Malati

Mathematics learning and teaching initiative

Statistics

Probability 2

Grades 8 and 9

Teacher Document

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Guidelines for Module: Probability 2

We would like learners who have worked through this module to display the following:
1. To see the need for a systematic representation
2. To be able to make the systematic representations (tree diagram, grid)
3. To use the tree diagram / grid to find probabilities.

Pre-requisite Knowledge: Working knowledge of fractions as described for Probability 1
Understanding of the notion of chance
The ability to express probability using fractions.

The Activities:
CORE:
Ice-cream Choices 1
Choosing Clothes 1
Choosing Clothes 2
Playing a Game with Dice
Another Game with Dice
Ice-cream Choices 2
Ice-cream Choices 3
Probabilities with Dice
Probabilities with Dice and Coins (assessment?)

CONSOLIDATION:
- Drawing tree diagrams: The teacher can change the numbers in existing activities, or set new activities using different contexts (see ideas in teacher notes)
- Using tree diagrams and grids to find probabilities – changing contexts / rules of the game (see teacher notes)

Catering for Diversity in the Classroom:
Recognising the need for a systematic representation:
It is important that a learner sees the need for using a systematic representation (whether this is in the form of a list or tree diagram of some sort). A learner who does not recognise this will struggle to construct an accurate tree diagram.

The first question in the activity "Ice-cream Choices 1" requires that learners find all the possible combinations to make an ice-cream "combo". The teacher should use learner responses to this second activity and the discussion that follows to assess whether learners have recognised the need for a systematic representation. Which learners are still listing the combinations randomly?

For those learners who do not see the need for systematic representation, it is recommended that the teacher use the same context (ice-creams), but change the numbers, for example, by increasing the number of cones, ice-creams or toppings. It is hoped that when faced with the greater number of possibilities, the learner will realise that random listing is not an efficient method.

The remaining learners can also benefit from these further activities as they will have to adapt their existing representations and, given further opportunities, might recognise a mathematical formula for calculating the number of representations.

Constructing tree diagrams and grids correctly:
In these materials the tree diagram and the grid are used to calculate probabilities, so it is important that a learner is able to construct these structures correctly. The activity "Choosing Clothes 1" requires that the learners construct a tree in a different context to the ice-cream activity. The teacher can set further activities using different contexts if it is felt that further information on learners needs to be gathered. Some contexts are sandwiches, pizzas and choices of school subjects and/or extra-murals.

Remediation: Some strategies for dealing with this problem are:
• Encouraging learners to compare the number of combinations obtained from a tree diagram with that obtained by other means, for example, a grid or by listing
• Encouraging learners to compare their representations.

Other Assessment: The learners' ability to calculate probabilities using the tree diagram or grid can be assessed at the end of this module and remediation provided where necessary. Extension activities can focus on counting problems (see Module 3).

