LEARNERS' UNDERSTANDING OF THE ADDITION OF FRACTIONS

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Research studies and experiences with primary school learners reveal that many of them have difficulties with fractions and in particular addition and subtraction operations involving fractions. This study explores some reasons for errors made by learners when adding fractions. We believe that these reasons are firstly a weak or non-existent understanding of the fraction concept, and secondly a very common limiting construction arising from learners' experience with whole numbers and the set algorithms which are taught for whole number arithmetic. We also report on strong indications that these problems can be prevented and overcome by appropriate teaching materials and a different approach to teaching and learning mathematics.

Introduction

This paper attempts to characterise learners' conceptions and limiting constructions when they have to add fractions. By limiting constructions we mean prior exposure to situations which give the learner a narrow view of a concept which hampers further thinking, for example, only dealing with halves and quarters for some time before introducing thirds (Murray, Olivier & Human, 1996). In this study, the limiting constructions originated from whole number schemes that completely blocked out the probably short and superficial introduction to the meaning of fractions that these learners may have received. We believe that the learners' errors reported on here can be traced back to these two causes:

- A weak or non-existent understanding of the fraction concept and in particular, no understanding of the symbolical representation of a fraction.
- The urge to use familiar (even if incorrect) algorithms for whole number arithmetic.

Two types of analyses will be used.

Firstly we analyse learners' responses to various tasks. The analysis of learners' errors will be the centre of our focus since they provide insight into learners' understanding of the concept in question. A second analysis will consider the learners' responses to an addition task after a teaching intervention.

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The problem of fractions in primary school is well documented in literature by many researchers such as Steffe and Olive (1991), Carpenter, Hiebert and Moser (1981), Carpenter (1976), Davydov and Tsvetkovich (1991), Newstead and Murray (1998) and Hasemann (1981). Some of the problems identified in the literature that make fractions so difficult are:

- The abstract way in which fractions are presented.
- Fractions do not form a normal part of learners' environment.
- The tendency to introduce the algorithms for the operations on fractions before learners have understood the concept.
- The abstract definition of the operations on fractions.
- The formulation and practising of computational rules receiving too much attention whereas the fundamental concept of fraction is ill-developed.

Background

The Mathematics Learning and Teaching Initiative (MALATI) approach to the learning and teaching of Mathematics in general and fractions in particular has been described by Newstead and Murray (1998) and Murray, Olivier and Human (1998). In this approach, "different meanings of fractions and operations with fractions are developed using a rich variety of carefully-selected problems, supported by a learning environment that encourages reflection and social interaction" (Newstead & Murray, 1998, 3, 301-302). The theoretical foundation for this approach is in line with the views of other researchers such as Streefland (1982) and Kamii and Clark (1995).

Streefland's (1982) approach can be described as follows:

- Developing the concept of a fraction through exploring distribution/sharing situations and performing equal distribution/sharing with an eye on the twin meanings of fractions.
- A multi-faceted approach towards the concept of a fraction, based on the frequent performing and describing of fractions-provoking problem situations; the careful development of language for fractions, aimed at the prevention of after-effects of the meaning of the symbols used due to both the figures and the operational signs having already acquired a definite meaning for the learners within the context of natural

numbers. All these are done through the use of contexts as source and domain of application for fractions.

- The postponement of fixed algorithmic procedures.

Streefland’s approach recognises and values the use of less sophisticated methods for solving problems involving fractions. Streefland’s approach describes not only the activities for developing the concept of fractions but also, at very early stages, addresses the limiting constructions that teachers might expect from the learners as they engage in the problems.

Kamii and Clark (1995) further describe this approach as follows:

1. Teaching that starts with realistic problems and encourages children to invent their own solutions so that fractions can grow out of children's own thinking. Encouraging children to logico-mathematize their own reality is much better than presenting a chapter titled “Fractions” with pictures of circles, squares, and rectangles that have already been partitioned.
2. Ready-made pictures or manipulatives are not given, and children have to put their own thinking on paper. Children may draw circles that look like those found in today's textbooks, but the figurative knowledge they put on paper represents their own work and understandings as opposed to the circles presented in textbooks, which represent someone else's thinking.
3. Equivalent fractions can be invented from the very beginning in relation to whole numbers. This is in contrast with traditional instruction that waits for a long time to present mixed numbers and addition with unlike denominators. Streefland’s approach involves halves and quarters, which are easy for children to invent.

