# ADDRESSING STUDENTS' CONCEPTIONS OF COMMON FRACTIONS

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*Grade 6 and 7 South African upper elementary school students' conceptions of and operations with common fractions were investigated before and after a year of exposure to materials which had been designed to challenge common limiting constructions identified in a previous study, and to a teaching approach of which reflection and interaction are essential components. There was a significant improvement in several of the items in the written tests, in line with the aims and extent of the materials used.*

## Introduction

This paper reports on the impact of an approach for the teaching of common fractions on Grade 6 and 7 students' conceptions of fractions. The approach was intended to challenge common limiting constructions (D'Ambrosio and Mewborn, 1994) and included materials that were designed to specifically address students' problems with fractions which had been identified in a previous study (Newstead and Murray, 1998) in a larger sample of which the current sample is a subset.

The approach and materials were used during the first year of implementation of the Mathematics Learning and Teaching Initiative (Malati) teacher and curriculum development project in schools, as a vehicle for introducing elementary school teachers to our approach. This approach requires a classroom culture as described below which originates from our theoretical orientation as reported in previous PME papers (e.g. Murray, Olivier & Human, 1996). Such an approach is based on the view that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "… we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of direct instruction," and "The central issue is not whether students are constructing, but the *quality and nature* of these constructions" (p. 28, our italics).

Based on the previous study (Newstead and Murray, 1998) and on the existing literature, and in line with the Malati philosophy, the fractions materials were designed according to the following basic principles:

- Students are introduced to fractions using sharing situations in which the number of objects to be shared exceeds the number of 'friends' and leaves a remainder which can also be further shared (e.g. Empson, 1995; Murray, Human & Olivier, 1996).
- Students are exposed to a wide variety of fractions at an early stage (not only halves and quarters) and to a variety of meanings of fractions, not only the fraction as part-of-a-whole where the whole is single discrete object, but also for example the fraction as part of a collection of objects, the fraction as a ratio, and the fraction as an operator.

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- Students are encouraged to create their own representations of fractions; pre-partitioned manipulatives and geometric shapes do not facilitate the development of the necessary reasoning skills and may lead to limiting constructions (Kamii and Clark, 1995).
- The introduction of fraction names and written symbols is delayed until students have a stable conception of fractions. Written, higher order symbolization is not the result of natural learning, and students struggle to construct meaning for such representations of fractions in the absence of instruction which builds on their own informal knowledge (Mack, 1995).
- Similarly, students can and should make sense of operations with fractions in a problem context before being expected to make sense of them out of context (Piel and Green, 1994).
- The materials repeatedly pose problems with similar structures to provide students repeated opportunities to make sense of particular structures. Fractions are taught continuously throughout the year, once or twice a week rather than in a concentrated 'block' of time.
- A supporting classroom culture is required in which learning takes place via problem solving, discussion and challenge and in which errors and misconceptions are identified and resolved through interaction and reflection. Teachers do not demonstrate solution strategies, but expect students to construct and share their own strategies and thus to gradually develop more powerful strategies.

The Malati elementary school material and approach was implemented in 4 traditionally disadvantaged schools near Cape Town in 1998. In addition to student worksheets and comprehensive teacher notes, the teachers attended workshops and reflection sessions and were visited regularly in their classrooms by Malati project workers.

## Methodology

To study the impact of the Malati materials and approach, research was conducted on both teacher and student change in two of the four elementary schools. The impact on Grade 6 and 7 students' learning was investigated using both written individual tests and observation of students interacting in groups to solve challenging problems. This paper reports on changes in students' responses to the common fractions items in the written tests only.

The written tests were designed by Malati researchers to 'cover' both the curriculum traditionally taught in the schools, and the intended Malati curriculum. Thus items in the Grade 6 and 7 tests differed in some cases according to the existing curriculum aims. All the Grade 6 and 7 students completed the three tests on the same day towards the end of the academic year in November 1997. For one of these tests, calculators were not allowed. The tests were administered by Malati project workers who encouraged the students to try their best, but assured the students that the results were not for school 'marks'. Exactly the same procedure was followed with the Grade 6 and 7 students in the same two schools in November 1998. The tests were coded by Malati project workers according to a coding schedule devised after an initial analysis of the first dataset. Out of every class of approximately 30 to 50 students, the first five complete tests were coded by two Malati project workers who then compared and discussed any discrepancies before continuing to code individually. The data was input and the analysis carried out using SPSS for Windows.

## Results

The response categories for the fractions items were collapsed for the purpose of a chi-squared analysis. In some cases, certain responses were coded as showing some understanding of the problem, in which case a third ‘semi-correct’ category is indicated. In these cases, the chi-squared analysis was conducted using the categories ‘no correct response’, ‘semi-correct response’ and ‘correct response’.

