TEACHING MATHEMATICAL PROBLEM SOLVING: IMPLEMENTING THE VISION

A Literature Review

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Preface

The Northwest Regional Educational Laboratory in Portland, Oregon, is a nonprofit educational research organization providing services and resources to educators and policymakers in the Northwest states of Alaska, Idaho, Montana, Oregon, and Washington. The Laboratory's Mathematics and Science Education Center provides K-12 mathematics and science teachers with technical assistance and professional development opportunities, classroom-based research, access to a lending library of teaching resources, and products such as videos, publications, resource kits, and instructional models to support standards-based teaching practices.

In 1999, the Mathematics and Science Education Center developed a mathematics problem-solving model that K-12 teachers can use to infuse their instruction and curriculum with open-ended problem solving. The model includes classroom tasks, a scoring guide for assessing students' performance of those tasks, and sample student work. The tasks engage students in active, hands-on investigations into such things as number theory, computation, geometry, estimation, probability, statistics, and algebra. The scoring guide informs instruction and helps teachers to assess students' performance of the essential traits of problem solving. These traits include conceptual understanding, strategies and reasoning, computation and execution, mathematical insights, and communication. The model also includes intensive professional development. This document reviews recent research and literature on the essential traits and processes of teaching and learning mathematics through open-ended problem solving. The literature and research on effective problem solving informed the design of the NWREL Mathematics Problem-Solving Model™.

Introduction

> "A problem is not necessarily solved because the correct answer has been made. A problem is not truly solved unless the learner understands what he has done and knows why his actions were appropriate."
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> — William A. Brownell, *The Measurement of Understanding* (1946)

It's been two decades since the National Council for Teachers of Mathematics (NCTM) recommended in its *Agenda for Action* (1980) that problem solving be the focus of mathematics education. In their oft-cited *Curriculum and Evaluation Standards for School Mathematics*, NCTM (1989) identified the central goal of mathematics education to be developing students' mathematical power: "an individual's ability to explore, conjecture, and reason logically, as well as the ability to use a variety of mathematical methods effectively to solve nonroutine problems."

Yet, judging by U.S. students' performance on recent tests like the Third International Mathematics and Science Study (TIMSS) (in which U.S. students scored below the international average at all grade levels on problem solving), our students are not yet meeting the expectations of the mathematics education community (Peak, 1996, 1997; Takahira, S., Gonzales, P., Frase, M., & Salganik, L.H., 1998). Despite calls for problem-solving approaches to teaching mathematics, the shift from teaching math facts and procedures to teaching with an emphasis on mathematical concepts and thinking skills has been slow and difficult. Many teachers are unconvinced that traditional methods should be abandoned. Of those who want to change, many are unsure how to go about it. Further complicating the issue is a lack of consensus about what is meant by *problem solving*—especially open-ended problem solving—and how it is best taught and assessed.

These issues are the topic of this monograph. Through a review of the current research and literature on effective methods for teaching problem solving, this paper provides an overview of the meaning and purpose of open-ended problem solving in school mathematics.

What is Open-Ended Problem Solving?

In open-ended problem solving, the problem will have multiple possible answers that can be derived by multiple solution methods. The focus is not on the answer to the problem, but on the methods for arriving at an answer. Genuine problem solving requires a problem that is just beyond the student's skill level so that she will not automatically know which solution method to use. The problem should be nonroutine, in that the student perceives the problem as challenging and unfamiliar, yet not insurmountable (Becker & Shimada, 1997).

In open-ended problem solving, students are responsible for making many of the decisions that, in the past, have been the responsibility of teachers and textbooks. To decide which method, or procedure, to undertake to solve an open-ended problem, a student will draw on her previous knowledge and experience with related problems. She might construct her own procedure, trying this and that, before arriving at a solution. She will then reflect on and explain to others her problem-solving experience, tracing her thinking process and reviewing the strategies she attempted, determining why some worked and others didn't. This period of reflection deepens her understanding of the problem and helps to clarify her thinking about effective solution methods, and how the problem and methods she used relate to other problems or areas of mathematics.

One of the teacher's key responsibilities is selecting and presenting "good" problem tasks. By choosing good problems, the teacher sets up optimal conditions for her students to be engaged in meaningful problem solving. This means that the problem will:

- Be open-ended, in that it presents multiple solution methods and answers
- Address important mathematics concepts
- Challenge and interest students
- Connect to students' previous learning

Why Teach Open-Ended Problem Solving?

To help young people be better problem solvers is to prepare them not only to think mathematically but to approach life's challenges with confidence in their problem-solving ability. The thinking and skills required for mathematical problem solving transfer to other areas of life. The writers of the groundbreaking report *Everybody Counts: A Report to the Nation on the Future of Mathematics Education* put it this way:

> Experience with mathematical modes of thought builds mathematical power—a capacity of mind of increasing value in this technological age that enables one to read critically, to identify fallacies, to detect bias, to assess risk, and to suggest alternatives. Mathematics empowers us to understand better the information-laden world in which we live (National Research Council, 1989).