Ice-cream Choices 1

MENU

Cones

Plain   Chocolate

Ice-cream Flavours

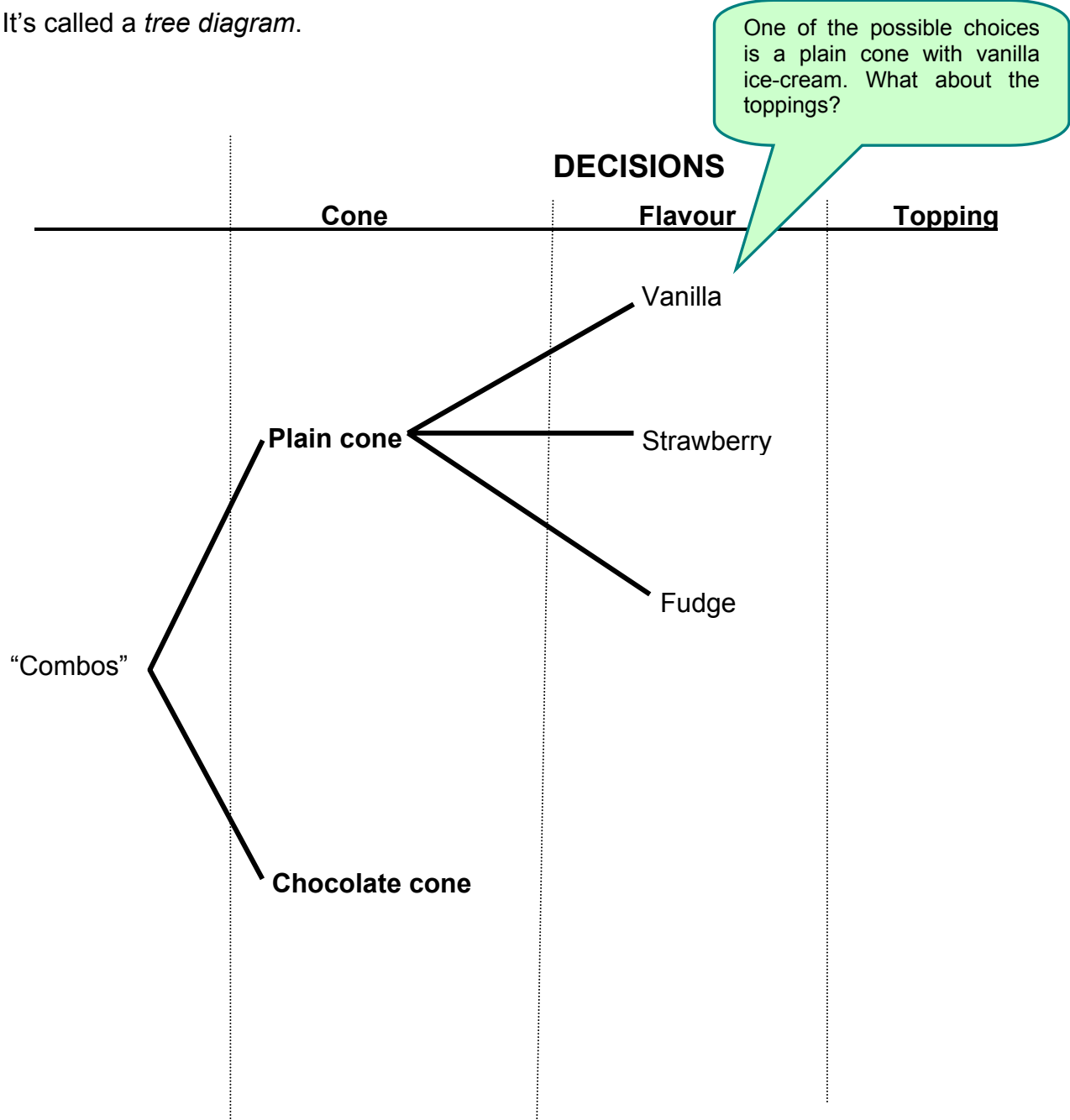Vanilla   Strawberry   Fudge

Toppings

Nuts   Jelly tots   Banana

This is the menu in an ice-cream shop. The shop sells "combos" which consist of one cone, one scoop of ice-cream and one topping. For example, you can order a "combo" with a plain cone, strawberry-flavoured ice-cream and nuts on top.

What is another possible "combo" you can order?

1. How many different "combos" can one order at this ice-cream shop? Show how you got your answer.

This is one way of representing all the possible choices of "combo".

It's called a *tree diagram*.

One of the possible choices is a plain cone with vanilla ice-cream. What about the toppings?

DECISIONS

| Cone | Flavour | Topping |

Vanilla

Plain cone

Strawberry

Fudge

"Combos"

Chocolate cone

2. Complete the tree diagram, adding all the missing branches and including the different toppings.

3. Use your tree diagram to answer the following questions:

   (a) How many different "combos" can one order which have banana on top?

   (b) Andile does not like chocolate. How many different "combos" can she order?

   (c) Denise likes neither strawberry nor fudge ice-cream. How many different "combos" can she order?

   (d) Phumeza does not like nuts. How many different "combos" can she order?

   (e) Lungi does not like chocolate or nuts. How many different "combos" can she order?

4. During the holiday season the owner of the ice-cream shop decided to add to the menu: She added lemon-flavoured ice-cream to the ice-cream flavours and Flake to the toppings.

   Explain how you can find the number of "combos" you can now order from the ice-cream shop. You may make a list or draw a tree diagram to check your answer.

Teacher Notes: Ice-Cream Choices 1

Question 1 of this activity deliberately avoids suggesting any form of systematic representation. **It is important that pupils are only given question 1 to start with and that they decide on their own form of representation before seeing the tree diagram.**
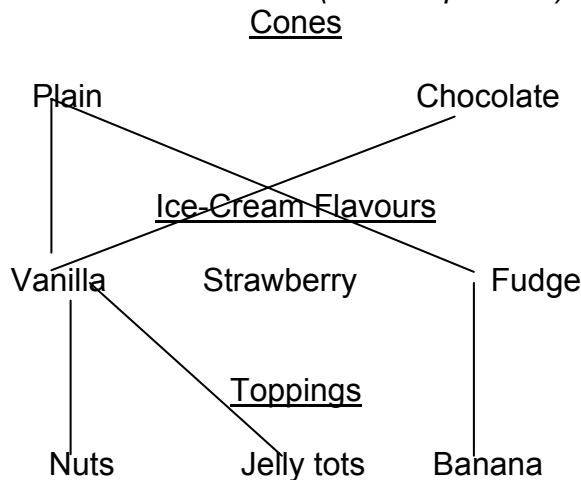
If pupils are struggling to understand the problem and/or interpret the menu, they should be asked to give examples of "combos" – what would they choose at this ice-cream shop?

Pupils will use different methods of representing all the different "combos". For example, pupils may simply write the "combos" randomly as follows:

> *plain cone, vanilla ice-cream, nuts*
> *plain cone, fudge ice-cream, banana*
> *chocolate cone, vanilla ice-cream, jelly tots etc.*

Pupils might get tired of writing out all the possibilities in full each time – this is time-consuming and they can be encouraged to abbreviate, for example, use PVN, PVJ etc.