The following tables show the success rate on the common fractions items. The items are numbered for convenience and not according to test item numbers. All the values in the cells are percentages of the total number of students who were tested. The number of students in each grade differ in 1997 and 1998 owing to the non-longitudinal research design, e.g. Grade 6 students in 1998 (who have been exposed to the intervention for a year) are compared to Grade 6 students in 1997 (who had not been exposed to the intervention).

Item 1 tested students’ part-of-a-whole conception of fractions:

*1. What fraction of the following figures is shaded? If it is not possible to say, explain why not.*

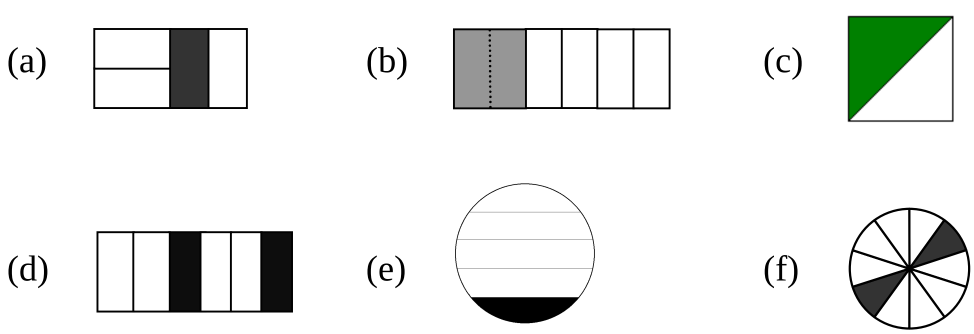


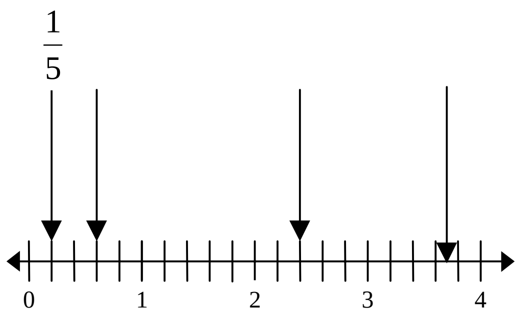

Table 1 shows the success rate on these items and significance of change in responses:

| Item | Grade 6 | | | Grade 7 | | |
|---|---|---|---|---|---|---|
| | 1997 (N=208) | 1998 (N=174) | Significance | 1997 (N=191) | 1998 (N=193) | Significance |
| 1a | 58 | 85 | $p < 0,001$ | 71 | 78 | $p > 0,05$ |
| 1b | 52 | 83 | $p < 0,001$ | 69 | 79 | $p < 0,05$ |
| 1c | 69 | 89 | $p < 0,001$ | 87 | 90 | $p > 0,05$ |
| 1d | 53 | 79 | $p < 0,001$ | 73 | 78 | $p > 0,05$ |
| 1e | 2 | 5 | $p > 0,05$ | 4 | 12 | $p < 0,01$ |
| 1f | 48 | 77 | $p < 0,001$ | 67 | 72 | $p > 0,05$ |

Table 1: Success in identifying fractions using pre-partitioned shapes

Although rational number lines were not taught in the Malati curriculum, the following number line items were included to test students' conception of fractions as rational numbers:

*2. What numbers are shown by the arrows? The first one has been done for you.*




*3. Show the following numbers on the number line below. The number $\frac{3}{4}$ has been done for you.*

(a) $1\frac{1}{3}$ (b) $\frac{12}{6}$ (c) $\frac{2}{3}$

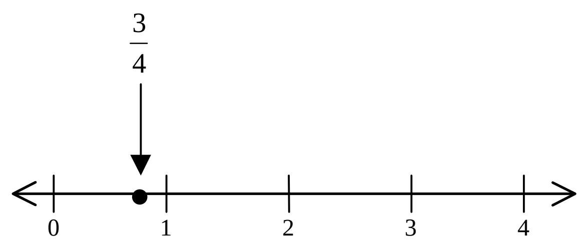



Table 2 below shows the success rate on these items and significance of change in responses. Blank cells in the table indicate that these particular items were not included in (in this case) the Grade 7 tests.

| | Grade 6 | | | Grade 7 | | |
|---|---|---|---|---|---|---|
| Item | 1997 | 1998 | Significance | 1997 | 1998 | Significance |
| 2a | 35 | 46 | $p < 0,05$ | 33 | 47 | $p < 0,01$ |
| 2b | 16 | 32 | $p < 0,001$ | 19 | 27 | $p < 0,05$ |
| 2c | 4 | 14 | $p < 0,01$ | 14 | 17 | $p > 0,05$ |
| 3a | 17 | 35 | $p < 0,001$ | | | |
| 3b | 6 | 18 | $p < 0,001$ | | | |
| 3c | 9 | 17 | $p < 0,05$ | | | |

