RECONCEPTUALISING CALCULUS TEACHING

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Note: **Error!**

If you print the Calculus Module 3 materials from the CD, you may find that some fractions are displayed as **Error!**

This happens because we have used the comma as list separator in the Equation field, and you have a different default setting. To have the fractions displayed correctly, you will have to (temporarily) change your default settings.
In Windows 95 go Start | Settings | Control Panel | Regional Settings | Number

Now mark the decimal symbol as a period (point) and change the List separator to a comma. (In Windows NT this is in the Number Format area of the International Control Panel).

RECONCEPTUALISING CALCULUS TEACHING

The rationale for searching for/developing an alternative approach to calculus, at tertiary as well as secondary levels, is multifaceted. It relates to:

- Fundamental changes in the practice of calculus, past and present.
- Poor outcomes of traditional approaches.
- Extending contemporary insights in learning and teaching to learning and teaching of calculus.

"Calculus" was chosen as one of the focuses for MALATI action for a variety of reasons, including the following:

- The notion of making the function concept a central focus of high school algebra, in spite of wide acceptance among mathematics educators internationally has not yet taken root in the practise of school mathematics in South Africa. Instead of being treated as a basic, illuminating and unifying concept providing a conceptual framework for high school algebra, the function concept is still treated in most South African school textbooks and classrooms as an isolated esoteric piece of work, often relegated to a late or last chapter.
- Logarithms became part of the school curriculum as a tool for computation in the pre-calculator age. In South Africa, apart from some graphing work, there is little or no emphasis on exponential and logarithmic *functions*, and on utilising such functions as models of real-life situations.
- The work on differential calculus that is currently done in South African schools is highly instrumental in nature and hence of questionable value, and although its inclusion in the curriculum was motivated in terms of providing learners with more experiences of authentic applications of mathematics, little real application is done.

The term 'calculus' more or less means 'computational system'. Some major computational systems are arithmetic (the calculus of numbers), vector algebra (the calculus of vectors), matrix algebra (the calculus of transformations as well as the calculus of simultaneous equations) and 'calculus proper' (the calculus of change). With respect to each of these mathematical domains, one may distinguish between:

- The fundamental 'problem types' or 'contextual structures' that define the agenda ('project') of the field.

- The elemental objects (numbers in the case of arithmetic, n-tuples in the case of vector algebra, vector to vector functions and simultaneous equations in the case of matrix algebra, functions in the case of calculus) that are mentally handled in the domain, and that form the building blocks of models of real-world situations).
- The basic operations or processes that are performed on the objects.
- The basic properties of the operations/processes (the 'algebra' of the domain).
- The operational (procedural) instrumentation of the domain.

The curriculum (content and practise of learning and teaching, including learning materials and assessment practises) for a mathematical domain continually faces a major pitfall and a major challenge:

- The pitfall is to allow the present or previous instrumentation to dominate the curriculum, even of *becoming* the curriculum, at the cost of learners being provided access to the other dimensions of the domain (instrumental learning, in Skemp's sense of the term), and more specifically at the cost of learners being enabled to make sense of the domain (e.g. to assign authentic meanings). This may lead to specific instruments being elevated to the status of fundamental concepts, definitions of operations/processes, as happened with a specific long-division algorithm to the extent that even some professional mathematicians in both South Africa and the United States of America equated it with the concept of division.
- The challenge is to accommodate the curriculum to contemporary changes in instrumentation. This challenge becomes aggravated when the curriculum is already in the pitfall of instrumentalism, and hence seriously caught up in the previous instrumentation.

Apart from possible gains in knowledge of a domain, and possible progress in the articulation and organisation of such knowledge, the most prominent change that occurs over historical time in these domains is changes in the operational instrumentation of a domain. In the domain of arithmetic, history has seen a variety of instrumentations, including pebble counting, abacuses, finger reckoning systems, sand tables, different genres of clay-and-stylus and pencil and paper algorithms, transformations and tables (trig tables, logarithmic tables) and mechanical (analog) reckoning machines, and we now have calculators and computer-based spreadsheets as the arithmetical instrumentation of our time. (As was probably the case with previous instrumentation of arithmetic, we now cannot imagine something better to replace this, in fact, we have some difficulties in evolving from the instrumentation of a previous epoch into which we were educated).