Learning mathematics by grappling with open-ended and challenging problems accommodates diverse learning styles. The active and varied nature of problem solving helps students with diverse learning styles to develop and demonstrate mathematical understanding (Moyer, Cai, & Grampp, 1997). Traditional teaching approaches involving rote learning and teacher-centered instructional strategies often do not meet the learning needs of many students who may be active learners or require multiple entrances into the curriculum.

Learning through open-ended problem solving helps students to develop understanding that is flexible, that can be adapted to new situations and used to learn new things (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier, & Human, 1997). "Things learned with understanding are the most useful things to know in a changing and unpredictable world," explains Hiebert and colleagues. Yet, usefulness is not the only reason to learn with understanding. To learn with understanding is to also grapple intellectually with mathematics as a subject. "When we memorize rules for moving symbols around on paper we may be learning something, but we are not learning mathematics," says Hiebert. "Knowing a subject means getting inside it and seeing how things work, how things are related to each other, and why they work like they do."

When students encounter mathematical ideas that interest and challenge them in an open-ended problem solving context, they are more likely to experience the kinds of internal rewards that keep them engaged, says Hiebert (Hiebert et al., 1997). Students who must resort to memorizing will lack understanding and will likely feel little sense of satisfaction, perhaps withdrawing from learning altogether. In fact, he says, evidence suggests that if students memorize and practice procedures repeatedly in a rote fashion, it's difficult for them to go back later and gain a deeper understanding of the mathematical concepts underlying those procedures. Researchers Jerry Becker and Shigeru Shimada (1997) concur: "Lessons based on solving open-ended problems as a central theme have a rich potential for improving teaching and learning."

Recognizing the centrality of problem solving to mathematics learning, education leaders have made it a focal point of standards reform for the past two decades. In the spring of 2000, the National Council of Teachers of Mathematics renewed its commitment to problem solving when it published *Principles and Standards for School Mathematics* (NCTM, 2000), an update of the council's 1989 statement about standards for teaching and learning mathematics, *Curriculum and Evaluation Standards for School Mathematics*. Like that seminal work, the council's updated standards identify problem solving as an essential component of math learning for all grade levels.

Furthermore, many states have adopted content and performance standards and assessments based on the NCTM standards that include an emphasis on problem solving. While the Northwest states are at different stages in the process of adopting standards and developing assessment systems, most are addressing the importance of teaching and assessing reasoning, communicating, making connections, and applying knowledge to problem situations—key tenets of problem solving.

The Role of Problem Solving in School Mathematics

Stanic and Kilpatrick (1989) identify three general themes that have historically characterized the role of problem solving in school mathematics: problem solving as context, problem solving as skill, and problem solving as art.

Problem solving as context. The authors divide problem solving as a context for doing mathematics into several subcategories. Problem solving has been used as *justification* for teaching mathematics. To persuade students of the value of mathematics, the content is related to real-world problem-solving experiences. Problem solving also has been used to *motivate* students, sparking their interest in a specific mathematical topic or algorithm by providing a contextual (real-world) example of its use. Problem solving has been used as *recreation,* a fun activity often used as a reward or break from routine studies. Problem solving as *practice,* probably the most widespread use, has been used to reinforce skills and concepts that have been taught directly.

When problem solving is used as context for mathematics, the emphasis is on finding interesting and engaging tasks or problems that help illuminate a mathematical concept or procedure. To use problem solving as context, a teacher might present the concept of fractions, for example, assigning groups of students the problem of dividing two pieces of licorice so that each gets an equal share. By providing this problem-solving context, the teacher's goals are multiple: to create opportunities for students to make discoveries about fraction concepts using a familiar and desirable medium (*motivation)*; to help make the concepts more concrete (*practice*); and to offer a rationale for learning about fractions (*justification)*.

Problem solving as a skill. Advocates of this view teach problem solving skills as a separate topic in the curriculum, rather than throughout as a means for developing conceptual understanding and basic skills. They teach students a set of general procedures (or rules of thumb) for solving problems—such as drawing a picture, working backwards, or making a list—and give them practice in using these procedures to solve routine problems. When problem solving is viewed as a collection of skills, however, the skills are often placed in a hierarchy in which students are expected to first master the ability to solve routine problems before attempting nonroutine problems. Consequently, nonroutine problem solving is often taught only to advanced students rather than to *all* students. When defining the learning objectives of a problem-solving activity, teachers will want to be aware of the distinction between teaching problem solving as a separate skill and infusing problem solving throughout the curriculum to develop conceptual understanding as well as basic skills.