Younger pupils are inclined to 'match' words (or even pictures) with lines, for example:

<u>Cones</u>

Plain   Chocolate

<u>Ice-Cream Flavours</u>

Vanilla   Strawberry   Fudge

<u>Toppings</u>

Nuts   Jelly tots   Banana

While solving Question 1, pupils should experience a <u>need to be systematic</u> about indicating these combinations. If pupils do not show any tendency towards the need for systematic representation, they can be challenged by asking them how they know that all the possibilities have been mentioned. A whole-class discussion should result in a systematic way of listing these combinations as pupils will start to notice repetitions and omissions.

It is unlikely that pupils, especially younger pupils, will use the tree diagram spontaneously. In this case the teacher can introduce it as a mathematical way of representing all the possible outcomes (Question 2). Then the pupils are encouraged to complete the diagram and count the 18 different "combos".

It is important that pupils realise that each 'level' on the tree diagram represents a different decision – for EACH choice of cone there are three choices of ice-cream flavour, and for EACH of these there are three choices of topping. Thus there are two groups of three groups of three combos (2 x 3 x 3) = 18. It is important that the teacher points out that the branches at each level do not represent separate "combos", but that one "combo" (e.g. plain cone, vanilla ice-cream and nuts) is represented across all three levels – the teacher should follow the branches across the various levels to demonstrate this. Pupils can then be asked to come and use colours to indicate on a tree diagram on the blackboard where, for example, the "combo" with chocolate cone, strawberry flavour and banana topping, is represented. They must show the <u>whole</u> series of branches and not just the last one (topping).

The effect of cancelling out different 'branches' of the tree diagram is then investigated … not having a chocolate cone cancels out one half of the tree diagram while not having banana on top cancels only a third.
By counting the "combos" on the tree diagram, pupils should complete the questions and validate their answers in the groups:
*(a) 6 with banana on top*
(b) excluding the combos with chocolate cone, there are 9 possible combos
(c) two groups of 3, or 18 – 12 gives 6 combos
(d) six groups of 2, or 18 - 6 gives 12 combos
(e) only the 'plain cone' half of the tree diagram can be considered, and only the combos without nuts, in other words (18 - 9) - 3 or 3 x 2 gives 6.

The teacher should encourage the pupils to reflect on the difference between eliminating two options at the same decision level (2(c)) and eliminating two options at different decision levels (2(e)).

In Question 3, pupils should be encouraged to predict the number of "combos" by finding a mathematical way of doing it, before verifying using the tree diagram or listing. While constructing the tree diagram to check their answers, some pupils might show 32 different combinations by looking at a new (extended) tree diagram, while others will add extra branches to their existing tree diagram.

If pupils have finished and are waiting for their peers, the teacher can ask them to add/take away an item of the menu at any of the levels and work out how many combinations are now possible.

Choosing Clothes 1
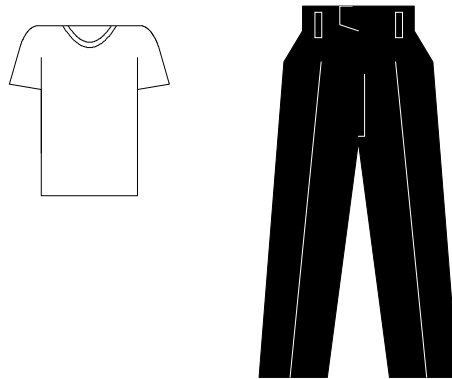
Siyanda has a white, a yellow and a red shirt and she has black, green and navy jeans.

1. Draw a tree diagram to show how in many different ways Siyanda can dress.

2. Siyanda loses her black jeans, and decides to buy two hats instead – a striped hat, and a spotty hat. In how many ways can she dress now?

Teacher Notes: Choosing Clothes 1

This activity provides an opportunity to create a tree diagram to represent all the 9 different combinations. Even if pupils immediately know the number of combinations, they should be encouraged to draw the tree diagram.

Question 2 can be calculated using the tree diagram and adding the hats while taking away the black jeans. Some pupils may formulate a mathematical method (3 x 2 x 2), which they should be encouraged to share. However, if no pupil has formulated this, the teacher should not be concerned as it is addressed in the next activity.

Choosing Clothes 2
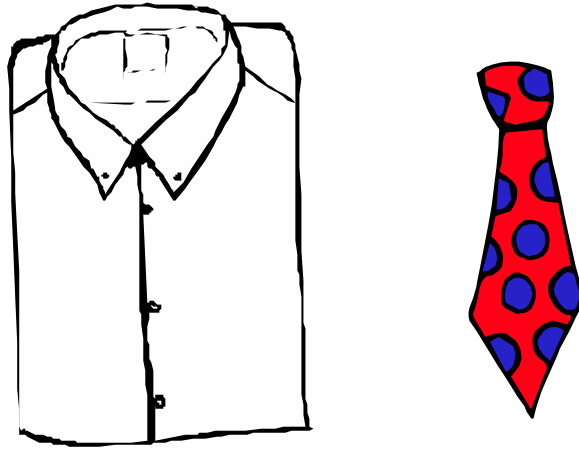
Mzwake has five different shirts and three different ties to wear to work.

His wife says she has an easy way of working out how many different combinations of a shirt and tie he can wear. She says it is

"the number of shirts" × "the number of ties"

But Mzwake disagrees. He says the number of different combinations is

"the number of shirts" + "the number of ties".