In the case of calculus, apart from other changes (importantly also in instrumentation), the fundamental problems types changed in a certain sense through the course of history. The ideas of the integral and differential calculus (in that order) originated form a preoccupation with two geometric/graphical problems: finding the area enclosed by a curve (Archimedes) and finding the 'extreme point' of a curve viewed from a certain orientation (Fermat). These occupations and the invention of 'calculus' methods to address them, took place before the invention of co-ordinate graphs, but can be elegantly articulated in terms of co-ordinate graphs. Hence, 'classical' (pre-Newton/Leibnitz) calculus was totally rooted/contextualised in geometry: It was not perceived to have any relation to functions, in fact the function concept did not play a prominent role in mathematics at that time and was hardly recognised and explicitly articulated. In spite of its geometric context, classical calculus utilised arithmetic, and in Fermat, early algebra, as its instrumentation.

Functions and their behaviour became a prominent focus of mathematical activity from about the fifteenth century. Integrated with this development, and possibly (partly) because its arithmetical instrumentation indicated its potential usefulness in the field, the ideas and methods of the calculus were quickly adopted in the analysis of functions, and the contextualisation of calculus was both expanded and fundamentally changed. In the geometric context, a curve was the object of analysis, differentiation producing one kind of information about the curve (how steeply it curves, where the curve changes direction from a certain perspective), and integration producing another kind of information (the enclosed area). Translated into the language of functions, this means that, contextually, both differentiation and integration are applied to the same function, producing two other functions so that we end up with a triad of related functions:

the 'original' function $f(x)$, its derivative $f'(x)$, and its integral F(x):

$$\mathrm{F}(x) \leftarrow \text{integration} \;\text{———}\; f(x) \;\text{———}\; \text{differentiation} \rightarrow f'(x)$$

the area under the graph   the graph of $f(x)$ the slope of the graph

In this schema, differentiation and integration are 'opposite' processes doing different things to the same function. Within this schema, it is a huge surprise, rather contrary to contextually-invoked intuition, that differentiation and integration are *inverse processes,* that 'integrating the slope gives the function' and 'differentiating the area gives the curve'.

The function context provokes and requires a rather different interpretation. A variable is a quantity that changes. A dependent variable changes when the independent variable changes. A dependent variable may change slowly or it may change fast. In fact, the speed (rate) of change may vary. *Differential calculus is the mathematical tool for analysing and describing such a variable rate of change*, while integral calculus is the tool for accumulating known changes in order to determine the total change (sum of changes) over an interval.

This contextualisation induces the following schema, in fact, making sense of calculus in this context requires the schema.

$$f(x) \underset{\leftarrow \text{ integration}}{\overset{\text{differentiation } \rightarrow}{\rightleftharpoons}} f'(x)$$

These two questions (what is the rate of change, what is the total change over an interval) can be phrased in graphical (geometric) terms, and can be asked of graphical (geometric) situations. But the questions can also be phrased in purely numerical-algebraic terms, and are most often asked of purely numerical situations.

A second historical change that affected calculus is in its instrumentation. Numerical methods of differentiation and integration were always known, in fact numerical approximation by taking finite intervals had always been the conceptual basis for limit processes. But with pre-electronic arithmetic numerical differentiation and integration was of limited value and calculus needed a different instrumentation than arithmetic. Limit processes was the instrumentation of classical calculus. The situation has changed. In the electronic age numerical approximation can be speedily done to any required degree of accuracy. The role of limit processes in calculus has changed. It is no longer the principal instrumentation, yet it is still extremely useful because of the elegance of the differentiation and integration formulae which are produced by limit processes, which greatly facilitate understanding of phenomena modelled by calculus.