Problem solving as art. In his classic book, *How To Solve It*, George Polya (1945) introduced the idea that problem solving could be taught as a practical *art*, like playing the piano or swimming. Polya saw problem solving as an act of discovery and introduced the term "modern heuristics" (the art of inquiry and discovery) to describe the abilities needed to successfully investigate new problems. He encouraged presenting mathematics not as a finished set of facts and rules, but as an experimental and inductive science. The aim of teaching problem solving as art is to develop students’ abilities to become skillful and enthusiastic problem solvers; to be independent thinkers who are capable of dealing with open-ended, ill-defined problems.

Challenges of Teaching Problem Solving

Although Polya presented the inquiry-based framework for teaching problem solving more than 50 years ago, there has yet to be widespread implementation of his ideas in U.S. classrooms. This suggests that there are a number of challenges to making this shift in mathematics teaching.

Teaching nonroutine problem solving is difficult. True problem solving is as demanding on the teacher as it is on the students. The art of teaching mathematical problem solving is best mastered over a long period of time (Thompson, 1989). Teaching problem solving is difficult, writes Schoenfeld (1992). Teachers:

- Must perceive the implications of students' different approaches, whether they may be fruitful and, if not, what might make them so.
- Must decide when to intervene, and what suggestions will help the students while leaving the solution essentially in their hands, and carry this through for each student.
- Will at times be in the position of not knowing; to work well without knowing all the answers requires experience, confidence, and self-awareness.

Burkhardt (1988, as cited in Schoenfeld, 1992) states even more succinctly that teaching problem solving is difficult for teachers mathematically, pedagogically, and personally. Teachers must have the mathematical expertise to understand the different approaches that students might take to a problem and how promising those approaches will be. Many elementary teachers are trained as generalists and often do not have the strong mathematical background required to teach from a problem-solving approach. Pedagogically, teachers must make complex decisions about the level of difficulty of the problems assigned, when to give help, and how to give assistance that supports students' success while ensuring that they retain ownership of their solution strategies. Personally, teachers will sometimes find themselves in the uncomfortable position of not knowing the solution. Letting go of the "expert" role teachers have traditionally played requires experience, confidence, and self-awareness. Often, teachers are asked to teach mathematics they never encountered in school and in a way that differs from how *they* were taught. For these reasons, teachers may need additional training in mathematical content and theory, as well as in methods for teaching problem solving.

Nonroutine problems are difficult for students. Nonroutine, open-ended problems are often, by their nature, difficult for many students. Shannon and Zawojewski (1995) conducted a ministudy that demonstrated the difficulty presenting problem-solving tasks without providing hints and procedural steps poses to students. In the study, two groups of students were presented with similar tasks. In one task, "Supermarket Carts," students were given a scale drawing of 12 shopping carts nested together and asked to create a rule to determine the length of storage space needed for any number of carts and the number of carts that would fit into a given space. This was essentially all the direction given. A second group of students was assigned the task "Shopping Carts," which included several prompts or subproblems to help guide them toward a solution. Students were asked to find the length of one shopping cart, find how much a cart sticks out when the carts are nested, find the total length of 20 carts, and find how many carts could fit into a 10-meter space. Then they were asked to find the two formulas that were asked for in the Supermarket Carts task.

The researchers reported that students attempting the Supermarket Carts task had difficulty knowing how to get started. Only a few students successfully derived the formulas required. On the other hand, none of the students working on the Shopping Carts task had any difficulty getting started, and all but one group successfully derived the requested formulas. The authors conclude that, "the sense of students' having to struggle was greater in Supermarket Carts than in Shopping Carts." Watching their students struggle in frustration is often very difficult for teachers. Knowing when to give hints and how much help to give requires striking a delicate balance that comes with experience and knowing students' capabilities.

Teachers are concerned about content coverage. The TIMSS research characterized the U.S. curriculum as "a mile wide and an inch deep" compared to the mathematics curriculum in other countries (Peak, 1996, 1997; Takahira, et al, 1998). Teachers in the U.S. are generally expected to cover large areas of content each year. Yet solving challenging, nonroutine problems takes time. Often a single problem can occupy a class for a whole period or more. Therefore, it's essential that content and skills be integrated within the context of problem solving. By selecting rich, engaging, and worthwhile tasks, teachers can ensure that time is well-spent.

Textbooks present few nonroutine problems. Although they are improving, many textbooks do not provide an adequate number of nonroutine problems from which teachers can choose. Many teachers are not comfortable straying from the scope and sequence the textbook provides, but they must develop the confidence to search out and develop other materials to supplement their texts.