1. Explain which of these rules you think is correct and why. You may make a list or draw a tree diagram to check your answer.

2. Does you rule work in Ice-Cream Choices 1 and Choosing Clothes 1?

Teacher Notes: Choosing Clothes 2

In this activity, pupils are encouraged to reflect on the mathematics behind the problem of "how many different combinations", and should be able to explain clearly why multiplication and not addition produces the correct number of combinations. Saying that "multiplication is correct because it gives the right answer" is NOT a sufficient response...the teacher should ask WHY is this the case.

For example, pupils might say that there are 15 combinations because for each shirt, there are 3 different choices of tie, and there are 5 shirts.

Pupils may use tree diagrams or other systematic representations to VERIFY their explanations.

The activity can be extended by changing the number of shirts and/or ties.

Playing a Game with Dice

This time you and your friend have two dice – each of you throws one of them, and then you add up the two scores.
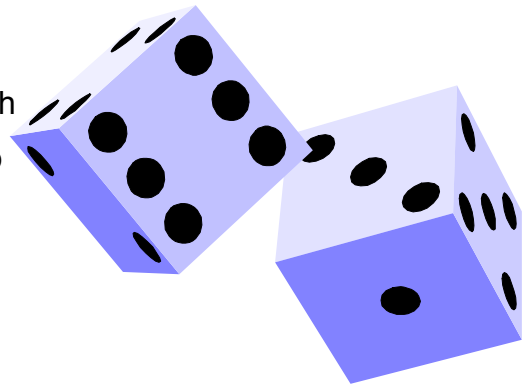
Player A scores a point if the sum is **odd**.
Player B scores a point if the sum is **even**.

1. We can use a grid to represent all the possible results (outcomes) of throwing two dice. Use this grid to represent the results that would lead to Player A winning, and the results which would lead to Player B winning. One winning total for 'Player A' is already shown on the grid …

Dice 1

|					
6					
5					
4					
3					
2		A			
1					

1 2 3 4 5 6

Dice 2
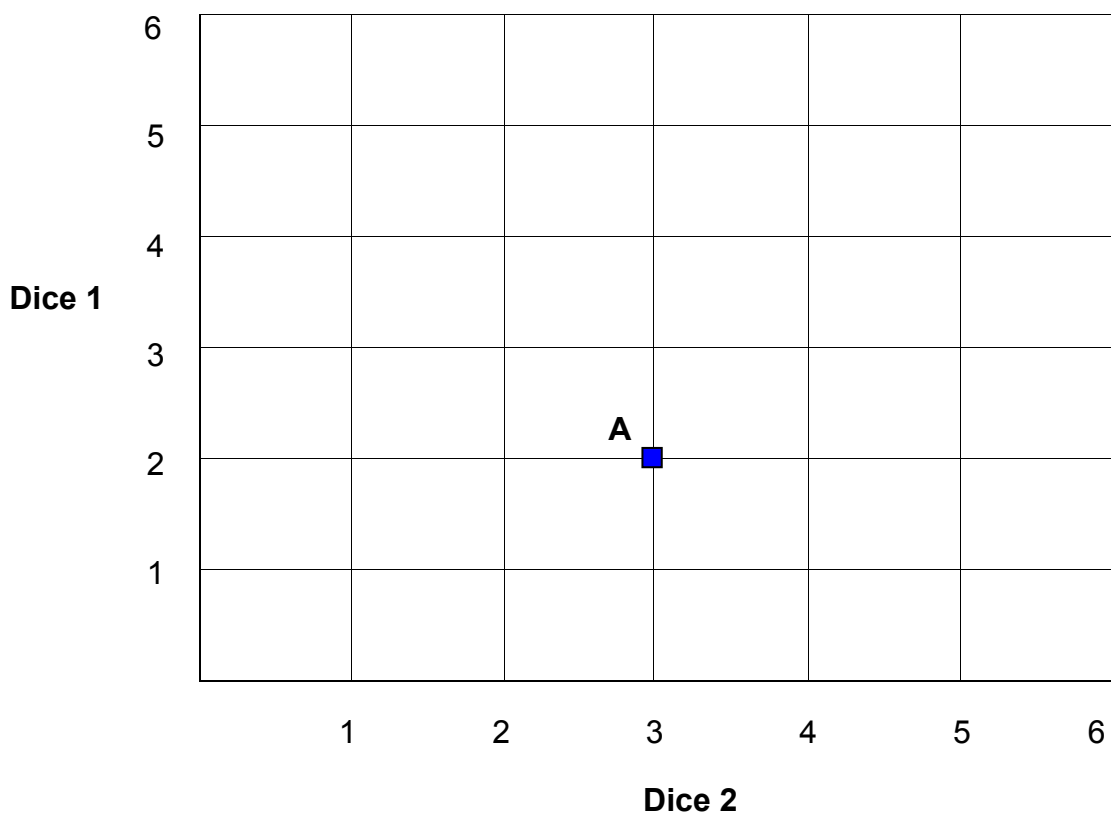
(a) Complete the grid.

(b) Which player would you want to be if you wanted to win this game? Explain.

(c) What is your chance/probability of winning if you are Player A? And if you are Player B? Discuss!

