RETEACHING FRACTIONS FOR UNDERSTANDING[1]

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This paper reports on the viability of a programme aimed at encouraging sixth grade students who have already been exposed to teaching practices leading to entrenched limiting constructions, to construct the concept of a fraction anew and to invent solution strategies for realistic problems involving fractions, in a school and classroom environment with serious practical and organisational problems.

Introduction

Much research has been done on the problems elementary school students experience with common fractions and on the design of teaching programmes for fractions at different grade levels (see Pitkethly and Hunting, 1996, for a review of the research).

An important issue is the effect that limiting constructions (D'Ambrosio & Mewborn, 1994) has on students' attempts to make sense of fractions. These include, for example, the influence of whole number schemes, which encourage the student to interpret the fraction symbol as two separate whole numbers, and limited part-whole contexts, where the student has had no or not sufficient experience of fractions as parts of collections of objects. Another issue is the possible adverse effect of rote procedures on students' attempts to construct meaningful algorithms for operations on fractions (Mack, 1990).

The above problems can be *prevented* by appropriate programmes for learning fractions in the lower elementary grades (e.g. Empson, 1995; Murray, Olivier & Human, 1996). However, when these limiting constructions are already firmly entrenched, it is to be expected that the task of encouraging students to develop strong and error-free conceptual and procedural knowledge about fractions will be much more difficult. Such attempts have already been made successfully (e.g. Bell, 1993; Kamii & Clark, 1995; Mack, 1990) in what we believe to be favourable learning environments.

In this paper we explore the possibility of implementing a programme for common fractions for Grade 6 students in less than favourable learning environments.

Theoretical framework

In line with our approach to the teaching and learning of whole number arithmetic (e.g. Murray, Olivier & Human, 1994, 1998), we believe that the teaching and learning of fractions should be based on eliciting and clarifying students' intuitions about fractions through posing realistic problems for which students have to invent their own solution strategies (cf. Empson, 1995; Kamii & Clark, 1995).

[1] Murray, H., Olivier, A. & De Beer, T. (1999). Reteaching fractions for understanding. In O. Zaslavsky (Ed.), **Proceedings of the Twenty-third International Conference for the Psychology of Mathematics Education: Vol. 3**. (pp. 305-312). Haifa, Israel.

The following aspects are crucial in our approach to the teaching and learning of fractions:

Choice of problems. Knowledge about fractions involves knowledge about the concept of fractions, of which two subconstructs are the part-whole relationship between the fractional part and the unit, and the idea that the fractional part is that quantity which can be iterated a certain number of times to produce the unit. The unit may be a single object or a collection of objects. Fractions are also used in different ways and have different meanings, for example the part-whole mentioned above, but also a ratio, a quotient, a measure, etc.

If the problems posed in a teaching programme do not include, and students do not experience, these different subconstructs and meanings within a reasonable time, limiting constructions are formed (e.g. Murray, Olivier & Human, 1998). For the same reason, the fractions addressed should also immediately include thirds, fifths, etc., and not only halves and quarters, as is common in many teaching programmes. For example, in a previous study we found that Grade 1 students freely constructed appropriate different sized fractional parts in response to realistic problems, whereas many of the Grade 3 students in the same school who had only been exposed to halves and quarters during their teaching programmes for fractions, could not conceptualise thirds and/or could not recognise the difference between halves and thirds when they were trying to solve the same problems (Murray, Olivier & Human, 1996).

Social interaction. Social interaction creates opportunities for students to *talk* about their thinking, and this talk encourages reflection. ``From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics … To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted." Also, ``… leading students to discuss *their* view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies" (Von Glasersfeld, 1991, pp. xviii, xix).

We therefore believe that we should not only provide opportunities for students to build on their informal knowledge, but that students should be encouraged to make explicit and *become aware of* the nature of their own personal constructions (Bell, 1993).

Students' own representations. Students are expected to create their own representations of fractions. This is achieved by confronting students with sharing situations where a remainder also has to be shared out, for example sharing four chocolate bars equally among three friends (Murray, Olivier & Human, 1996). Prepartitioned materials are not used until later and the introduction of written symbols for fractions is delayed until the need for fractions and some conceptions of fractions have been developed by the students themselves.

This study

This study forms part of the Mathematics Learning and Teaching Initiative (MALATI) project aimed at curriculum and teacher development. The project teachers received student worksheets and teachers' guides which were studied during workshops, and were supported by regular classroom visits of project workers during the 1998 academic year. The student worksheets consist of two packs of activities, an introductory pack and a further pack.