Changing One's Practice: Teacher Readiness

A teacher's approach to teaching mathematics reflects her beliefs about what mathematics is as a discipline (Hersh, 1986). If she characterizes mathematics as involving correct answers and infallible procedures consisting of arithmetic operations, algebraic procedures, and geometric terms and theorems, chances are, her instructional approach will likely emphasize the presentation of mathematical concepts, procedures, facts, and theorems with a focus on student practice and memorization. The meaning and context associated with many of these theorems and procedures may be relegated to the fringes of her curricular focus. On the other hand, if she views mathematics as an active, creative endeavor involving inquiry and discovery, she will likely emphasize activities that involve students in generating and uncovering meaning and making connections. She will view her role as a facilitator, challenging students to think and to question their findings and assumptions.

Ernest (1988) outlines three conceptions of mathematics, each of which prompts a different emphasis in instruction:

> First of all, there is a dynamic, problem-driven view of mathematics as a continually expanding field of human creation and invention, in which patterns are generated and then distilled into knowledge. Thus mathematics is a process of enquiry and coming to know, adding to the sum of knowledge. Mathematics is not a finished product, for its results remain open to revision (**the problem-solving view**).
>
> Secondly, there is a view of mathematics as a static but unified body of knowledge, a crystalline realm of interconnecting structures and truths, bound together by filaments of logic and meaning. Thus mathematics is a monolith, a static immutable product. Mathematics is discovered, not created (**the Platonic view**).
>
> Thirdly, there is the view that mathematics, like a bag of tools, is made up of an accumulation of facts, rules and skills to be used by the trained artisan skillfully in the pursuance of some external end. Thus mathematics is a set of unrelated but utilitarian rules and facts (**the instrumentalist view**).

Each of these views perceives the essence of mathematics differently. The instrumentalist view sees mathematics as a set of tools. Teachers with an instrumentalist view can be expected to stress rules, facts, and procedures in their classes. Their classes tend to be teacher-directed and emphasize routine drill and practice. The Platonic view sees math as a body of knowledge. Teachers who ascribe to the Platonic view of mathematics focus on the interrelationships, underlying concepts, and internal logic of mathematical procedures. The problem-solving view focuses on the process of inquiry. Teachers with a problem-solving view tend to be more learner-focused and constructivist in their teaching style, actively involving students in exploring mathematical concepts, creating solution strategies, and constructing personal meaning in a problem-rich environment (Thompson, 1992).

Students' beliefs about the nature of mathematics are greatly influenced by their teacher's beliefs. Surveys of student beliefs about mathematics reveal that most students think there should be a ready method for solving problems and that that method should quickly lead to an answer (Schoenfeld 1989, 1992). Schoenfeld (1992) cites a 1983 survey conducted by the National Assessment of Educational Progress (NAEP) in which half of the students who responded agreed that "learning mathematics is mostly memorizing facts." Three-quarters agreed that "doing mathematics requires lots of practice in following rules," while 90 percent agreed with the statement, "There is always a rule to follow in solving mathematical problems." Students holding such beliefs may not even attempt to solve a problem that involves too much complexity or does not appear to offer a clear-cut algorithmic approach.

Furthermore, Schoenfeld (1992) notes that most students believe that all problems have an answer; that there is only one right answer and one correct solution method; and that ordinary students cannot expect to *understand* mathematics but can merely memorize and apply mathematical procedures in a mechanical fashion. These beliefs largely develop out of the experiences students have in mathematics classes and from the attitudes and beliefs passed on by their teachers.

A problem-solving approach to teaching mathematics helps broaden students' perception of mathematics from a rule- and fact-based discipline to one that involves inquiry, uncertainty, and creativity. But first, the teacher must make his own paradigm shift, and this requires him to come face-to-face with deeply held personal beliefs about teaching and learning, and to face his own propensity for risk and initiative (Dirkes, 1993). Many teachers feel unprepared to take a problem-solving approach to teaching mathematics. Few teachers learned math themselves in this way. Even if they encountered problem solving in their college methods courses, once in the classroom, they often conform to the conventional methods that hold sway in most schools. Being an agent of change, when one is surrounded by deeply ingrained beliefs about teaching and learning, is a difficult role to perform. Teachers today are often caught between daily pressure from colleagues, parents, and others to uphold tradition in the classroom, and pressure from policymakers to employ standards-based practices (with the conflicting expectation that students will perform highly on standardized tests that measure basic skills, not performance of standards-based material).

A teacher's path to change must begin with an acknowledgment of her previous experience. She will build on her past experiences by reflecting on them in light of new ideas about effective teaching strategies (Richardson, 1990). Broadening teachers' conceptions of the nature of problem solving and its potential as an instructional tool requires that they, too, engage in solving open-ended problems. This means spending time solving a wide variety of problems and reflecting on their attempts to solve them.