Most traditional (current) calculus courses are characterised by:

- Strong instrumentalism, sometimes countered by rigorous proofs about limits and limit processes.
- Taking limit processes as the dominant instrumentation, often excluding any substantial work on numerical methods.
- Initial contextualisation in co-ordinate graph articulations of the two classical geometric contexts, with the recontextualisation in the function, rate of change context being introduced only later (in applied calculus).

The basic problems are here normally articulated **as problems about graphs**.

In emerging approaches:

- The function/rate of change contextualisation is utilised from the outset. Graphs may or may not be used to represent/illuminate **non-graphical problems**.
- Numerical methods is taken as the dominant mode of instrumentation, limits being introduced later.
- A strong focus on understanding the two fundamental problem types and their intimate relationship.
- Care is taken about understanding crucial sub-constructs, e.g. the concept of "average".

THE MALATI CALCULUS INITIATIVE

The aim of the MALATI Calculus intervention is that learners should be provided with a conceptual background which empowers them to make rational sense of elementary differential calculus.

Differential calculus deals with the problem of determining the rate of change of a function at any given point. This is really a problem for non-linear functions. While the average (effective) rate of change of a function over an interval can easily be determined by **dividing** the total change over the interval by the width of the interval, the rate of change at a point cannot be calculated in this way.

However, the (precise) effective rate of change over the small interval including the point can be used as an *estimate* of the rate of change at the point. If the function is well-behaved, like polynomial functions are, we have a situation where *the smaller the interval over which the affective rate of change is calculated, the closer the effective rate of change over the interval approximates the rate of change at a point.*

If the effective rate of change over a small interval approach a limit when the width of the interval approaches zero, we assume that the rate of change at the point is equal to this limit. So, if we can establish that the effective rate of change over an interval approaches a limit when the width of the interval approaches zero, and if we can determine this limit, we have a method of determining the rate of change at the point.

For polynomial functions, this can be done by writing a formula for the effective rate of change over an interval and simplifying this formula. For instance, for the function $y = x^2$, the effective rate of change over the interval (a, a+*h*) is $\frac{(a+h)^2 - a^2}{h}$. It is not possible to see whether this quantity approaches a limit when *h* approaches zero, or what the limit is. However, this formula for the rate of change over the interval can be simplified to 2a+*h*. It is clear that the smaller *h* is, the closer the value of 2a+*h* gets to 2a. So 2a is the limit, and hence the rate of change of the function $y = x^2$ at x = a.

To be empowered to understand the above, learners need an understanding of the following:

1. **The concept of average**

Conceptual understanding:

single value of a variable that produces the same aggregate over an interval

vs

Procedural understanding:

Sum of numbers divided by their number

'Average is maintained' does not mean that the particular value is maintained. If the variable is discontinuous it may in fact never be attained. Learners need to understand that the 'average value' of a continuous function over an interval is not

a sum of scores (numbers) divided by the number of scores

but

the total change in the function value divided by the width of the interval.

Example: Two parachutists A and B are dropped from an aeroplane. The following table shows the distance each person had fallen after different periods of time, in seconds. The fall distances are given in metres.

Time	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
Distance of A	5	20	45	80	125	180	245	320	405	500	511	522	533	544	555	566
Distance of B	5	20	45	80	125	180	245	320	405	500	605	720	845	980	1125	1280

Estimate at what speed each of the parachutists are falling at (after) 16 seconds.

We have found that both Grade 12 learners and their teachers typically give the following answers:

Parachutist A: $\frac{566}{16} = 35{,}375 \text{ m/s}$

Parachutist B: $\frac{1280}{16} = 80 \text{ m/s}$

2. **The difference between 'effective rate of change over an interval' and 'the rate of change at a point'**

We have found that, typically, learners (and teachers) do not have the concept

- *average (effective) value of a continuous function over an interval as something different from the average value of a set of numbers*

  nor the concept of

- *rate of change at a point as something different from the effective rate of change over an interval.*

The early activities in the MALATI calculus materials are designed to provide learners with opportunities to form these two concepts. After this groundwork is done, teachers may either use their standard textbook materials or the further MALATI materials.

Of course, underlying these concepts of rate of change of a function is the concept of function itself, and different representations of functions. We make a few brief remarks on further fundamental aspects of functions:

The definition of a function

A function is a relationship between two quantities where the first quantity determines the second.

