PROBING CHILDREN'S THINKING IN THE PROCESS OF GENERALISATION

Marlene Sasman, Liora Linchevski, Alwyn Olivier & Rolene Liebenberg

Mathematics Learning and Teaching Initiative (Malati)

This paper reports on part of a study of students' ability to handle algebraic generalisation problems. In this paper we focus and elaborate on moments when students grapple with deciding about the validity of their generalisations. We interviewed ten students near the end of grade 7. During the interviews we tried to create cognitive conflict by challenging the students' justification for the methods they used and then documented their attempts to resolve such conflicts. We found that most students' justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation.

INTRODUCTION

Number patterns, the relationship between variables and generalisation are considered important components of algebra curricula reform in many countries. It is evident that South Africa shares this view as can be seen in the curricular changes to the Junior Secondary syllabus (AMESA Western Cape Region Inservice Curriculum Material for Algebra, 1995; Syllabus of Western Cape Education Department, 1996). Our observations indicate that although teachers were keen to try an approach to algebra via generalisation, many of them very quickly reverted to the traditional way. According to them children experience major difficulties with the new approach and they do not have the tools to address these difficulties. Moreover, they claim that most of the available learning materials do not correlate with the proposed changes to the syllabus.

In our work at the Mathematics Learning and Teaching Initiative, a project aimed at informing curriculum development and designing learning materials, we are attempting to address these difficulties. We found, in the current literature, that indeed some research has been done on algebraic generalization (e.g. Garcia-Cruz and Martinon, 1997; Taplin, 1995; Orton and Orton, 1994, 1996; MacGregor and Stacey, 1993). However, there is not enough in these reports to enable us to delve into the students' thinking in the processes of generalization and validation. For example, do students view their efforts at generalising as hypotheses? Do they realise the necessity to validate their strategies and answers, i.e. do they reflect on their strategies and answers? How do they become convinced that their generalisations are correct or wrong? Garcia-Cruz and Martinon (1997) for example, report that most children they interviewed checked their rules. This was done either by counting or drawing or extending the numerical sequence. It is not clear from their report, however, whether their students spontaneously checked their answers because they felt the need for validation, or how they became convinced of the validity of their strategies and answers.

In this paper we focus and elaborate on such moments where students grapple with deciding about the validity of their generalisations. During interviews with children, we tried in several ways to create cognitive conflict by challenging their justification for the methods they used and then documented how they tried to resolve such conflicts.

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RESEARCH CONTEXT

The project enlisted eight schools in the suburbs of Cape Town as project schools. Seven of the eight schools are in traditional black townships. All the children interviewed in this research came from one of these seven schools. As a baseline study for the project's diagnostic purposes, we have been collecting data on children's performance in mathematics using various tools. One of these tools is a written baseline test.

RESEARCH METHODOLOGY

As a first stage we wanted to gather data on the most mathematically competent students. The students were chosen on the basis of performance in the baseline test and the teacher's evaluation. We interviewed ten students near the end of grade 7. Each student was interviewed three times in 45-minute sessions by two of the researchers, twice individually and once in pairs. A fourth session took place in which the students were given two generalisation problems to do individually. All interviews were videotaped. In addition to the video protocols, written transcripts of the subjects' verbal responses as well as their paper-and-pencil activities were used in the analysis.

THE PROBLEMS

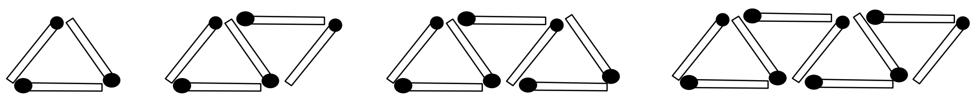
We presented the students with a series of eight generalisation problems in which we varied the representation of the problems. Some problems were formulated in terms of numbers only (in the form of a table of values), some were formulated in terms of pictures only (in the form of a drawing of the situation) and some problems were formulated in terms of both pictures and numbers.

The questions were in each case basically the same, namely given the values of f(1)[1], f(2), f(3), and f(4), we asked students to find the values of f(5), f(20) and f(100) and to explain and justify their answers and strategies. Six of the functions were linear functions of the form $f(n) = an + b$, and two functions were simple quadratic functions of the form $f(n) = n^2$. Here are two examples, "cans" and "matches":

(B1): Cans are packed to form pyramids.
The table shows how many cans are needed for different pyramids. Complete the table.

Pyramid number	1	2	3	4	5		20		100
Number of cans	1	4	9	16					

(C3): Matches are used to build pictures like this:



Picture 1 Picture 2 Picture 3 Picture 4

The table shows how many matches are used for the different pictures. Complete the table.

Picture number	1	2	3	4	5		20		100
Number of matches	3	5	7	9					

[1] Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for reporting on the students.