2. Let's change the rules …

Player A now gets a point if the *product* of the two scores (the two scores multiplied) is **odd**, and **Player B** gets a point if the product of the two scores is **even**.

(a) Use this grid to represent the results that would lead to Player A winning, and the results which would lead to Player B winning. One winning total for Player B is already shown.

Dice 1

6

5

4

3

B

2

1

1   2   3   4   5   6

Dice 2
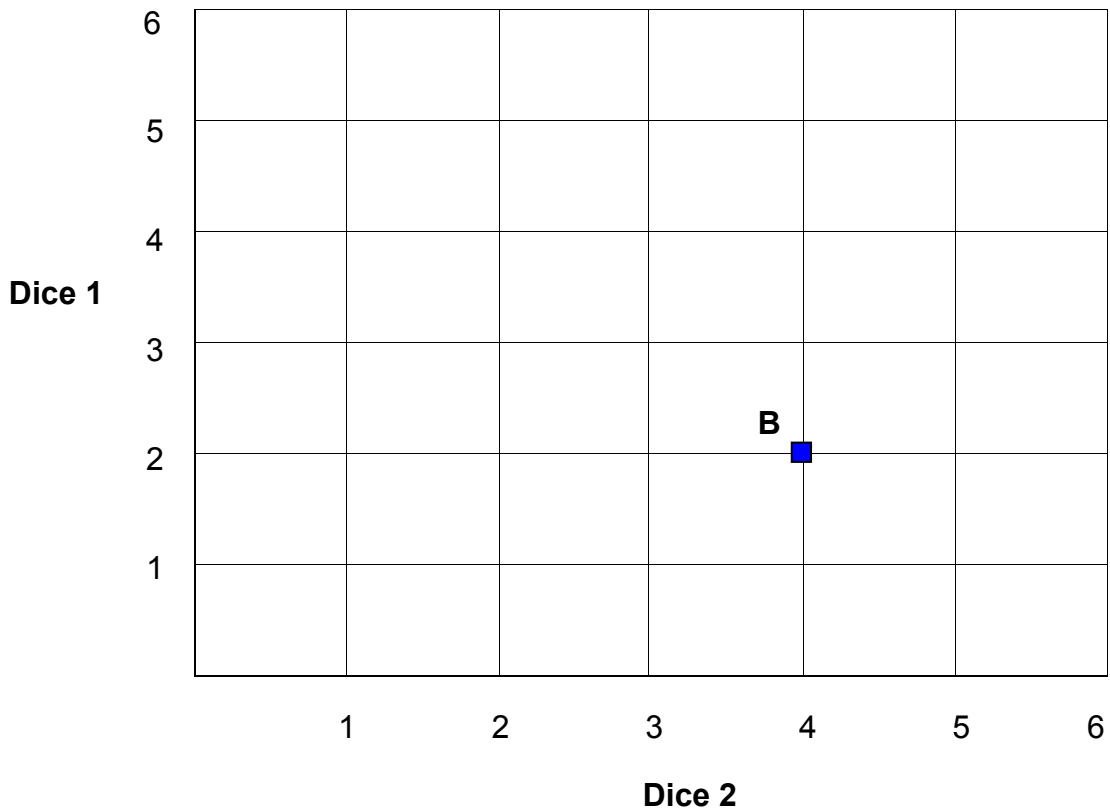
(b) Is this a fair game now? Which player would you want to be to win this game? Explain.

(c) What is your chance/probability of winning if you are Player A? And if you are Player B? Discuss.

3. Invent another game with dice which is *fair*, in other words in which their is *equal likelihood* of Player A and Player B winning? What would be the new rules of your game? How do you know that this is a fair game?

4. Now invent a game with dice in which Player B has no chance of winning …
   Player B's winning must therefore be an *impossible outcome*. What would be the new rules of your game?

Teacher Notes: Playing a Game with Dice
This game is based on predictions only and dice should only be used if pupils are unfamiliar with the dice or if they do not understand that the chances of a dice landing on each of the sides is equally likely. Pupils do NOT actually play these games.

It is essential that pupils read the rules of the game before commencing. One way of ensuring this is to ask pupils to explain these rules to each other.

*In this activity it is necessary to record the outcomes of events systematically. Pupils are given the **grid** to do this, but they should be encouraged to also use other systematic ways of representing the outcomes, if they feel more comfortable calculating probability in a particular way. It is, however, important that all pupils understand how the grid works. They should be encouraged to reflect on the grid in Question 1 and discuss why they think the "A" has been placed in that spot before completing it. The grid provides an important link with co-ordinate geometry.*

It is also acceptable if they fill in the A or B in the blocks, if the values on the axes are moved accordingly.

*Having completed the grid, the pupils should count that there are an equal number of A's and B's. Both have and $\frac{18}{36}$ or 50% chance of winning. Thus the players have an equal chance/likelihood of winning. In Question 2, the pupils will have to read the question carefully again. Having completed the grid, they will see that Player B has a greater chance of winning ($\frac{27}{36}$ as compared to Player A's $\frac{9}{36}$). It is thus not a fair game. The concepts **possible/impossible events** are revisited. Pupils should be encouraged to convince their friends that their games are suitable, by for example, using a grid to show possible outcomes. The teacher should encourage them to check whether their invented games are fair (Question 3) or give Player B 0% chance of winning (Question 4).*

Pupils should be encouraged to reflect on the use of the grid as a tool for working out probabilities.