Example 1: People claim that eggs laid by a hen is a function of the weather, meaning that on any given day the number of eggs laid by a hen is dependent on the weather.

Example 2: The number of learners at a school depends on the number of teachers at the school, meaning that learner numbers is a function of teachers.

Example 3: The cost of a car depends on the year the model was brought out, meaning that car cost is a function of the year model.

Example 4: The number of litres of paint to paint a house depends on the size of the house, meaning that the number of litres of paint is a function of the area to be painted.

Variables

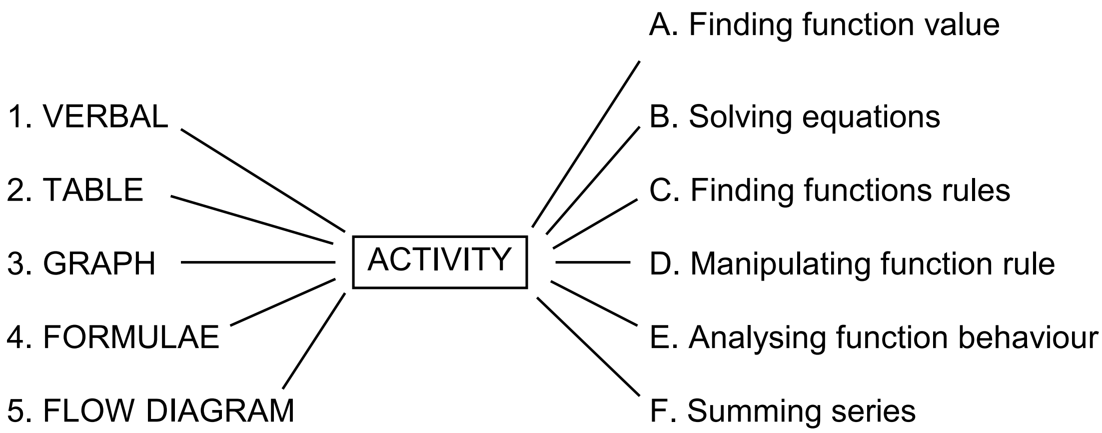
The quantities described by this dependent relationship are called variables, often represented by x and y. 'Saying one thing is a function of another', amounts to saying the first thing depends on the second. In the first example above, the weather is the independent variable (x), and the number of eggs laid is the dependent variable (y).

Function notation

A notation can describe the relationship between the variables. Every element in the set (independent variable, the domain) is related to exactly one element in the dependent variable (the range). The notation used is f(x) = y

Representing functions

- **Representation of functions and problem types**



- **Functions**

Calculus deals with *problem types* where you move from the function to the rate of change of the function and vice-versa. To be able to do this, it is essential that learners are able to analyse and manipulate functions, i.e. *finding function values* via extrapolation and interpolation -- this ensures an understanding of the rate of change and its uses, and *finding the function rules* – to be able to find the relationship between two variables and *solve equations* – to be able to decompose a function and find an inverse function.

Functions are represented in four ways namely, in words; as a formula; in tables and using graphs. This module tries to ensure that learners will be able to use all four representations and be able to translate between the methods.

♦ **words/verbal**

This is in the form of words or described in a verbal way, for example due to the weak performance of the rand, the interest rates increases or when one buys items at a shop at a certain price.

