YOUNG STUDENTS' CONSTRUCTIONS OF FRACTIONS[1]

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Grade 4 and 6 South African students' concepts of and operations with fractions were investigated using written tests. Their responses to some of the items were analysed in terms of success and misconceptions. In line with international studies, these students had several limiting constructions regarding fractions. Some of these may be interpreted as the result of the current teaching approach for fractions, while others might be the result of students' incorrect intuitions which have not been resolved or clarified in the classroom.

Introduction

This paper reports on the first phase of a project that addresses the learning and teaching of fractions in the elementary grades. The project involves the development and selection of materials and the in-service training of teachers. In order to inform this development, and to make evaluation of the success of the implementation of these materials and training possible, the project began with a base-line study of students' present understandings in four large government schools in the Western Cape, South Africa.

The results of this baseline study are interpreted within our existing theoretical framework of the learning and teaching of mathematics in general, as reported in previous PME papers e.g. Murray, Olivier & Human (1996). Such an approach is based on the view that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "…we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of direct instruction," and "The central issue is not whether students are constructing, but the *quality and nature* of these constructions" (p. 28, our italics).

There are many factors that may contribute towards elementary school students' poor understanding of common fractions. Based on the research results reported by, for example, Baroody & Hume (1991); Streefland (1991) and D'Ambrosio & Mewborn (1994), as well as local projects e.g. Murray *et al*. (1996), there appear to be three main possible causes:

- The *way and sequence* in which the content is initially presented to the students, in particular exposure to a limited variety of fractions (only halves and quarters), and the use of pre-partitioned manipulatives;
- A *classroom environment* in which, through lack of opportunity, *incorrect intuitions and informal (everyday) conceptions of fractions are not monitored or resolved*; and

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- The inappropriate application of *whole-number schemes*, based on the interpretation of the digits of a fraction at face value or seeing the numerator and denominator as separate whole numbers. This can be seen as a special case of the previous problem, but is commonly reported in the literature and will thus be considered separately.

This paper reports on our investigation of our first hypothesis, namely that if these were the main causes, they would be evident in our analysis of students' present understandings of fractions.

Methodology

In South African primary schools, fractions are usually introduced by presenting halves and quarters using pre-partitioned geometric shapes or other manipulatives. It is expected that by Grade 4, students should have been introduced to selected fractions in the mathematics classroom in this manner. By the beginning of Grade 6, according to the syllabus, students should have been exposed to equivalent fractions, comparisons of the sizes of fractions, and then addition and subtraction of fractions.

Thus Grade 4 and Grade 6 were selected as important age-groups at which students' current understandings of fractions should be investigated in the base-line study. Written tests were designed to evaluate these students' concept of what a fraction is (Items 1 to 3 in Table 1), comparison of the size of fractions with different denominators (Items 4 to 6) and their operations with fractions (Items 7 to 13). It can be seen from Table 1 that the tests for Grade 4 and Grade 6 had several items in common, in order to investigate the effect of age and teaching. Although multiplication and division with fractions are not covered in the syllabus at either level, such items were also included in the Grade 6 test, in particular division items that challenge common experience-based ideas, e.g. that 'division makes smaller'.

The tests included context-free items and items in context. Some items were adapted from previous studies (e.g. Baroody & Hume, 1991; Pirie & Kieren, 1992; D'Ambrosio & Mewborn, 1994). The items were refined and adapted after an analysis of students' responses in a pilot study in three schools.

370 Grade 4 students and 382 Grade 6 students participated in this study, representing all three main language groups in the Western Cape, namely Xhosa, Afrikaans and English. Tests were available in all three languages. The tests were administered and coded by the Fractions Working Group of the Mathematics Learning and Teaching Initiative.

Results

The following table gives a summary of the success rate on each of the items, as well as the most common misconceptions identified.