Further activities for enrichment and consolidation (and/or homework) can be designed based on the pupils' own games, or on other games using, for example, the sum of the two numbers. Pupils can also be asked to reflect on the effect of changing one of the dice using different geometrical shapes, for example to create a tetrahedron – how would this affect the grid and the probabilities of winning?

<u>*Source of ideas:*</u>
*Burns, M. (1992) **About Teaching Mathematics: A K-8 Resource**. Math Solutions Publications*

Another Game with Dice!

This time you have two dice, and you throw both of them. You really want to win this game. Which set of rules would you prefer?

1. You win if both the numbers thrown are even.
2. You win if at least one of the numbers thrown is even.
3. You win if only one of the numbers thrown is even.

What is the probability of each of these outcomes?

Teacher Notes: Another Game with Dice!

Pupils should be encouraged to represent the different outcomes systematically in each case. They can use tree diagrams, grids or any other systematic method.

This activity addresses composite events:
1. *probability of even AND even*
2. *probability of even and odd OR odd and even OR even and even*
3. *probability of even and odd OR odd and even.*

Pupils should be encouraged to reflect on ways of calculating the various probabilities. For example, they may notice that, in the case of "AND", the individual probabilities are multiplied, while in the case of "OR", the individual probabilities are added. They should not, however, be asked to memorise this at this stage.

Ice-cream Choices 2

Remember this ice-cream shop menu?

A customer can't make up his mind what "combo" to have, so he closes his eyes and chooses a cone and a flavour by pointing blindly. The shop owner then presents the customer with a "combo" with this cone and flavour.

Refer to your tree diagram in "Ice-Cream Choices 1" showing all the possible different "combos" the customer could order, or draw a new tree diagram to help you answer the following questions:

1. What is the probability that the customer orders an "combo" with a chocolate cone? Explain how you got your answer.

2. What is the probability that he orders fudge-flavoured ice-cream? Explain how you got your answer.

3. What is the probability that he orders a plain cone with vanilla ice-cream? Explain how you got your answer. (Hint: Count all the possible different ice-creams he could order).
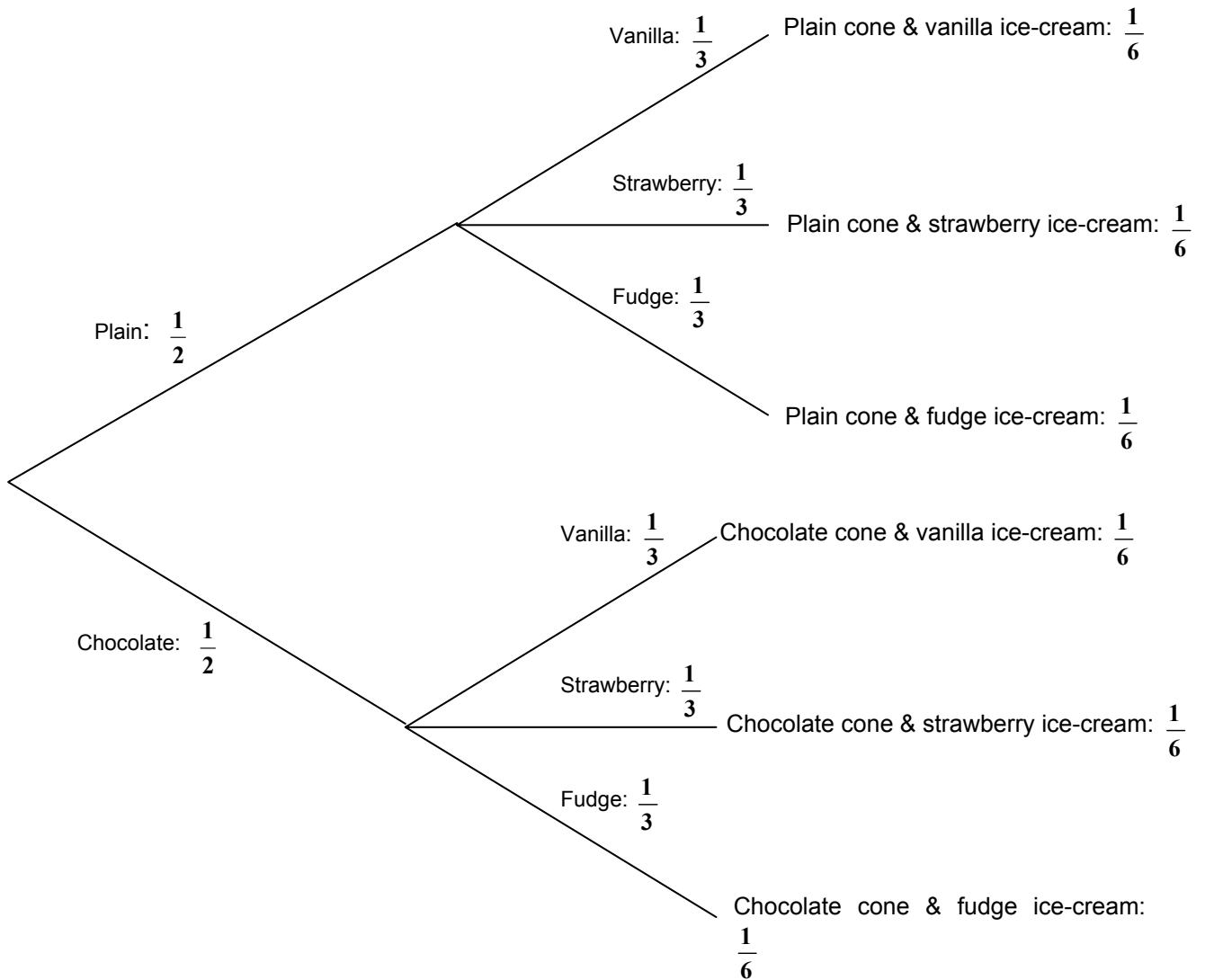
Teacher notes: Ice-Cream Choices 2