Table 2: Success on items using rational number lines

Additional items that were included in the Grade 6 and 7 tests are shown in Table 3 and 4 respectively. For Item 5, a 'semi-correct' response indicates that students correctly chose Amina as having spent more, but did not supply a reason. For Item 6, a 'semi-correct' response indicates that students gave a response of 6 or 7 rather than the precise correct answer of $6\frac{1}{4}$. 'SC' indicates the frequency of semi-correct responses (where such a category was coded), while 'C' indicates the frequency of correct responses.

| Item (Grade 6 test only) | 1997 | | 1998 | | Significance |
|---|---|---|---|---|---|
| | SC | C | SC | C | |
| 4. Jackie spends $\frac{3}{4}$ of her pocket money and Piet spends $\frac{1}{2}$ of his pocket money. Could Piet have spent more money than Jackie? How? | | 3 | | 2 | $p > 0{,}05$ |
| 5. Anwar and Amina each received R30 pocket money. Anwar spent $\frac{5}{8}$ of his pocket money and Amina spent $\frac{7}{10}$ of hers. Who spent more? Explain your answer. | 11 | 17 | 19 | 17 | $p > 0{,}05$ |
| 6. Mrs Brown wants to cook porridge for 10 people. She normally uses 5 cups of oats for 8 people. How many cups of oats does she need for 10 people? Show your calculations. | 16 | <1 | 33 | 4 | $p < 0{,}001$ |

Table 3: Success on comparison of fractions items and the use of fractions as a ratio

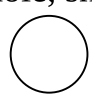
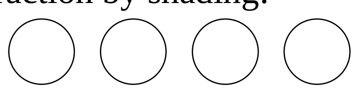
| Item (Grade 7 test only) | 1997 | 1998 | Significance |
|---|---|---|---|
| 7a. If the diagram below represents a whole, show by means of a suitable drawing how you would represent $\frac{7}{5}$.  | 9 | 6 | $p > 0{,}05$ |
| 7b. If the diagram below represents a whole, show by means of a suitable drawing how you would represent $\frac{1}{2}$.  | 47 | 59 | $p < 0{,}01$ |
| 8a. Four pizzas were bought: $\frac{1}{3}$ of the pizzas was eaten. Show this fraction by shading:  | 13 | 24 | $p < 0{,}01$ |
| 8b. How many pizzas were left? | 7 | 17 | $p < 0{,}01$ |
| 9. $\frac{1}{3}$ of a man's salary is R3200. What is his salary? | 39 | 37 | $p > 0{,}05$ |
| 10. After a party $\frac{1}{5}$ of a cake is left. The next day John eats $\frac{3}{4}$ of the leftover cake. What fraction of the cake is left then? | 6 | 4 | $p > 0{,}05$ |

Table 4: Success on items in context testing concepts of and operations with fractions

The following items tested operations with fractions out of context. Students were not permitted to use calculators for these items. For Items 11 and 12, a 'semi-correct' (SC) category was included for responses in which a suitable common denominator was used but the correct answer was not obtained. In the case of Item 17, a SC category was also included for responses of '30 min + 15 min' and '$\frac{1}{2}+\frac{1}{4}$'. Table 5 shows the frequencies in the various categories and significance of change in responses. In some cases, a chi-squared analysis could not be conducted as some of the cell frequencies were too low.

| Item | Grade 6 | | | | | Grade 7 | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1997 | | 1998 | | | 1997 | | 1998 | | |
| | SC | C | SC | C | Significance | SC | C | SC | C | Significance |
| 11. $\frac{5}{8}+\frac{4}{5}$ | 1 | <1 | 5 | 2 | - | 10 | 11 | 9 | 8 | $p > 0,05$ |
| 12. $3\frac{1}{4}+1\frac{4}{5}$ | 1 | <1 | 6 | 2 | - | 8 | 11 | 2 | 8 | $p > 0,05$ |
| 13. $\frac{1}{6}\times 26$ | | | | | | | 5 | | 9 | $p > 0,05$ |
| 14. $\frac{7}{8}\div\frac{1}{4}$ | | 1 | | 1 | - | | 6 | | 6 | $p > 0,05$ |
| 15. $\frac{3}{4}$ of $\frac{1}{5}$ | | | | | | | 19 | | 9 | $p < 0,05$ |
| 16. $\frac{3}{4}$ of R120 | | 11 | | 16 | $p > 0,05$ | | | | | |
| 17. half of $1\frac{1}{2}$ hours | 1 | 17 | 6 | 34 | $p < 0,001$ | | | | | |

Table 5: Success on context-free items testing operations with fractions

## **Discussion**

*Fraction as part-of-a-whole* The results show a significant and substantial improvement in many of the items that reflect the students' conception of the fraction as part-of-a-whole, which is considered to be an essential foundation to the understanding of fractions. This improvement was most evident at the Grade 6 level in all Item 1 questions, except 1e (see Table 1). There was not a similar significant improvement in the Grade 7 responses, which could probably be ascribed to the relatively high success rate in 1997.