The introductory activity pack of 33 worksheets is aimed at

- developing the fraction concept through sharing situations
- introducing realistic problem situations for operations involving fractions (e.g. division by a fraction)
- comparison of fractions
- equivalence of fractions
- introducing the fraction notation

The further pack attempts to make explicit students' informal procedures for the operations developed in the introductory pack.

Teachers were requested to firstly spread out the work over the year, and secondly to use the worksheets in the suggested sequence, with all the students in the class working on the same worksheet during a particular lesson period. The reason for this is that students are encouraged to solve a problem at whatever level they feel comfortable, and that students share their conceptualisations with the class to the benefit of all. Since our materials repeatedly pose problems with the same structures, it provides students with repeated opportunities to make sense of particular structures.

In this paper we limit ourselves to describing the effect of this intervention on a specific sixth grade class in one of the MALATI elementary project schools in a black township near Cape Town. The circumstances in the school and in the community do not support learning. Many students have transport problems to school and come from extremely unstable and impoverished homes. Absenteeism is not only high, but malnutrition and chronic health problems prevent some students from functioning optimally even when they are at school. The school organisation is poor; the lesson timetable is not followed and there are continual unscheduled interruptions in the form of staff meetings, celebrations and outings during school time. It is difficult to persuade teachers to attend workshops after school hours.

There were 42 students in the class. The teacher, Nolo, was eager to co-operate and although she started the year in a very traditional way (in spite of the initial workshop), she had after a few weeks established a culture of enquiry, argument and discussion among most of the students. However, even by the end of the year some students were unwilling to share their ideas through fear of being wrong.

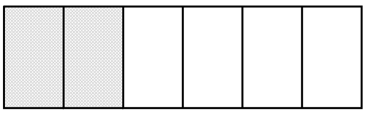
By the end of the year, this class had only completed the introductory pack. This reflects their low level of knowledge at the beginning of the year as well as the above-mentioned practical and organisational problems during the year.

Results

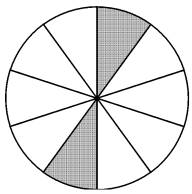
We have available two sets of written tests and all the students' written work during the year as well as several videotaped classroom episodes.

Test Set A comprises a pre-test completed in November 1997 (the end of the previous academic year) by all the sixth graders of the school under discussion, and a post-test completed in November 1998 by the sixth graders of Nolo's class. These are therefore not the same students, but students in the same grade in successive years in the same school.
Test Set B comprises a pre-test completed at the beginning of the 1998 academic year by Nolo's class and a post-test (a different one from Set A) completed by Nolo's class at the end of the academic year.

Test Set A. The items testing students' part-whole conceptions showed a very substantial gain in 1998. For example, these are the percentages of students correctly stating the fraction shaded in the following figures:



1997: 40%
1998: 90%



1997: 33%
1998: 89%

The following item testing the comparison of fractions also showed a substantial gain from 18% in 1997 to 31% in 1998:

Anwar and Amina each received R30 pocket money. Anwar spent $\frac{5}{8}$ of his pocket money and Amina spent $\frac{7}{10}$ of hers. Who spent more? Explain your answer.

Test Set B. The pre-test at the beginning of the year revealed that students had very little knowledge of fractions. The post-test showed definite gains. For example, the success rate for the item ``*Which is bigger, $\frac{3}{5}$ of a cake or $\frac{3}{4}$ of the cake?*" had a success rate of 4% in the pre-test, and a similar item had a success rate of 45% in the post-test. The success rate of the item ``$\frac{3}{5}$ *of 10*" increased from 4% to 31%.

Both pre-tests identified the following main problem areas in students' understanding of fractions:

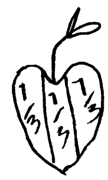
- a very weak understanding of the fraction concept
- strong interference from whole number schemes
- strong interference from rote algorithms

For example, this type of error occurred frequently as a response to ``Which is bigger, $\frac{3}{5}$ of a cake or $\frac{3}{4}$ of a cake?"

$$\frac{3}{5} - \frac{3}{4} = \frac{0}{1}$$

Although both post-tests showed substantial gains, we felt the success rates to be low considering that students had solved similar and more difficult problems during their lesson periods.