This activity encourages pupils to use their tree diagram to COUNT all the desired outcomes and all the possible outcomes, in order to calculate the probabilities. Pupils may give various equivalent answers, and should be encouraged to discuss their reasoning.

1. *there are 9 out 18 possible combos ($\frac{9}{18}$), or 3 out of 6 possible combos if one disregards the toppings ($\frac{3}{6}$), or 1 out of 2 possibilities ($\frac{1}{2}$ or 50%).*

2. *there are 8 out of 18 possible combos , or 1 out of 3 possible combos if one disregards the toppings ($\frac{1}{3}$ or 33,3%)*

3. *there are 3 out of 18 possible combos ($\frac{3}{18}$), or 1 out of 6 possible combos if one disregards the toppings ($\frac{1}{6}$)*

Pupils should be encouraged to reflect on the difference between probabilities of events at each level (i.e. cone, flavour, cone AND flavour).

Ice-Cream Choices 3

We can also show the various *probabilities* on the tree diagram. For example, this tree diagram shows the probabilities for all the possible choices of cone and ice-cream.

Vanilla: $\frac{1}{3}$ — Plain cone & vanilla ice-cream: $\frac{1}{6}$

Strawberry: $\frac{1}{3}$ — Plain cone & strawberry ice-cream: $\frac{1}{6}$

Fudge: $\frac{1}{3}$

Plain: $\frac{1}{2}$

Plain cone & fudge ice-cream: $\frac{1}{6}$

Vanilla: $\frac{1}{3}$ — Chocolate cone & vanilla ice-cream: $\frac{1}{6}$

Chocolate: $\frac{1}{2}$

Strawberry: $\frac{1}{3}$ — Chocolate cone & strawberry ice-cream: $\frac{1}{6}$

Fudge: $\frac{1}{3}$

Chocolate cone & fudge ice-cream: $\frac{1}{6}$

We have said that the probability of choosing a plain cone with vanilla ice-cream is $\frac{1}{6}$ because there are SIX possible choices of ice-cream flavour and this is ONE of them.

We can now also see on the tree diagram that this choice is $\frac{1}{3}$ of the possible choices with a plain cone, in other words $\frac{1}{3}$ **of** $\frac{1}{2}$ of all the possible choices:

$$\frac{1}{3} \times \frac{1}{2} = \frac{1}{6} \text{ !}$$

1. Extend the tree diagram to include the toppings, and fill in the probabilities on these branches. Use the tree diagram to answer the following questions:

   (a) What is the probability that the customer orders a chocolate cone?

   (b) What is the probability that he orders strawberry ice-cream?

   (c) What is the probability that he orders a chocolate cone with strawberry ice-cream? Explain how you got your answer.

   (d) What is the probability that he has strawberry ice-cream OR vanilla ice-cream? Explain how you got your answer.

   (e) What is the probability that he orders a plain cone AND vanilla ice-cream AND banana topping?

2. (Enrichment):
   What is the probability that the customer orders a plain cone OR vanilla ice-cream?

Teacher notes: Ice-Cream Choices 3

In this activity, pupils are encouraged to go beyond counting the branches of the tree diagram in order to calculate probabilities. Pupils' attention is focused on the fact that the probability $\frac{1}{6}$ can be calculated by <u>multiplying</u> $\frac{1}{2}$ by $\frac{1}{3}$. This may be a surprising result and pupils should be given the opportunity to explore and discuss. Pupils are given the opportunity to use the tree diagram to calculate probabilities.

To calculate the probability of a plain cone AND (with) vanilla ice-cream (1(c)), one multiplies the probability of a plain cone (1(a)) and the probability of strawberry ice-cream (1(b)). Pupils should be encouraged to check by counting branches if they are unsure, and to compare with their answers in Ice-cream Choices 2.

To calculate the probability of strawberry OR vanilla ice-cream (1(d)), one adds the probability of strawberry ice-cream ($\frac{1}{3}$) and the probability of vanilla ice-cream ($\frac{1}{3}$). Once again, pupils can confirm by counting 2 out of 3 possibilities of ice-cream flavour or 4 out of 6 possible cone-flavour combinations.