Item	Success[2] Grade 4 (n=370)	Success Grade 6 (n=382)	Most common misconceptions (Percentages given as Grade 4; 6)
1. What does $\frac{4}{5}$ mean[3]?		15%	a) 9, 20 or 'four fives' (3%); b) 'recipes': numerator, denominator (7%) c) shaded geometric shapes (7%)
2. Show $\frac{3}{4}$ in at least 3 different ways.	11%	14%	a) 12; 1; 7; 'three fours'; 3-4 (or 4-3); 3+4; or 3×4 (8%; 3%) b) incorrect shaded geometric shapes (12%; 10%)
3. Mother has 10 smarties[4]. She says you can have $\frac{3}{5}$ of the smarties. How many smarties will you get?	2%	10%	
4. Put these fractions in order from smallest to biggest: $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{9}$	5%	23%	a) order $\frac{2}{3}$, $\frac{2}{5}$, $\frac{2}{9}$ (16%; 38%) b) 2;2;2;3;5;9 (14%; 3%); c) $\frac{6}{17}$ or other whole-number procedures (22%; 5%).
5. Would you rather have $\frac{3}{5}$ or $\frac{3}{4}$ of a pizza? Why?	10%	26%	a) $\frac{3}{5}$ is larger (18%; 30%)
6. Jean spends $\frac{1}{4}$ of her pocket money. Piet spends $\frac{1}{2}$ of his. Could Jean have spent more than Piet? How?	2%	3%	a) No, $\frac{1}{2}$ is bigger than $\frac{1}{4}$ (22%; 33%) b) $\frac{1}{4}$ is bigger than $\frac{1}{2}$ (8%; 18%)

[2] In the cases where the cell is blank, the item concerned was not given to students in this grade.
[3] For Items 1 and 2, the percentage given is the percentage of students with at least one correct response.
[4] Smarties are small, round candy-coated chocolates.

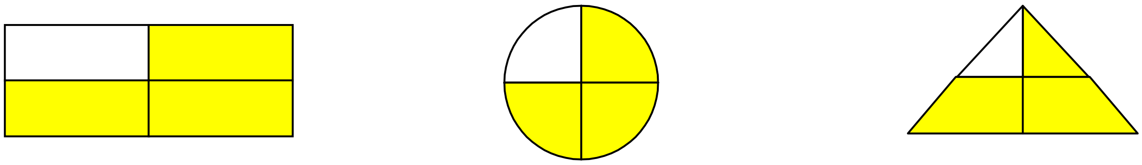
Item	Success Grade 4	Success Grade 6	Most common misconceptions
7. $\frac{7}{8}+\frac{7}{8}$	12%	31%	a) 30, ‘15+15’, ‘14+16’ or ’14,16’ (25%; 9%) b) $\frac{14}{16}$ (2%; 32%) c) $\begin{array}{r} {}^{1}\frac{7}{8}+\frac{7}{8} \\ \hline \underline{1\ 6\ 5} \end{array}$ (2%; 2%)
8. $\frac{2}{3}+\frac{4}{5}$		11%	a) $\frac{6}{8}$ (43%) b) Partial procedure for finding equivalent fractions, e.g. $\frac{6}{15}$ (5%) c) 5 as the LCD, e.g. $\frac{6}{5}$ (5%)
9. $\frac{3}{4}\times\frac{2}{5}$		38%	a) Equivalent fractions procedures not leading to successful solution (12%)
10. $2 \div \frac{1}{2}$		8%	a) 1 or $\frac{2}{2}$ (21%)
11. $4 \div 8$	5%	7%	a) 2 (32%; 50%)
12. Some friends go to a restaurant and order 3 pizzas. The waiter brings them the pizzas, sliced into eighths. Each person gets $\frac{3}{8}$ of a pizza. How many <u>people</u> will get pizza?	12%	11%	
13. We need $\frac{1}{2}$ metre of material to make a scarf. How many scarves can we make if we have 2 metres of material?	20%		

Table 1: Success Rates and Misconceptions by Item: Grade 4 and Grade 6 Students

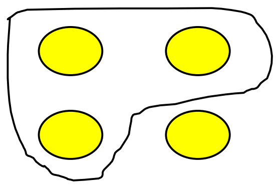
Discussion

It was not always possible to identify common misconceptions, as in many cases students omitted items or responded with various fraction or whole-number answers with no explanation.