The materials used in the MALATI programme cover:

- developing the fraction concept through sharing situations
- introducing realistic problem situations for operations involving fractions (e.g. division by a fraction)
- comparison of fractions
- equivalence of fractions

- introducing the fraction notation
- the informal addition and subtraction of fractions and mixed numbers.
- the different meanings of fractions (e.g. fraction of a collection, fraction of a whole, etc.)

Method

Subjects

The subjects were 95 Grade 5 and Grade 6 learners in the same school. The school was one of the MALATI project schools in a township near Cape Town. At the time the children were tested in February 1998, no MALATI instruction on the fraction topic had been given, but the learners had been exposed to the fraction teaching in their previous grades as required by the curriculum. We therefore expected that the difficulties which many primary school learners traditionally encounter with fractions would also show up in this sample.

The pre-test

At the beginning of the school year in February 1998, a pre-test was given to the Grade 5 and 6 learners. The test consisted of problems placed in a context which might resemble a real-life situation, for example chocolates, bottles of milk, etc., as well as context-free questions. The word "context" has a very general meaning, but allow us to use the words "context" or "contextualized" in this paper to signify problems which are posed in semi-real life situations. The pre-test was given prior to any engagement with either the teachers or the learners to assess learners' understanding of basic fraction concepts, representations of fractions (e.g. area models, set models, number lines), relationship between fractions (e.g. ordering, equivalence), and procedures (e.g., converting improper fractions to mixed numbers and vice versa, generating equivalent fractions, addition of fractions and mixed numbers).

None of the learners in Grades 5 and 6 (95 learners in all) were successful with any of the 12 contextualized questions posed in the pre-test. Some learners were successful in the context-free questions. The results of one of these questions are summarised in the table below.as a typical case. None of the seven context-free questions stood out as significantly different from the selected question namely, $\frac{7}{8}+\frac{7}{8}$.

The results for $\frac{7}{8}+\frac{7}{8}$ in the pre-test are given in Table 1 below.

		Fraction Answers							
Grade	**No of Learners**	Correct		Incorrect $\frac{a+c}{b+d}$ [a]		Incorrect - Any other Fraction [b]		Whole Numbers [c]	
5	57	01	1.75%	7	12,28%	9	15,79%	40	70,18%
6	38	11	28,95%	7	18,42%	13	34,21%	7	18,42%

Table 1

For example:

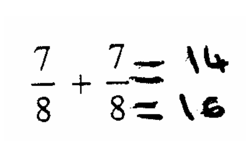
a. $\frac{7}{8}+\frac{7}{8}=\frac{14}{16}$

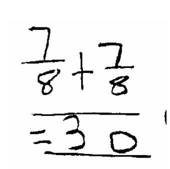
b. $\frac{7}{8}+\frac{7}{8}=\frac{8}{11}$

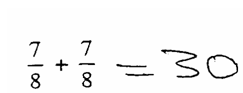
c. $\frac{7}{8}+\frac{7}{8}=165$

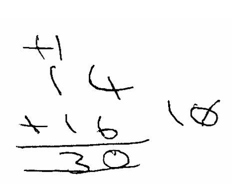
All the learners attempted the context-free problem cited above ($\frac{7}{8}+\frac{7}{8}$). They manipulated the numbers using algorithms they know, such as adding or subtracting whole numbers (numerators only and denominators separately or vertical addition), finding the LCM (but not knowing where it should be and what to do with it), etc. It was clear that learners did not understand the basic concept of fractions and therefore could not evaluate (or have a feeling for) the answers they produced.

We present some examples of learners' responses to the pre-test item of $\frac{7}{8}+\frac{7}{8}$.









Many learners approached these problems from a whole number point of view, implementing the whole number addition process without regarding the fractions as fractions. One common error resulting from this point of view is adding denominators to denominators, and has been identified and documented by many researchers, e.g. Carpenter (1976) and Howard (1991). It can be expressed symbolically as $\frac{a+c}{b+d}$.