Students' written work. At first, there were strong signs of previous teaching. Mangaliso offered this incorrectly partitioned *apple* as solution to a problem involving *chocolate bars* (apples are frequently used to demonstrate fractions in many teaching programmes).

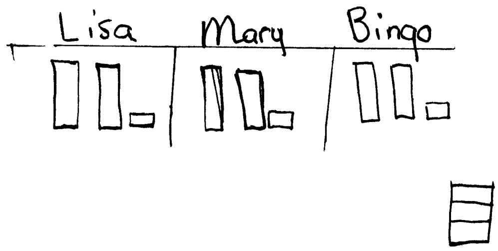


Because A Hole divided by three is a third

After a while, this type of response did not occur again in Nolo's class.

Most students produced their own representations of fractions in response to the initial sharing problems. For example, Thomboloxole solved the following problem as follows:

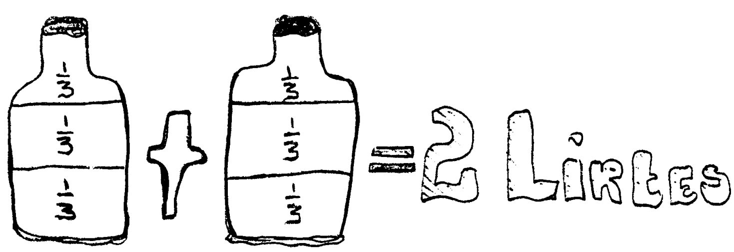
Lisa, Mary and Bingo have 7 bars of chocolate that they want to share equally among the three of them so that nothing is left. Help them to do it.



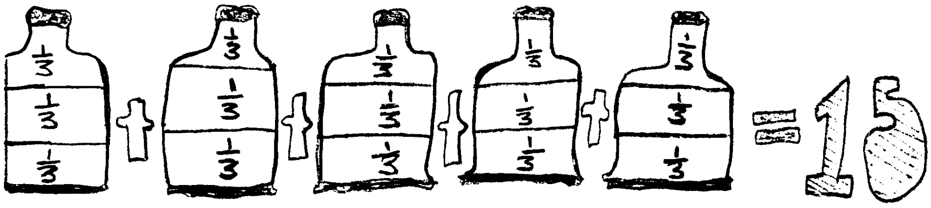
They also solved problems which prepared the way for operations with fractions successfully through their own representations of the physical situations. For example, Worksheet 9 poses an addition and a division-type problem. Zanele solved the two problems like this:

Peter and Anna prepare soft porridge for breakfast. For each bowl they use $\frac{1}{3}$ of a litre of milk.

- *If they make 6 bowls of porridge, how many litres of milk do they use?*



- *They have 5 litres of milk. How many bowls of porridge can they prepare?*

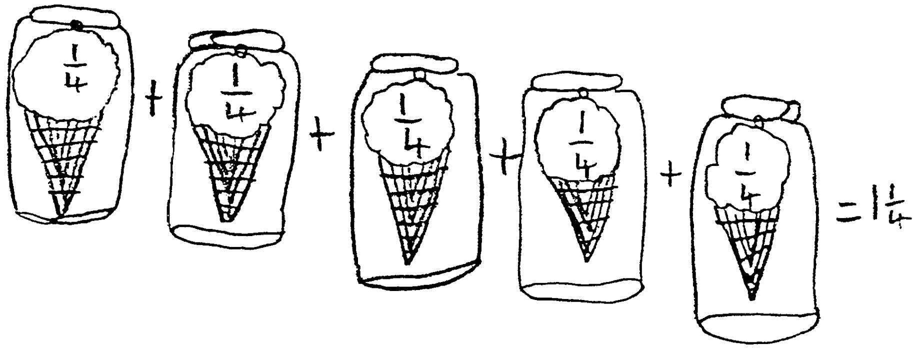


Although all the students worked on the same worksheet during a particular lesson, they functioned at different levels of abstraction. For example, Dumisani solved the previous problem like this:

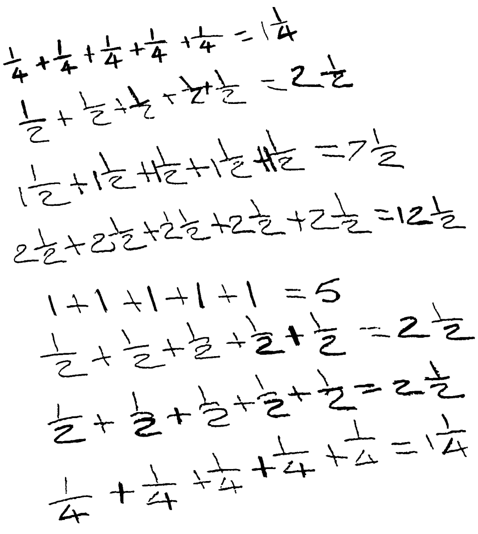
$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$$
$\frac{1}{3}$ = 15 bowls porridge