Misconceptions arising from initial exposure to fractions at school Some students produced memorised 'recipes' for fractions, for example "numerator, line segment, denominator" (1b). There was also evidence of the reproduction of pre-partitioned illustrations (1c, 2b). For example, some students responded to Item 2 by drawing the following three shapes:



Some students even wrote "square, rectangle, circle", particularly in one class where such an illustration of fractions was displayed on the wall. The generalisation of the partitioning of a shape into four parts to a triangle indicates a limited concept of a fraction that does not include equal partitioning. The introduction of fractions using mainly a continuous area model in which the fraction represents part of the whole[5], was also evident from the fact that while 20% of Grade 4 students and 35% of the Grade 6 pupils produced such illustrations of shaded geometric shapes, only 2% of Grade 4 students and 3% of Grade 6 pupils represented $\frac{3}{4}$ as a part of a collection of objects, for example:



The 'smartie' problem (Item 3) which addresses this particular meaning of the fractions, had a particularly poor success rate, especially at Grade 4 level (2%). It is probable that the students had not previously been exposed to problems that address the fraction as part of a collection of discrete objects.

The poor success rate on Item 6 (2%; 3%), in which students did not consider the role of the whole (6a), could also be interpreted as the consequence of being exposed to a limited range of problems. In this case, the problems that the students have solved in the past probably required them to compare the size of different fractions in cases where the whole is *always the same* and is usually a single continuous shape. Similarly, exposure to division problems which can *always* be interpreted as *sharing* and in which the divisor is thus always a whole number and always smaller than the dividend, may also be a cause of the poor success rate on Items 10 (8%) and 11 (5%; 7%).

The results also suggest that some of the students had been exposed to a procedure for generating equivalent fractions, but with little understanding of the purpose of this procedure (9a) or the reasoning behind it. The latter resulted in some students

[5] It is of course true that the given responses might not be the students' *only* interpretation of fractions, but the first or easiest response or that which the students believe is expected of them.

remembering only part of the procedure, namely finding the lowest common denominator (8b). Other students took 5 as the lowest common denominator (8c). The latter can be interpreted as the result of the traditional sequence for teaching addition of fractions: first addition with like denominators, then with denominators of which one is a multiple of the other (of which one then chooses the larger), and finally with unlike denominators. In this case, students may have generalised their current rule (for the second type of problem) to the unfamiliar situation of fractions with unlike denominators.

Misconceptions arising from students' own incorrect intuitions and informal experiences Students' inability to interpret the item "$2 \div \frac{1}{2}$" as 'how many $\frac{1}{2}$'s are there in 2?' provides an example of a limiting construction arising from their own intuitions and real-life experiences. The division of fractions out of context conflicts with students' deep-seated idea*s* about division, as it produces an answer larger than the number to be divided and, unlike whole-number division, cannot be interpreted as a 'sharing' situation (Baroody & Hume, 1991). However, exposure to a wider variety of division situations at school would have provided an opportunity for this conflict to be resolved, as mentioned above.