Howard (1991) mentions a few factors that entrench this response, for example, the teaching of the multiplication of fractions before other operations. The reasons given by Carpenter (1976) concur with Howard's observation. According to Carpenter (1976), the major reason for learners committing this error is the introduction of multiplication of fractions before addition. When learners are introduced to multiplication where the "top × top over bottom × bottom" is correct, they then transfer that process to addition.

> One episode of learners reasoning about the "top × top over bottom × bottom" was observed in a group discussion. One girl argued that her teacher told her that multiplication is repeated addition. So if 7 + 7 is the same as 7 × 2, then $\frac{7}{8} + \frac{7}{8}$ is the same as $\frac{7}{8} \times 2 = \frac{14}{16}$.

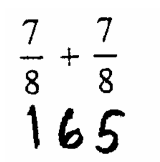
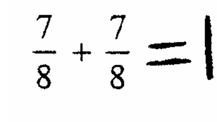
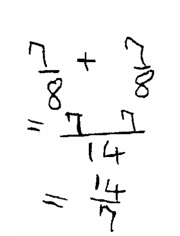
In this school, however, Carpenter's reason for this type of error does not hold because the learners had not been taught the multiplication algorithm. We believe the main reason for this error lies in the learners' lack of understanding of the concept of a fraction; they therefore do not see the numerator and denominator as representing a single "idea" or "object" (the fractional part), but simply as two whole numbers.

Learners who commit the above error may be described as follows:

- When they look at a problem involving fractions, it seems to fit their existing schemas for whole number operations. They then approach the problem using the (whole number) mathematical resources that they have.
- They operate on the symbols without an adequate, quantitative basis for their thinking, and in particular without understanding the differences between whole number and fraction symbols.

The data in the table above show that many learners in both grades view the fraction as two separate whole numbers. If two fractions are involved, the learners view them as four separate whole numbers that need to be combined in some way or other to give a whole number. Although this error occurred less frequently in Grade 6, it still appears to be a dominant thinking pattern. While there is no way of knowing exactly what the respondents in a written test think, we suggest firstly that the reason for this error is the natural inclination of learners to operate with whole numbers (since the whole number scheme was formed first), and secondly that learners do not see the fraction as a single "object". The learners had therefore not developed an understanding of the concept of fraction.

This problem underlies the denominator plus denominator over numerator plus numerator error, but it also manifests itself in other ways, as shown by the following examples.

The errors learners made on the task provide some insight into the reasons for relative task difficulty and characterising of the learners' constructions of fractions. According to D'Ambrosio and Mewborn (1994), learners' errors can be characterised as follows:

- Numerators and denominators are viewed by learners as independent in making meanings for fractions, and learners often see very little or no relationship between them in a fraction representation.
- Counting is a foolproof method for determining fractional amounts.

The data also show that some learners do find fractional (even if incorrect) answers. It is difficult to identify the reasons for these answers, because of the diversity of the answers and the individualistic nature of the strategies. However, these answers further underscore the importance of understanding how algorithms work and why certain steps are necessary. Learners will, for example, show that they have learnt a process of finding common denominators, but it is strictly a mechanical process and they do not know what to do with it.

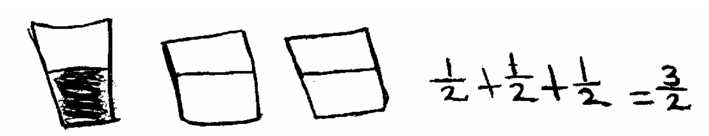
The incorrect strategies in the examples above show that learners have learnt the operations and algorithms for whole numbers and think that the same processes can be applied to the operations with fractions. According to D'Ambrosio and Mewborn (1994), fraction concepts should emerge from learners own constructions which should be carefully used to develop algorithms.

The MALATI intervention

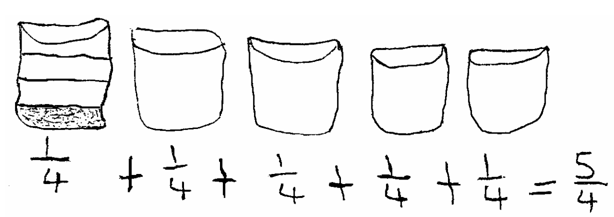
The test was followed by the implementation of the MALATI programme. The teachers attended workshops where they familiarised themselves with the materials to be implemented and the kind of environment conducive for its implementation. The learners were confronted with a variety of problem situations that they had to solve through group effort. During these lessons, the researcher assumed the role of observer. Field notes were taken from observed lessons for analysis. All the learners' written work was collected for analysis. During instruction, learners worked in groups of 4 to 6 as they attempted the problems. After engaging with this material for about five months, some evidence of success in solving addition problems was observed for both contextualized and context-free questions. An example of this can be observed in Zukiswa's work below.