In Grade 7 there was a significant improvement in Item 1e which may indicate that some Grade 7 students are reflecting more successfully on the necessity of *equal* parts – In 1998, 78% of the Grade 7 students gave the answer as $\frac{1}{5}$, as opposed to only 44% in 1997. In Grade 6, the percentage of students who gave the answer as $\frac{1}{5}$ changed from 69% to 66%. However, the poor success rate in both grades on this item indicates that more emphasis needs to be given to the necessity of equal parts.

*Fraction as rational number* It is interesting to note the significant and substantial improvement on most of the questions of Items 2 and 3 that require students to make use of the rational number line. The Malati material does not make use of any such number lines, so this may be ascribed to an improvement in the ability to make sense of fractions as rational numbers. The exception to the significant improvement on these items is the Grade 7 response to Item 2c, although 'semi-correct' responses like $\frac{18\frac{1}{2}}{5}$ and $3\frac{3\frac{1}{2}}{5}$ were included as correct responses. It is expected that students will respond to this item with greater success once they have been exposed to decimal fractions, which were not taught in 1998.

However, the lack of success on Items 4 and 5 indicates that the Grade 6 students were not able to use fractions as rational numbers within the problem solving context. According to Watson, Collis & Campbell's (1995) classification, this particular use of fractions within the problem solving context represents the most complex level of fraction items. Indeed, our data indicates that students need more experience with fractions as abstract rational numbers in a problem solving context.

*Other meanings: Fraction as part of a collection, as operator and as ratio* We had expected that the materials had addressed fractions as part of a collection of objects sufficiently. Indeed, there was a significant improvement in the responses to Item 8a. The disappointing success rate on Item 9 could be ascribed to the fact that the materials neglected to address fractions as operators sufficiently. The significant change in the responses to Item 6 (the fraction-as-a-ratio) was, on the other hand, unexpected as we had felt that this meaning of fractions was not sufficiently addressed in the materials. However, the significant change is in fact in the semi-correct responses and not in the correct responses, and could thus be ascribed to better reasoning and/or estimation, or simply to an increased willingness to attempt the problem (see below).

*The role of the whole* The lack of success on Item 4 may be attributed to the fact that comparing fractions of which the whole is not necessarily the same was not addressed at all in the existing material. The conception of the relationship between the fraction and the whole will need to be further addressed in 1999, as indicated by the lack of success on Item 7a. There was however increased success on Item 7b.

*Operations with fractions* The students did not show a significant improvement on the non-calculator items (Items 11 to 17) which involved context-free operations with fractions. This can be explained by the fact that thus far in the materials they have only been required to make sense of operations in the context of problems. Context-free operations with fractions are expected to be revisited and consolidated in further fraction materials. The Grade 6 students' significant improvement in Item 17 could be ascribed to the fact that this item is not really context-free although it was included in the non-calculator test.

It is of concern that the students did not achieve more success on Item 10 which concerns multiplication of fractions in context. However, there was a significant improvement in subtraction of fractions in context (Item 8b), which can be expected as students had much experience with such problems.

*Willingness to try* Significantly more Grade 6 students (at least $p<0{,}05$) attempted Items 1a, 1b, 1c, 1d, 4, 5, 6, 16 and 17 in 1998 than in 1997. Significantly more Grade 7 students (at least $p<0{,}05$) attempted Items 1b, 2b and 2c in 1998 than in 1997. The only item in which fewer Grade 6 students attempted a response in 1998 than in 1997 was Item 11, and significantly fewer Grade 7 students attempted a response to Items 11 and 15 in 1998 than in 1997. Items 11 and 15 involved context-free operations with fractions, of which the learners had little experience.

*Conclusion* The items on which the students performed poorly in 1997, and on which there was no significant improvement during 1998, may indicate insufficiencies in our curriculum design principles and implementation. For example the role of the whole needs to be more thoroughly addressed. In spite of our attempts to address various meanings of fractions, more attention needs to be paid to some of these such as the fraction as operator. As intended according to our design principles, operations with fractions need to be covered out of context now. Another aspect of fractions which may not have been sufficiently addressed in our materials was the transition from unit fractions to related non-unitary fractions. This transition is not one which occurs naturally (Davis, Hunting & Pearn, 1993). This may account for the lack of success on several of the items.

However, in this study design principles to facilitate improved learning of fractions were developed based on research on students' understandings and limiting constructions. The resulting materials and classroom culture of reflection and discussion helped to facilitate a better basic understanding of the conception of fractions, and a greater willingness on the part of the student to try.

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