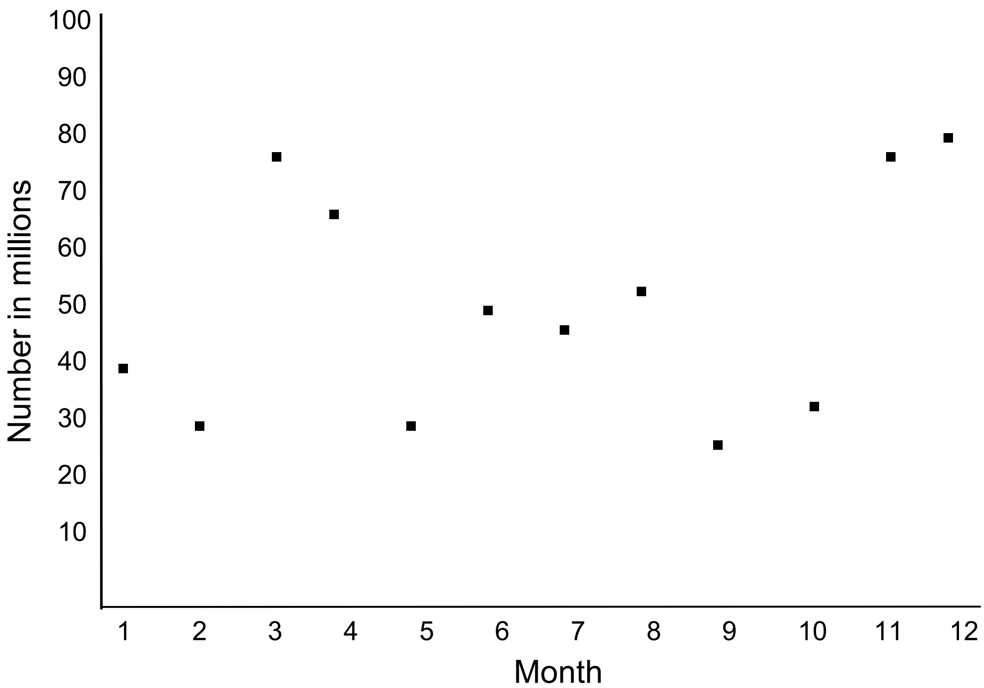
♦ **table**

This could be any relationship described in table form, for example:

Book no.	1	2	3	4				100	
No. of pages	215	100	1522	700				1000	

♦ **graphs**

This is information given by the relationship in graph form, for example, sales of cars in South Africa in 1997.



♦ **formulae**

This is normally a rule or formula, but need not be, between the dependent and independent variable showing the relationship between this association. This rule acts like a machine where there is an input value and an output value as illustrated:

Input → $\boxed{\text{Formula}}$ → Output

Function do not necessarily always have a formula connecting the input-output values. For example, if learners write a test and score the following marks,

Ken	90%
Kate	47%
Karen	25%
Bingo	36%

each learner can be associated with only one mark (so the situation is a function, here represented as a table), and yet there is no formula to predict or calculate the mark.

CONTENT OBJECTIVES IN A CALCULUS LEARNING PROGRAMME

In light of the previous discussion, we can make the following list of objectives (possible content outcomes) in learning activities for Introductory Calculus:

1. Discrete table completion
2. Interpolation/extrapolation in bare tables and contextual problems as accessing context for both gradient and summing.
3. Emphasis on juxtaposing 'function → gradient' and 'gradient → function' problems.
4. Emphasis on integration as process of finding function values/formula
5. Entry at discrete functions but careful provision for extension to continuous functions
6. Work on summing series, earlier than at present and extended to more types of series.
7. Best effort at developing knowledge and understanding of graphs.
8. Developing modelling know-how
9. Wide and relevant range of contexts
10. Concept of 'average'/effective value of a continuous function over an interval.
11. Emphasis on messy/linear/exponential/periodic/power functions.
12. Utilising gradient at a point as measurable/knowable quantity before introducing methods to calculate such gradients.
13. No limits in 'pre-calculus'
14. Emphasis on better approximations rather than early formal introduction of limits
15. Emphasis on sandwiching and error clarity
16. Special provision for algebraic manipulation insights/skills relevant in deriving nice formulae for approximations (e.g. sophisticated common factoring)
17. Concerted development of knowledge of important and powerful contexts (e.g. biological growth, predator prey, infrastructure/production cost, economics of scale, kinematics, periodic phenomena)
18. Inclusion of functions with other independent variables than time and development of understanding of rational powers.