Whatever responses the children gave, we asked them to *explain* their answers by posing questions like: "Can you explain how you got this answer?", or "Convince me that your answer is correct", or "Show me how you got this answer". If the students' explanations were based on the information given in the problem (the *database*), we accepted it as a satisfactory answer.

SOME RESPONSES

Some general observations

Concerning children's use of different representations for the problems, it is interesting that all but one of the children worked exclusively in the number context and did not use the structure of the pictures at all. In the problems that were formulated in terms of pictures, children immediately constructed a "table" of values and then used only the table of values in their solutions and explanations.

Concerning children's strategies, it is interesting that for the simple quadratic problems, nearly all the children recognised the functional rule $f(n) = n^2$ from the database and used it to find the values of f(5), f(20) and f(100). For example, Thandi explains: "I say 2 times 2 is 4, 3 times 3 is 9, 4 times 4 is 16, 5 times 5 is 25". However, in *all* the linear problems all students correctly used recursion to find f(5) as f(4) + d where d is the common difference between successive terms. For example, Thandi explains how she finds f(5) = 36 in the table for the function f(n) = 8n + 4: "I say 4 plus 8 is 12, 12 plus 8 is 20, 20 plus 8 is 28, 28 plus 8 is 36".

It is also interesting that when they had to find f(20) and f(100), most children abandoned their successful recursive strategy because they were trying to find a "shortcut" to calculate f(20) and f(100). These short methods were mostly *not* based on the database and were seriously prone to error. None of our students felt the need for any kind of validation. Although they offered some kind of explanation for the method they used in the extended domain, they were not aware of the role of the database in the process of validation.

Some alternative interpretations

In several cases children's answers were far from what we would expect, yet still based on the database. For example, in the cans-problem Roy wrote that f(5) = 23, f(20) = 28 and f(100) = 31, because he was using a symmetry structure (3; 5; 7; 7; 5; 3):

$$f(2) = f(1) + 3; \quad f(3) = f(2) + 5; \quad f(4) = f(3) + 7$$

$$\therefore \quad f(5) = f(4) + 7; \quad f(20) = f(5) + 5; \quad f(100) = f(20) + 3.$$

He ignored the shaded columns that we intended as representing several "missing" columns in the table.

Also in the cans-problem Sipho wrote: f(5) = 20, f(20) = 100 and f(100) = 600. This seemed to us rather arbitrary, but he was in fact using the rule that f(n) is a multiple of n, without specifying which multiple:

Interviewer: Can you explain how you got 20? [*for f(5)*]
Sipho: I took pyramid .... [*pause*] .... I saw that each doesn't have a remainder.

In the same problem Vusi wrote f(5) = 25. We were, of course, sure that he was using the functional rule $f(n) = n^2$. However, then he wrote f(20) = 120 and f(100) = 800, explaining that "you multiply each number in the upper row by the number of the column". Closer questioning revealed that he misinterpreted the shaded columns. For Vusi n = 20 was in the 6^{th} column, so f(20) = 20 × 6, n = 100 was in the 8^{th} column and his rule therefore produced f(100) = 100 × 8 = 800. He ignored the first shaded column and then counted the next 3 columns as the 6^{th}-, the 7^{th}- and the 8^{th} column.

Some mistakes

Students made various mistakes, for example, to concentrate only on the relationship between a single pair (n ; f(n)) and then to use it as a general rule. For example, in the cans-problem Roy saw that f(3) = 3 × 3 = 9 and then used the rule f(n) = 3n to find f(5) = 3 × 5 = 15.

However, the most common, nearly universal mistake children made in their efforts to find a manageable method to calculate larger values, was to use the proportionality property that if $x_2 = k \times x_1$, then $f(x_2) = k \times f(x_1)$. For example, in the matches-problem, Mathole, after finding f(5) = 11, calculates f(20) as 4 × 11 = 44. This mistake was also found by Taplin (1995) and Garcia-Cruz and Martinon (1997).

CREATING CONFLICT

When we were not convinced that the students' responses reflected awareness of the role of the database in the justification process, we tried to create a cognitive conflict, using three different strategies as described below. (Because children were not using the pictures, we did not use a strategy of drawing pictures to check their answers.)

Strategy 1: The first strategy we used was to confront the answer driven from the recursive approach with the one obtained by the mistaken approach. This strategy was used when the child had in front of him/her a table he/she had formed in order to find some f(n) through recursion.

For example, Vusi and Thandi, working as a pair, used the recursive method to correctly determine f(20). In order to determine f(100), they systematically continued using the recursive method. However, when they reached f(50) they changed to the multiplication method, claiming that f(100) = 2 × f(50). We wanted them to reflect on the incorrect multiplication method. For this purpose we challenged them to apply their multiplication method on the domain between 1 and 50 since they had already obtained these values by the recursive method. Vusi was asked to find f(20) using f(5) and the multiplication method.

Vusi: Its 72 [*multiplying 18 by 4*]. I got 63 [*the result he obtained by the recursive method*].