Changing one’s practice is further facilitated when effective teaching techniques are modeled in the classroom by a practitioner who is skilled in problem-solving instruction. This modeling should be followed by a discussion among the teachers about the selection and use of strategies. Modeling and discussion provide concrete illustrations of the teachers' role in teaching problem solving (Richardson, 1990). Reading literature on the theory and practice of problem-solving instruction can also influence teachers to make changes in their practice (Thompson, 1989). “Examining research inquisitively and skeptically,” writes Ball (1996), “teachers can seek insights from scholarship without according undue weight to its conclusions. They can use the broadly outlined reforms as a resource for developing inspired but locally tailored innovations.”

The truth is, teachers are constantly making changes to meet the changing needs of their students and to try out ideas they've heard from other teachers. Teachers establish their own voice of authority in defining what takes place in the classroom. The notion of authority plays a critical role in conceptualizing and advancing mathematics teacher change (Wilson & Lloyd, 2000). Teachers themselves must be involved in making judgments about what change is worthwhile and significant (Richardson, 1990).

In pursuing reform goals, teachers often feel anxious about their effectiveness and knowledge. Moving in the direction of math reforms means confronting up close the uncertainties, ambiguities, and complexities of what “understanding” and “learning” might really mean. When we ask students to voice their ideas in a problem-solving context, we run the risk of discovering what they do and do not know. Those discoveries can be unsettling when students reveal that they know far less than the teacher expected or far more than the teacher is prepared to deal with (Ball, 1996).

Inquiry- and problem-based teaching requires qualities beyond mathematics knowledge and skill. Personal qualities, such as patience, curiosity, generosity, confidence, trust, and imagination, matter a great deal. Interest in seeing the world from another's perspective, enjoyment of humor, empathy with confusion, and concern for the frustration and shame of others are other important qualities that can help a teacher create a learning environment that fosters students’ problem-solving abilities (Ball, 1996). “As teachers build their own understandings and relationships with math, they chart new mathematical courses with their students. And, as they move on new paths with students, their own mathematical understandings change,” Ball writes.

Problem Solving as an Instructional Strategy

Polya (1945) suggests that problem solving consists of four phases: *understanding the problem, devising a plan, carrying out the plan,* and *looking back*. Lajoie (1992) defines mathematical problem solving as: “modeling the problem and formulating and verifying hypotheses by collecting and interpreting data, using pattern analysis, graphing, or computers and calculators.” This definition focuses on the processes of formulation, investigation, and verification, but it does not encompass the important elements inherent in Polya’s *looking back* phase, which involve evaluating and interpreting methods and results. The *looking back* phase includes such activities as:

- Verifying the result
- Checking for alternative methods of solution
- Determining the validity of an argument
- Applying the result or method of solution to other problems
- Interpreting the result
- Generalizing the solution
- Generating new problems to be solved

Looking back may be the most important aspect of teaching problem solving because it provides students the opportunity to learn about problem-solving processes and how a problem is related to other problems. Schoenfeld (1985) and others have shown that the principal traits that separate expert from novice problem solvers are their ability to see past the surface features of problems to their common underlying structures, and their ability to self-monitor and recognize when an approach or tactic is not being productive. Although teachers and researchers report that it is difficult to develop a willingness in students to continue past finding the correct answer to a problem, the development of self-awareness and reflection are critical for improving problem-solving ability.

Teaching the Key Traits of Problem Solving

According to current research and literature on problem solving, there are some key traits students will exhibit when they are performing at a high level of problem solving. The NWREL Mathematics Problem-Solving Model™ is based on the following research-based traits: conceptual understanding, strategies and reasoning, communication, computation and execution, and mathematical insights. The traits are described more fully below, followed by a key question teachers can reflect on when assessing students’ problem-solving abilities, as well as prompts teachers can give students to support their problem-solving attempts.

Conceptual understanding—Students demonstrate conceptual understanding by interpreting the mathematical principles in a problem and translating those ideas into a coherent mathematical representation using the important facts of the problem. Students show good conceptual understanding of the mathematics in a problem when they choose appropriate representations, use relevant information, use mathematical terms precisely, and select applicable mathematical procedures (Hiebert et al., 1997; NCTM, 1989, 2000; Schoenfeld, 1989, 1992; Greenwood, 1993).

Key question: Does the student's interpretation of the problem, using mathematical representations and procedures, accurately reflect the key mathematical concepts?

Prompts a teacher might give to help students with interpreting the problem's information include:

- What is the problem about? Rewrite the problem in your own words.
- What do you know?
- What is the problem asking you to find?
- What are the important facts and numbers in the problem? Is some of the information unnecessary in solving the problem?
- What math terms will help you understand and solve the problem?
- What will the answer look like (units of measure, level of accuracy required, form of the answer)?

Prompts to help students understand the problem's mathematical concepts might include:

- What types of computation will be required to do the problem?
- How can you represent the problem to help make it easier to understand?
- What mathematical ideas and skills could help you to represent and solve the problem (e.g., graphing, identifying patterns, adding fractions, etc.)?