We envisage that these different aspects of the necessary foundations for an understanding of calculus can be developed over time as follows:

PHASE (GRADE)	SYLLABUS	COMMENT
Grade 6-9	1. Introduction to the *concept of a function* 2. Functions *as a machine* 3. Functions *in table form* 4. Functions *as graphs* 5. Exploring the concrete *concept of rate of change* using functions 6. Approximate the *area under the curve using functions*	• Learners become acquainted with functions linked to real life experiences • Exploring relationships, interpret graphical information • Simple problems
Grade 10-11	1. Functions: *definition using machine diagram* 2. Function *as a rule* 3. *using arrow diagram* 4. *Independent and dependent variable* 5. *Graphical functions* 6. *Inverse functions* 7. *Rate of change* 8. *Approximate areas for instance by measurement, grid paper*	• Exploring, analysing and interpreting relationships • Studying slopes and their meaning
Grade 12	1. *Differential calculus*: find rate of change when function is given 2. *Integral calculus*: find function when rate of change is given 3. *Limit of a sequence*	• Problem solving approach • Growth, interest, bacteria, population, exponential, etc. and approximate areas

MALATI MATERIALS

We distinguish the following aspects of the learning and teaching of calculus:

- Conceptual foundation
- Contextual foundation
- Skills foundation (prerequisite skills)

Traditional (pre-)calculus courses tended to focus on the skills foundation only. We want to develop conceptual and contextual foundations as well. In conjunction with emerging approaches, the MALATI Introductory Calculus materials include:

1. Activities to develop familiarity with the two basic problem types of the calculus, i.e. in some way or another, finding the gradient of a 'given' function, and finding the function, of which the gradient is 'known' as well as a situation in which the two types are actually combined. Some information about the gradient of a function can be given, and this information can then be used to produce more information about the function.
2. Activities to develop knowledge of other things that are important for calculus, e.g. 'average', graphs.
3. Activities to develop knowledge of and familiarity with a range of types of functions, i.e. linear, non-linear and messy non-linear, exponential, polynomial, hyperbolic and periodic functions.

Learners should have a wide variety of experiences designed to allow them to investigate the central ideas of calculus informally.

The MALATI package of materials consists of

- **A grades 6-9 functions package**, which is part of the algebra materials
- **A grade 10-11 functions package (Module 1 and Module 2)**
- **A grade 12 Introductory Calculus package (Module 3)**, which is adapted from the EMSCEP second-chance program. The EMSCEP materials have been developed, reviewed and extended on the basis of feedback from three years' application in second-chance programs.

We provide an index of these packages on the following page.

INDEX OF ACTIVITIES

Grade 6–9	
See Algebra materials	
Grade 10–11	
Module 1 Activities *Activity 1:* What do graphs tell us? *Activity 2:* Stories *Activity 3:* Sheds on a farm *Activity 4:* Mountain climbers *Activity 5:* Different but the same! *Activity 6:* Salary increase *Activity 7:* Storm water *Activity 8:* Inflation *Activity 9:* St Lucia estuary 1 *Activity 10:* St Lucia estuary 2 *Activity 11:* Extension activity	These activities focus on words, tables, graphs and formula as different representations of functions, while solving problems relating to the five fundamental problem types in work with functions (See also the Algebra package)
Module 2 Activities *Activity 1A*: Broken tachometer *Activity 1B:* Broken speedometer *Activity 2A*: Electricity consumption *Activity 2B:* Electricity cost *Activity 3*: Plant growth *Activity 4*: Heart rate *Activity 5*: Tides *Activity 6*: Growth and Decay *Activity 7:* Rabbits and Jackals *Activity 8:* Parachute Jumping	These activities focus and develop the following aspects: ▪ Known rate of change and this has to be used to predict function values and vice versa. ▪ To analyse the rate of change (messy and non-linear functions) ▪ The concept of average ▪ Solving equations and finding function values ▪ Known function, analyse the rate of change ▪ Heart rate as a dependent variable where the age is an independent variable ▪ Study of a periodic function ▪ Discrete gradient → function value work with bare tables and contextual problems ▪ Effective rate of change
Grade 12	
Module 3 Activities *Activity 1:* How fast to travel *Activity 2:* Slow and fast growing *Activity 3:* Effective speeds over small intervals *Activity 4:* Approximating the effective speed at a certain moment *Activity 5:* Varying gradients *Activity 6:* Making sure *Activity 7:* The gradients of functions of the form ax^3 *Activity 8:* Looking at the gradient from the other side, finding the derivatives of some other functions *Activity 9:* Things about graphs *Activity 10:* Tangents and the slope of curves *Activity 11:* More about derivatives and graphs of functions *Activity 12:* Differentiating functions *Activity 13*: Extreme values of functions and turning points on graphs	These activities could be used to develop the necessary concepts as an *introduction* to the Calculus work. These activities focus on: 1. Differentiation as in current syllabus. 2. Gradual development starting in grade 10-11, 3. Non-graphical numerical entry, 4. Sound understanding of effective gradient over interval, 5. Some gradient → function work to promote sense-making