Inappropriate application of whole-number schemes One of the most common general errors identified in this study was the students' inability to see a fraction as a quantity - a quotient relation between two numbers – rather than two separate whole numbers. In fact this is an example of a misconception or a limiting construction based on students' own intuitions and previous experience. In this study, it affected the students' success on the items investigating their concept of a fraction, comparison of the size of fractions and addition of fractions. Students responded to the face-value of the denominator and numerator, separating them and carrying out inappropriate operations with them (Misconceptions 1a, 2a, 4b, 4c). When it came to adding fractions, some students mechanically combined the two denominators and numerators (7a, 7b, 7c, 8a). It is interesting that the application of a similar whole-number scheme leads to success when it comes to multiplying fractions (Item 9). The interference of whole-number strategies, which results in students simply adding the numerators and the denominators, has been widely reported in the literature (Baroody & Hume, 1991; Streefland, 1991; D'Ambrosio & Mewborn, 1994). Even if students had not previously been exposed to the addition of fractions, it is disturbing that they were unable to judge that 30 and 165 are not reasonable answers for "$\frac{7}{8} + \frac{7}{8}$" (Item 7).

Comparing the size of fractions by considering only the size of the denominator can also be considered a case of regarding the numerator and denominator as two unrelated whole numbers (4a, 5a, 6b). This misconception was actually more prevalent in the case of the Grade 6 students than in the case of the Grade 4 students, although the success rate on these problems increased across these grades. The tendency to choose as the larger fraction the one with the larger denominator has also been reported in the literature (e.g. Baroody & Hume, 1991).

Also in the case of division by a fraction, students responded by attempting to apply a previous whole-number scheme (10a). Similarly, changing $4 \div 8$ to $8 \div 4$ may be a misapplication of the commutative law that applies in the case of addition but not in the case of division (11a).

In summary This research has found that, in line with research in other countries, these students have, after their first few years of school, limited and *limiting* understandings of fractions which persist into the upper elementary grades. These meanings could in some cases result from pupils' own intuitions or might be a direct result of aspects of the teaching approach currently used for the introduction and development of the fractions concept in these schools. In both cases the findings presented here suggest that the current teaching approach has not been successful in challenging students' incorrect intuitions and in preventing the development of additional limiting constructions.

Testing our approach to teaching and learning fractions Although this study suggests that students have some intuitions and experienced-based ideas which are incorrect, there is evidence that very young children are able to understand and solve sharing problems involving fractions (e.g. Empson, 1995; Murray *et al*., 1996). It was also clear from the findings of this study that students can use non-mathematical interpretations to make sense of unfamiliar fraction situations in the context of problems; It can be seen from Table 1 that Items 12 and 13 were solved with greater success than many of the other items in spite of the fact that they could be classified as division with fractions. Both items were given in the context of a problem. For example, the success on Item 13 was 20%, compared with that on Item 10 which was not given in a context and was solved with less success (8%) by older students. There was also other evidence of students responding sensibly to the contexts provided, for example the pupil who chose $\frac{3}{5}$ rather than $\frac{3}{4}$ of a pizza (Item 5) "because I don't eat a lot", or the pupils who said "I would like that because I have no pizza. I would like both (pieces) of fraction pizza" and "No, because is it not a hole but me I want a hole"!

Based on the analysis of our findings in this base-line study, and on the evidence that pupils can make better sense of unfamiliar situations with fractions in a problem context, we have developed material and a programme for in-service training in our four schools. The approach is problem-centered (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990). Different meanings of fractions and operations with fractions are developed using a rich variety of carefully-selected problems, supported by a learning environment that encourages reflection and social interaction. Teachers do not demonstrate solution methods for problems, but expect students to construct their own strategies, and depend on peer collaboration for error identification and the development of more powerful strategies. Written symbols and the introduction of symbolic algorithms for operations with fractions are delayed until students have had the opportunity to conceptualize fractions as single quantities (Baroody & Hume, 1991; Empson, 1995). There is already some evidence of the success of such an approach with young children (e.g. Empson, 1995; Murray *et al*., 1996). Our next hypothesis, which we are currently investigating, is that this approach contains at least some elements which directly address the causes of misconceptions discussed in this paper, by encouraging students to construct their own knowledge and by attempting to establish social procedures like discussion and justification to monitor and improve the nature and quality of those constructions.

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