Strategies and reasoning—Students demonstrate their ability to use strategies and reasoning by investigating and selecting appropriate problem-solving strategies and conducting a logical, well-planned, and supported process that leads to a reasonable solution. All forms of representations are consistent and integrated into their solution, progress is self-monitored and adjustments are made as needed, and work is verified or a proof of its correctness is provided (Hiebert et al., 1997; NCTM, 1989, 2000; Schoenfeld, 1989, 1992; Greenwood, 1993; Becker & Shimada, 1997; Polya, 1945, 1962-65; Stacey & Groves, 1985).

Key question: Is there evidence that the student proceeded from a plan, applied appropriate strategies, and followed a logical and verifiable process toward a solution?

Prompts to help students get started might include:

- Would drawing a picture or diagram or making a model be helpful in solving this problem?
- Would it be helpful to organize your information in a chart or table or in an organized list?
- Would guessing, checking, and adjusting be helpful for solving this type of problem?
- Should you be looking for patterns in your information?
- Would it help to first change the problem using simpler numbers?

- Can you work backwards from where you want to end up to where you want to begin?

Prompts to help students think about their solution might include:

- Is the strategy you used efficient? If not, could you find a more efficient way to solve the problem?
- Can you give examples to support your solution?
- Do you understand your plan/strategy well enough to explain it to someone else?
- Are there other ways to approach this problem that might work?
- Is this problem like others you have solved? Can you apply what you've learned to other problems?

Communication—We understand something if we see how it is related or connected to other things we know. Communication works together with reflection to produce new connections and relationships. Students who reflect on what they do and communicate with others about it are in the best position to build useful connections in mathematics (Hiebert et al., 1997). Students demonstrate good communication when they describe clearly what they did and why they did it; their explanation flows in a logical, fluent sequence. Good communication is direct, purposeful, and well-organized. The reader is not required to make inferences because the explanation is clear and contains no gaps (Hiebert et al., 1997; NCTM, 1989, 2000; Schoenfeld, 1989, 1992; Manouchehri & Enderson, 1999; Van Zoest & Enyart, 1998; Buschman, 1995).

Key question: Can one easily understand the student's thinking, or is it necessary to make inferences and guesses about what the student was trying to do?

Good communication depends on having a well-organized and clearly articulated solution plan or strategy. Having students explain their strategies orally before writing them can be helpful in developing skill in the trait of communication.

Prompts to help students communicate their thinking might include:

- Should you use tables, graphs, pictures, words, or a combination of these in explaining or expanding your thinking?
- What did you do first? Why? What did you do next? How did that help you toward your goal?
- How did you figure out ____________? What did you learn from doing this problem?
- Did you show how you verified your answer?
- Read your explanation to someone else to make sure it explains your process clearly and is easy to understand.

Computation and execution—Basic skills and conceptual understanding should develop together. To learn skills so that one remembers them, can apply them when needed, and can adjust them to solve new problems, one must learn them with understanding. If students are asked to develop their own procedures for calculating answers to arithmetic problems and to share their procedures with others, for example, their mathematical understanding will be fostered through executing, discussing, and reflecting on each other's ideas (Hiebert et al., 1997; NCTM, 1989, 2000; Schoenfeld, 1989, 1992). Students show mastery of computation by accurately executing all procedures, correctly applying and labeling all visual representations of the problem (charts, tables, graphs, etc.), and demonstrating the correct use of available technology or manipulatives.

Key question: Given the approach that the student took to solve the problem, is the solution (including all steps of the process) performed in an accurate and complete manner?

Prompts to aid students in improving their computational skills might include:

- Did you double-check your calculations as you went? (Remember to always estimate your answer when using a calculator.)
- Did you show the rule or formula you used?
- Did you verify your answer was correct either by solving the problem in a different way or by plugging your answer into the problem to check if it makes sense?
- If applicable, did you check your graphs and charts to make sure they are properly labeled?
- Are you competent with the calculations and algorithms required by the problem? If not, you should review and practice them before attempting the problem.
- Did you check to make sure your answer fits what the problem was asking for?

Mathematical insights—Students show insight into a problem when they recognize the significance of the problem in its relationship to other problems or in its connections to other disciplines or "real-world" applications. By recognizing patterns embedded in the problem, discovering multiple approaches and/or solutions, or creating a general rule or formula, students demonstrate insight into the underlying structure of the problem (Hiebert et al., 1997; NCTM, 1989, 2000; Schoenfeld, 1989, 1992; Becker & Shimada, 1997; Dirkes, 1993; Polya, 1945, 1962-65).

Key question: Does the student grasp the deeper structure of the problem and see how the process used to solve this problem connects it to other problems or "real-world" applications?

The key to developing students' insights is to get them to go beyond the solution of the problem and to think about the implications of the problem in other situations.