FURTHER NOTES ON THE GRADE 10-11 MODULES:
TEACHING THE CONCEPT OF CALCULUS BEFORE LIMITS

Given the potential for technology to eliminate much of the procedural work of calculus, for example finding complicated derivatives and applying obscure techniques of integration, the conceptual underpinnings of the subject serve as the core of the calculus curriculum.

Ferrini-Mundy and Lauten, 1994

The current program for school Calculus is limited to an essentially instrumental approach of the differentiation of polynomials relying on an incomplete understanding of the limit process. The derivative is defined in terms of graphs and learners become reliant on this visual idea of the derivative. Consequently, learners:

1. Develop a purely procedural approach and thus find it difficult to apply their knowledge to non-routine applications.
2. Develop a biased view of Calculus seeing it as a one way process from the function to the rate of change of the function. Learners are not exposed to the other side of Calculus, namely integration, which deals with problems where the rate of change of the function is known and from this they have to determine the function.
3. Are restricted to polynomial functions, however the only polynomial function properly studied at school is the quadratic function. Several other types of functions, for example exponential, trigonometric and others are studied at school. This restriction denies learners the experience of investigating functions with which they are familiar and they are thus failing to obtain further insight into the behaviour of these functions.
4. Are denied access too many meaningful real-life problems.

We are thus proposing that prior to introducing limits, the school program explores the following several function types.

1. Approximate and effective rates of change as a conceptual introduction to differentiation;
2. Finding function values for a function of which the rate of change is known at some points by extrapolation, as a conceptual introduction to integration;
3. The contrast and relation between finding rates of change of a given function and estimate values of a function with a given rate of change, as a conceptual introduction to the fundamental theorem of Calculus;
4. Refinement of estimates by reducing intervals as an alternative to limits.

Strang (1990) proposed these techniques as one approach to the understanding and teaching of the fundamental theorem of Calculus. Strang dealt with discrete models of four types of functions.

- Linear functions, f(t) = a.t, where a is constant
- Periodic or oscillating functions, f(t) = cos t
- Exponential functions, a^{nt}
- Step functions, $f(t) = a$ where $t \geq 1$, otherwise, $f(t) = 0$

We are attempting in the Grade 10-11 modules to implement the proposed alternative approach for the simple quadratic and exponential functions. The majority of the activities in this package consist of *applied problems* that are used to *introduce* concepts and recap sections which attempt to give mathematical structure to the concept. Using the proposed techniques for introducing differentiation and integration on the exponential function, we hope to illuminate one of its important features namely, that it is directly proportional to its derivative. This feature is used extensively in the solution of differential equations.

Activities are used to:
1. Deal with real-life problems;
2. Act as vehicles to promote the understanding of some of core concepts in Calculus;
3. Provide insight into the behaviour of functions which are familiar to learners.

When these aims are achieved, it is hoped that learners will understand and see the need for the progression to formal differentiation and integration using limits.

Quadratic functions

The quadratic function is dealt with extensively at school both in algebra and Calculus. The derivative is determined to be linear and occasionally there are questions that suggest that the anti-derivative of a linear function is quadratic. *Activity 6* of Module 1 (see Index of activities) is used to illustrate the idea of the fundamental theorem of Calculus using discrete mathematics. The facts that the derivative of a quadratic function is a linear function and that the anti-derivative of a quadratic function is a linear function will be by-products of the activity.