In 1(a) to 1(d), some pupils may include the toppings when counting the various options in order to calculate the probabilities. They should be encouraged to compare their answers and methods with those who used only the cones and flavours – the answers should be equivalent and the pupils can discuss why they think this is the case.

The probability required in 1.5 is $\frac{1}{18}$ which can be obtained by multiplying $\frac{1}{2}$ by $\frac{1}{3}$ by $\frac{1}{3}$. It is also clear from the tree diagram that this is ONE of 18 possible choices.

Pupils should not lose sight of the interpretation of the multiplication rule...the probability in 1(e) is derived from the fact that this choice is one third OF one third OF one half of all the possible choices.

Question 2 can be solved logically by including all the combinations with a plain cone (9) and another 3 combinations which have vanilla ice-cream but a chocolate cone. In terms of the mathematical rule, one adds $\frac{1}{2}$ and $\frac{1}{3} \times \frac{1}{2}$ to obtain $\frac{12}{18}$ or $\frac{4}{6}$ or $\frac{2}{3}$.

Probabilities with Dice
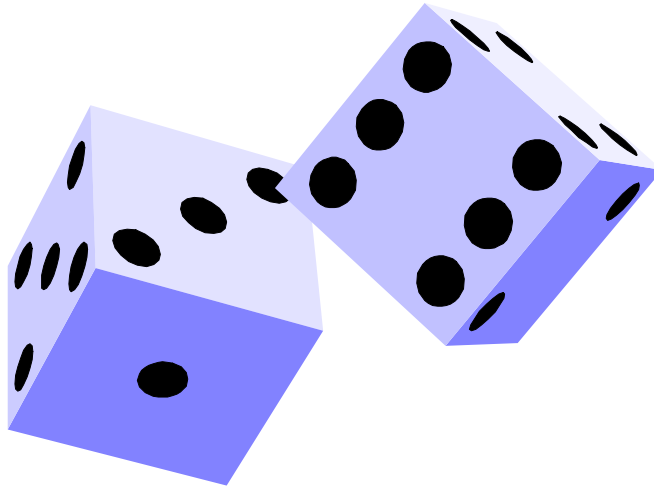
You throw two dice. One possible result is that you throw a '1' and a '4'.

1. Use a tree diagram to show all the possible results. How many different results could you get?

2. We can use a grid like the one we used in 'Playing a Game with Dice' to show *all the possible outcomes* when throwing two dice (also called the *sample space*). What do you think are the advantages and disadvantages of grids and tree diagrams?

3. Two dice are thrown together. What is the probability that both dice are sixes (that is, both dice land with a 6 on top)?

4. Give the following probabilities:
   (a) the probability that at least one of the dice is a 6.
   (b) the probability that neither dice is a six. What do you notice?
   (c) the probability that the numbers on the two dice are different.
   (d) the probability that the numbers on the two dice are the same. What do you notice?
   (e) the probability that the sum of the numbers on the two dice is an odd number.
   (f) the probability that the sum of the numbers on the two dice is an even number. What do you notice?

Teacher notes: Probabilities with Dice
This activity refers to the grid used for representing the sample space. It is important that pupils also do the tree diagram, so that they can compare using the different representations to work out probabilities.

The pupils may refer to events (such as 4(a) and 4(b) being 'opposite'. The teacher can then introduce some important terminology as soon as s/he feels that pupils are ready. The probabilities of throwing an even total or an odd total would add up to 1 because there are no other possible outcomes...thus these events are **complementary***. Complementary events are always* **exhaustive***: there are no other outcomes possible. These events are also* **mutually exclusive***, because the total cannot be both odd and even at the same time.*

These concepts will be addressed further in the Probability 3 package.

If you were throwing three dice, how many possible results would there be? Discuss why you have chosen your method to list all the possible results.

Probabilities with Dice and Coins

You throw a dice AND toss a coin.

1. One possible result is that you throw a '2' and that the coin lands on "tails". Represent all the possible results systematically. How many possible results/outcomes are there?

2. What is the probability of throwing a '6'?

3. What is the probability of the coin landing on "tails"?

4. What is the probability of throwing a '6' and the coin landing on "tails"?

5. What is the probability of throwing an even number and "heads"?

Teacher notes: Probabilities with Dice and Coins

This is a good activity for assessment or consolidation of understanding of systematic representation and probability. **A very good diagnostic assessment would involve omitting questions 1, 2 and 3 and ONLY asking question 4 and/or 5. This would illustrate to the teacher which pupils are able to see the use of systematic representation in calculating probability, which pupils have understanding of the multiplication rule, and which pupils still need more practice.**

This activity reinforces systematic representation. Pupils may use systematic listing, grid or tree diagrams to represent the 12 possible outcomes. Using their representations, they should reach the following probabilities:

2. $\frac{1}{6}$ or $\frac{2}{12}$

3. $\frac{1}{2}$ or $\frac{6}{12}$

4. $\frac{1}{12}$ (they may calculate this using the thinking developed in Ice-Cream Choices 3, namely $\frac{1}{6}$ x $\frac{1}{2}$)

5. $\frac{3}{12}$ or $\frac{1}{4}$