Prompts to improve insights might include:

- Is this problem like any other problem you have seen? If so, how are they similar?
- Did you discover any patterns while you were solving the problem?
- What assumptions did you make in solving the problem?
- Could you create a problem that was like this problem in some ways and different in others?

- Is your solution the only one that will work for this problem?
- Can you find a process (or formula) that could be used to solve all forms of this problem?
- How is this problem similar to other problems you have seen or to situations in real life?

Selecting "Good" Problems

NCTM and the research community consistently stress that students should grapple with challenging and unfamiliar problems (NCTM, 1989; Schoenfeld, 1985, Hiebert et al., 1997; Smith & Stein, 1998). Choosing "good" problems is key to effective problem-solving instruction, but teachers are often confused about what constitutes a good problem. The *Oxford American Dictionary* (Ehrlich & Flexner, 1980) gives the following definitions for the word *problem*:

Definition 1: Something difficult to deal with or understand.
Definition 2: An exercise in a textbook or examination.

Both of these meanings are in common usage. A teacher might tell a class to do problems one through 60 at the end of the chapter for practice. This is clearly an example of the use of the second definition because, if the teacher were referring to *problem* in the sense of the first definition, he would likely have a mutiny on his hands, since doing 60 truly difficult problems would be an unrealistic homework assignment.

Another important aspect of the definition of *problem* is its relative nature. What constitutes a problem for one student may be merely an exercise for another. Schoenfeld (1985) captures this relative nature of problems when he says:

> [B]eing a "problem" is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual and the task that makes the task a problem for that person. The word problem is used here in this relative sense, as a task that is difficult for the individual who is trying to solve it. Moreover that difficulty should be an intellectual impasse rather than a computational one.

Because of the relativity of problems, it is important that when teaching and assessing students' problem solving, tasks should be carefully chosen so that they are at the correct level of difficulty for the students. Problems need to be hard enough to present a challenge without being so difficult as to be a total mystery.

Polya stresses that to know mathematics is to be able to do mathematics (1965-69), meaning solving difficult problems. The aim of schooling is to improve students' ability to think. While solving routine problems can serve to teach students how to apply specific procedures, only through the judicious use of nonroutine problems "requiring some degree of independence, judgement, originality, creativity" can they develop their problem-solving abilities. Halmos (1980) states in his article "The Heart of

Mathematics," that "the mathematician's main reason for existence is to solve problems and that, therefore, what mathematics really consists of is problems and solutions." This view is far removed from the vision typically portrayed in school mathematics classes, that mathematics is a static, finite (though very large) set of facts, rules, and procedures that are meant to be memorized and practiced.

Teachers will want to distinguish between 1) using problem solving as a context for teaching concepts and procedures, 2) teaching problem-solving strategies, and 3) teaching and assessing the problem-solving abilities of their students. When assessing students' problem-solving abilities, students must be presented with unfamiliar tasks, those for which they have not learned a predetermined procedure or algorithm. The tasks should be within the students' ability to solve, but difficult and challenging to them.

In fact, tasks form the foundation of problem-solving instruction. Reflection and communication are possible only when the tasks are appropriately problematic. According to Hiebert (Hiebert et al., 1997), appropriate tasks have three features:

- ♦ Tasks make the subject problematic for students; students see the task as an interesting problem.
- ♦ Tasks must connect with where students are at in their understanding; students must use the knowledge and skills they already have to begin developing a method for completing the task.
- ♦ Tasks must offer students the opportunity to reflect on important math ideas.

Students develop mathematical understanding as they invent and examine methods for solving mathematical problems. The task makes all the difference in a student's ability to construct understanding (Hiebert et al., 1997).

"Good" problem-solving tasks:

- ♦ Engage and interest students
  - ✓ Apply to the real world
  - ✓ Connect to student interests
  - ✓ Are equitable in that they appeal to *all* students
  - ✓ Promote active involvement

- ♦ Contain important mathematical content
  - ✓ Connect to other problems and mathematics concepts
  - ✓ Align with current mathematics curriculum
  - ✓ Integrate other subject areas

- ♦ Are open-ended and nonroutine
  - ✓ Allow multiple approaches and solutions
  - ✓ Are not readily solvable by using a previously taught algorithm

- ♦ Are challenging but accessible to students

- ✓ Require persistence
- ✓ Allow entry to the problem

- ♦ Are well-crafted
  - ✓ Contain clear and unambiguous wording
  - ✓ Describe expectations
  - ✓ Elicit responses that can be scored

Classroom Environment

Classroom environment shapes students' beliefs about math, as do cultural beliefs and interactions with others. “If we are to understand how people develop their mathematical perspectives, we must look at the issue in terms of the mathematical communities in which students live and the practices that underlie those communities,” writes Schoenfeld (1992). Three features of a social culture that encourage students to treat tasks as interesting and worthwhile mathematical problems are:

- ♦ Ideas are the currency of the classroom, having potential to contribute to everyone's learning and warranting respect and response.
- ♦ Students have autonomy with respect to the methods used to solve problems, recognizing that there are a variety of methods to do the job. The freedom to explore alternative methods and to share their thinking with their peers encourages creativity and increases motivation.
- ♦ Mistakes are seen by the students and teacher as places that afford opportunities to examine errors in reasoning, raising everyone's analysis. Mistakes are to be used constructively as opportunities to learn.