The concepts like velocity and acceleration problems are avoided to make the problem accessible to all learners – not just to those studying physical science. However, Strang (1990) deals with velocity and distance as he claims that these concepts are easily accessible to most learners. Obviously, in a course covering this material several different contexts should be used.

Exponential functions

At school, we briefly mention compound interest in the Standard Grade syllabus and fairly often an exponential growth question sneaks into the geometric series problems. However, links between geometric series and exponential functions are seldom made explicit, nor are the many real applications of these topics explored. Exponential functions occur in a broad spectrum of contexts. below are a few such examples:

Biology: A year culture grows at a rate proportional to the amount of yeast present

History/science/geography: Radiocarbon disintegrates at a rate proportional to the amount of radiocarbon present (half-life 5568 years). William Frank Libby (1908) noted that radiocarbon is continually replaced in the atmosphere while animals and plants are alive but begin to decay when they die.

Business: The amount of interest earned in a blank account with a fixed interest rate is proportional to the amount in the account.

Physics: The tare of change of the temperature of a body is proportional to the difference between the temperature of the body and the surrounding temperature (Newton's Law of cooling).

Chemistry: The concentration of salt in a water tank into which fresh water is flowing and well mixed water is flowing out, decreases at a rate proportional to the concentration.

Technology: The number of companies adopting a new technical development is proportional to the number of companies utilising the development.

With all the examples above, the rate of change of a given quantity is proportional to the amount of quantity q, present and thus are of the form $\frac{dq}{dt} = k.q(t)$, where k is a constant of proportionality.

A point of discussion: Is there a function which satisfies the above condition if k = 1? This could provide an introduction into *e* and the natural logarithm, *ln*. This fundamental property of exponential functions could be discovered and understood by learners if the links between exponential functions and geometric series were made explicit.

In the current Grade 9 syllabus, learners are expected to develop a concept of compound interest through step by step calculation. This is a function that requires us to multiply the current value by a fixed amount in order to get a future value. Since we are multiplying by a fixed amount, the rate of change at any given time step depends on the current value. Thus as time progresses arithmetically, the amount increases geometrically. It was this precise link between an arithmetic series and a geometric series that originally led to the establishment of logarithmic tables (Confrey and Smith, 1995).

In order to utilise this link, we have used a geometric function which can be described as follows

$$f(x) = a.b^x \text{ , where a and b} \in \text{R, b} > 0 \text{ and } x \in \text{Z.}$$

Arithmetic and geometric series are taught in the current grade 11 syllabus. It is noted that consecutive terms in an AS differ by a constant amount whilst the ratio between consecutive terms in a GS remains constant. However, the difference between consecutive terms in a GS is not explored. In the following table we have two series both with an initial value of 2. The first is an AS with a constant difference of 3 and the second is a GS with a common ratio of 3.

Time	Value of an arithmetic series	Rate of change of an arithmetic series $q(t_{n+1}) - q(t_n)$	Value of a geometric series	Rate of change of a geometric series $q(t_{n+1})-q(t_n)$
t_0	2		2	
T_1	5	3	6	4
T_2	8	3	18	12
T_3	11	3	54	36
T_4	14	3	162	108

From the table, it is clear that in an AS, the rate of change remains constant with the time whereas with a GS, this rate of change varies with time. It is the behaviour of the rate of change that is of particular interest. Not only does the rate of change, in this case, increase with time, it forms another GS with a common ratio of 3. This similarity between the rate of change of the function and the function itself is the precise property of geometric functions that learners should understand and activities try to extend this concept to exponential functions by reducing the interval between consecutive function values. It is hoped that through understanding this property, learners will cope better with the exponential function when it is dealt with more rigorously in a Calculus course.

To sum it all up: the activities in the Grade 10-11 modules are designed to show how some of the fundamental concepts of Calculus could be introduced without clouding the issue with limits. If learners understand these concepts, they will cope more effectively and see the need for the progression to formal differentiation and integration using limits.

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