Lisa wants to know what she has left over after baking for a party. How much of each does she have?

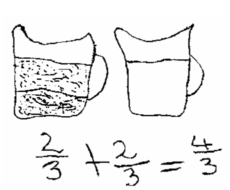
3 packets of chips, each $\frac{1}{2}$ *full.*



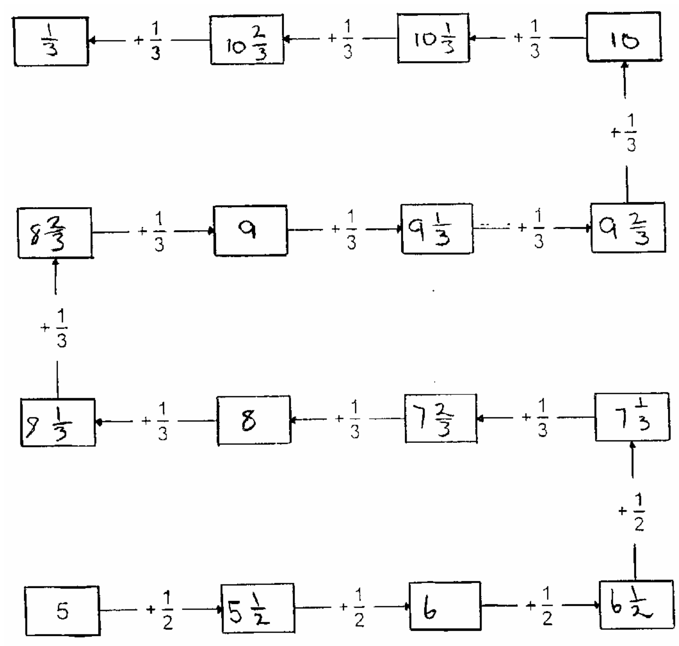
5 containers of ice-cream each $\frac{1}{4}$ *full.*



2 jugs of milk, each $\frac{2}{3}$ *full.*



Complete this chain:



In the pre-test, no learners in either grade were successful in solving any of the contextualized questions. However, in both grades learners showed improvement after exposure to the MALATI programme, achieving success in both the contextualized and context-free questions.

Although Zukiswa struggled in the pre-test, the impression is that she now shows some understanding of fraction concepts. Her success in solving the above problems seems to show that her problem-solving strategies are now based on some understanding of the fraction concept and operations involving fractions.

The success and understanding shown by Zukiswa was evident in most of the learners who received instructions with the MALATI approach. The Grade 5s' understanding at the end of the year was also better than the Grade 6 performance in the pre-test. This also suggests that the MALATI approach contains some elements that can address some of the errors identified in this paper.

After the MALATI intervention, learners responded to the structure of the problem posed, and showed no tendency to grasp at unsuitable and sometimes partly remembered algorithms.

Conclusion

The different individual responses of the learners both in the pre-test and in the MALATI activities suggest that these learners do not apply the method as taught by the teacher. They have constructed their own strategies, whether wrong or right, and they use them to solve various problems even where they are not applicable.

This study therefore highlights the fact that incorrect strategies are further entrenched by the traditional teaching of fractions, that is, by showing learners the different algorithms, we (the teachers) impose arbitrary definitions on learners that make no sense to them. By giving them algorithms to find the LCM, we impose rules on learners that do not make sense to them. Learners' knowledge of algorithms is often faulty and frequently interferes with their thinking in two ways: firstly, knowledge of number procedures often keeps learners from drawing on their informal knowledge of fractions from the context of real-world situations. Secondly, learners often trust answers obtained by applying faulty procedures drawn from traditional instruction without questioning whether the answers are suitable or not, because they are not used to making sense of mathematics.

This study suggests that the building of a strong fraction concept could provide learners with the ability to think about and deal successfully with the addition of fractions in ways which make sense to them without grasping at rules and algorithms that they do not understand.

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