Much research on the effectiveness of so-called alternative programs that emphasize problem solving has taken place at the primary-grade level in arithmetic. Hiebert (1999) analyzed this research and identified common features that characterize many of the programs. The programs:

- ♦ Build on students' prior knowledge and skills
- ♦ Provide opportunities for both invention and practice
- ♦ Focus on the analysis of multiple methods for solving problems
- ♦ Ask students to provide explanations
- ♦ Emphasize conceptual development without sacrificing skill development
- ♦ Emphasize learning new concepts and skills while solving problems

Rather than being the primary evaluator of correctness and source of information, the teacher focuses on selecting and sequencing appropriate problems, sharing information, and fostering a classroom culture in which pupils work on novel problems individually and interactively. The teacher relies on the reflective and conversational problem-solving activities of the students to drive their learning (Hiebert et al., 1997).

The dilemma of when to intervene without interfering in students' constructing their own understanding will always be an aspect of teaching, especially when using a problem-solving approach. Teachers can intervene in ways that stimulate and push students' thinking forward and, at the same time, promote students' autonomy by:

- Selecting tasks with goals in mind (knowledge of mathematics and understanding of students' thinking are essential to selecting appropriate tasks)
- Providing relevant information (math conventions, alternative methods, articulating ideas in students' methods)
- Guiding the development of classroom culture (focus on methods, adopt appropriate position of authority) (Hiebert et al., 1997)

Conclusion

Changing mathematics education will require restructuring the culture of the mathematics classroom and the epistemological perspective of mathematics teachers. Widely held beliefs—such as that mathematical ability is innate and therefore beyond the reach of some people, that coverage of content and accumulation of mathematical facts are more important than teaching students to think and solve problems, and that the most we can expect of students is that they *do* mathematics, not *understand* it—profoundly influence what and how mathematics is taught in this country. Students and teachers still share the belief that problems need to be solved quickly and that, in order to solve a problem, a student must have seen similar types of problems being solved. Seldom are students required to "invent" a process for solving a problem or to pose their own problems based on their evaluation of a situation or data. To do so would represent a significant raising of standards and expectations, aiming to develop higher-order thinking skills and confidence in students' abilities to solve authentic problems.

Often, teachers mistakenly correlate problem solving with word problems. But presenting a word problem designed to provide an application or context for learning mathematical operations, procedures, or concepts is unlikely to involve students in meaningful problem solving. While word-problem exercises can help prepare students to solve problems, they do not provide actual practice in what Polya (1945) refers to as the art of problem solving. Students seldom have opportunities to experience the creativity of investigation and discovery inherent in rich, nonroutine, and intellectually challenging problems. As a result, many students view mathematics as a routine, mundane, static set of facts and rules to be learned primarily through memorization rather than as an evolving, expanding science of inquiry and experimentation that is discovered and created through experimentation and conjecture. The latter view is both more inspiring, relevant, and interesting to most students, as well as less imposing and intimidating.

As Stacey (1990) points out, "good problem solvers need to be resourceful, flexible, confident and willing to explore." They also must learn persistence and the ability to tolerate a certain amount of frustration. To develop these abilities, students need to experience the frustration and exhilaration of struggling with and overcoming a daunting intellectual obstacle.

Students in mathematics classes that do not emphasize problem solving are being deprived, as well, of the feelings of exhilaration and empowerment that come from mastering a difficult problem. They are not developing the tools and the confidence they will need to tackle the types of problems that will occur in their working and personal lives. They often fail to gain a deeper conceptual understanding that comes from constructing one's own mathematical truths through deep thinking.

Kyle Forman, an eighth-grade student in Portland, Oregon, who is working in a problem-centered curriculum, expresses insightfully the benefits of learning math from the problem-solving perspective:

> Learning math is a journey full of freeways, detours, dead ends, and side trips. The journey is what we learn from, where we gain the confidence and knowledge needed to succeed as mathematicians. The journey is more important than the destination. In fact, the journey never ends. Any idea can be expanded and we can always search further.
>
> Our teacher trusts us as capable mathematical thinkers who can find our way … She knows that difficulties can temporarily impede our progress and she knows that we learn from the experience. The next time we venture down a path may be quicker or may take longer because of new side trips and discoveries (Foreman, 1998).

For a student whose metaphor for mathematics is a journey and who views its difficulties as merely temporary setbacks, there are no limitations to where the journey might take him or her.

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