *cubes!* And the exact *form* of our conjecture should be clear as a generalization:

One dimension:	How many rectangles?	$1 + 2 + 3 + 4 + \ldots n$
Two dimensions:	How many squares?	$1^2 + 2^2 + 3^2 + 4^2 + \ldots n^2$
Three dimensions:	How many cubes?	$1^3 + 2^3 + 3^3 + 4^3 + \ldots n^3$

And this is the point of the activity: We immediately have a conjecture before we even start working at the problem!!!!

Cubes



Our further work is completely different – our strategy is guided by our *conjecture*, and is basically only to *confirm* the answer we already have, not to "discover" it! In this sense generalisation is therefore a very powerful conjecturing strategy!

Size of cube: *n*	1	2	3	*n*
# Cubes	1 $=1^3$	1+8 $=1^3+2^3$	1+8+27 $=1^3+2^3+3^3$	$S(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3$

Further analysis will lead to the conjecture that

$$S(n) = 1^3 + 2^3 + 3^3 + \ldots + n^3 = (1 + 2 + 3 + \ldots + n)^2 = \left[\frac{n(n+1)}{2}\right]^2$$

LOOKING BACK
The problems discussed here are non-routine problems, and to "do" the mathematics require many important underlying skills that we should nurture in class, e.g. to introduce appropriate symbols and notations, to draw sketches, to systematically list cases, to organise our results into a table, to *specialise* (e.g. to take simple special cases), to *generalise* by describing the pattern in our generated data, etc. – all skills that are seldomly if ever used when we solve routine, known problems by applying learned procedures.

References:

Schoenfeld, A. H. (1992). Learning to think mathematically: Problem solving, metacognition, and sense making in mathematics. In D. A. Grouws (Ed.), **Handbook of research on mathematics teaching and learning** (pp.334–371). New York: Macmillan. http://www-gse.berkeley.edu/Faculty/aschoenfeld/LearningToThink/Learning_to_think_Math.html

Department of Education (2002). **Revised National Curriculum Statement for Grades R-9 (Schools) – Mathematics**. Government Gazette, Vol. 443, No. 23406. Pretoria. http://education.pwv.gov.za/content/documents/11.pdf

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5 $S_n = \dfrac{n(n+1)(2n+1)}{6}$