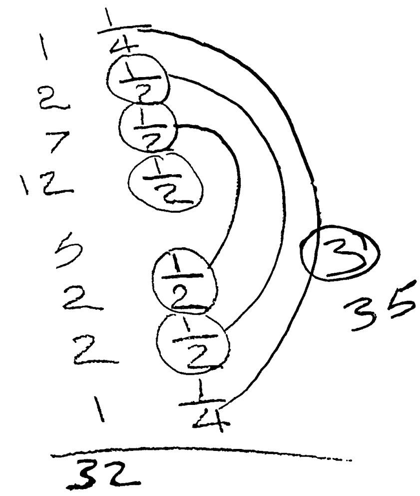
Students frequently used drawings which seemed inappropriate to us. After the initial realisation that drawings were acceptable as solution methods and as means of communication, some students came to believe that they *had* to make a drawing. For example, this drawing by Bonziwe as a solution to the following problem seems to be more decorative than functional:

How much ice-cream in total does Lisa have if there are 5 containers of ice-cream, each $\frac{1}{4}$ full?



Other classrooms in the same school produced a peculiar marriage of sense-making and rote learning. For example, Worksheet 8 lists the ingredients and their amounts in cups and teaspoons for one cake, and requires students to calculate how much of each ingredient would be needed for five cakes. Some students successfully calculated the quantities of each ingredient needed for five cakes, and then (correctly, but inappropriately) added up all the quantities in order to produce a *single* numerical answer. When challenged whether this was the *sensible* thing to do, the response was ``no, but in mathematics we have to give only one answer to a sum". A solution such as this never appeared in Nolo's class (these are the different ingredients):





Discussion

The results show that in this class the intervention achieved decided success in addressing students' conception of fractions. Students' written worksheets show that inappropriate whole number schemes had disappeared, and that students were able to invent procedures to solve realistic problems involving fractions. It is also quite clear that in an approach like this, the traditional idea that division by a fraction is the most difficult problem type does not hold. Zanele's and Dumisani's solutions for the second problem of Worksheet 9 ($5 \div \frac{1}{3}$) illustrate this. We suspect that division may be the easiest of the four operations for which to invent informal operations as long as the context is sensible to the student (Mack, 1990).

On the negative side, many students could not manage realistic problems in the test situations although they had solved similar problems successfully as collaborative groups without the help of the teacher. It is possible that because many students missed worksheets through absenteeism, their understanding of a particular problem structure could not become stable.

We tried to help the teacher to identify students who were behind so that they could be given additional learning opportunities, but this proved to be organisationally unmanageable. We still believe that students solving problems collaboratively, combined with whole class discussions, is a better approach than individualised learning (compare Bell, 1993).

It is also possible that language caused a problem during the tests. English is the second language of these students, and in class the groups spent a significant amount of time talking about the problem situation before they started to solve it. Although the teacher explained the wordings of problems during the tests, this was probably not sufficient. It is possible that one or two written tests *in the course of the year* might have improved their performance in the final test. (We originally strongly rejected the idea of regular written tests because the then-existing school culture depended heavily on written evaluations.)

The question we posed at the beginning of the paper was whether it was possible to develop anew a stable conception of fraction in students who had already formed limiting constructions, using an approach which expects students to make sense of realistic problems and invent their own procedures in an atmosphere of discussion and argument, and whether this could be done under difficult conditions. This has proved to be possible.

An intervention in extremely adverse learning environments also has positive effects: An approach only reveals its essential aspects in difficult teaching and learning environments, whereas a competent teacher in a supportive environment unconsciously anticipates and copes with possible problems. As an example, we cite the case of the students who did indeed solve the problem where they had to calculate the quantities of ingredients needed for five cakes competently and sensibly, but then offered solutions which they considered to be mathematically correct but which made no sense at all. It has therefore become clear that encouraging students to build on their informal knowledge, and solve problems through their own inventions, without also changing their beliefs about the *nature* of mathematics, is not sufficient.

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