Vusi is puzzled but still unconvinced that his method is wrong. He decides to recheck his multiplication method on the database:

Vusi: Lets try this one [*looking at f(2) and f(4) in the database*]. If 2 goes 2 in 4, so I must multiply 9 [*the value for f(2) in the given database*] by 2 is 18, but its 15 [*the value for f(4) in the given database*].

Interviewer: So what do you say when I ask you about 100?

Vusi: I said 20 times 5 so its 100. So 63 [*the value he obtained for f(5)*] times 5.

Vusi is sure that his answer for f(20), 63, he obtained by the recursive method is correct and the other answer for f(20), 72, obtained by $4 \times f(5)$ is wrong. He is sure the method to get f(4) by $2 \times f(2)$ is incorrect but at the same time he is not willing to give up his multiplication method when it comes to f(100).

Strategy 2: The second strategy was to create a conflict by choosing a take-off point different from the one the child had used when applying the multiplication method. Choosing different take-off points led to different answers for f(n). For example, Vusi spontaneously evaluates f(100) as $2 \times f(50) = 2 \times 147 = 294$. The interviewer prompts him to use different take-off points. He takes f(10) and f(20) and obtains $f(100) = 10 \times f(10) = 330$ and $f(100) = 5 \times f(20) = 315$.

Interviewer: Oh, so who is right?
Vusi: Now we have three plans.
Interviewer: Ok, I understand three plans, but I also have three answers, 330, 315 and 294. Are they all right?
Vusi: Yes, they are all right.

Thandi and Vusi are sure about the values they obtained for f(10), f(20) and f(50) since these values were obtained by the recursive method. f(100), however, is an abstract entity for them. The fact that the three different take-off points led to three different answers for f(100) did not lead them to question the *method* they used.

Mathole, when confronted with different answers for different take-off points is also not prepared to abandon the multiplication method, but attempts to give a justification for the different answers:

Interviewer: And now you said that in shape number 20 we have 144 okay? Because you took this 5 ... you divided 20 by 5 and timesed 36 [*the value for f(5)*] by 4. We are sure about it. Okay, let's say that your friend goes to shape number 4 and he now divides 100 by 4. To divide 100 by 4 gives 5, so he goes and times 28 [*the value for f(4) in the given database*] by 5, do you follow me?
Mathole: Yes
Interviewer: So he multiplies 28 by 5, how much is it? [*works on calculator*] 140. So what is the correct one, 144 or 140?
Mathole: 144
Interviewer: Why?
Mathole: Because ... here by the fourth shape you got 28 matches and fifth shape is 36 matches, so if he goes back to the ... to the ... 28 he'll have to add 4 and if he goes back to the third shape he'll have to add 8, it's like you tax a person for going back, you let him pay for going back, so he'll have to pay 4 for going back, then you'll have to add a 4 there, then you'll get the 144.

Strategy 3: The third strategy was to implement the method the child used in the extended domain on the domain given in the original table.
For example, Thandi obtained an answer for f(5) by correctly using the recursive method $f(5) = f(4) + 8 = 36$. For f(20), however, she wrote 28, explaining:

Thandi: I count to shape 5, and I count to 20 and then I add this top numbers [*refers to the shape number in the table*] by 8.

While f(5) was obtained correctly using the recursive rule f(n) = f(n – 1) + 8, she now changes her rule to find f(20) by using the function rule, f(n) = n + 8. She is then taken back to f(5) and asked how she obtained 36. She adds 8 to 5 (the shape number) and gets 13, not 36. She now realises that there is a contradiction. Thandi now no longer accepts her answer for f(20).

STRUGGLING FOR CONVICTION

It was clear that conviction about the role of the database in the process of validation develops *slowly*. Despite our efforts to create conflicts in order for them to reflect on the proportionality multiplication error and on the process of validation in early interviews, the same children repeatedly made the same mistake in later interviews. We follow below Sipho's struggle to come to terms with the proportionality multiplication error.

In the first problem given, Sipho obtained 36 for f(5) by using recursion correctly. However, for f(20) he abandoned recursion and used the multiplication method explaining: “5 goes four times in 20 so I multiply 36 [*the value he obtained for f(5)*] by 4 to get the number of matches in shape 20.” The interviewer challenged him to apply his multiplication method on the domain 1 to 5, to obtain f(4) as 2 × f(2) and f(5) as 5 × f(1). Sipho was sure that the answer for f(5) he obtained by recursion was the correct one and not the answer obtained by the multiplication method. However, when asked again about f(20) and later on f(100) he consistently used the multiplication method. This happened again in the next problem.

In the second interview Sipho was working with David. Both of them used recursion to obtain f(5). However, for f(20), David continued systematically with recursion, finding f(20) = 63, while Sipho used the multiplication method, finding f(20) = 4 × f(5) = 4 × 18 = 72.

Interviewer: I do not follow, shape number 20 is 63 or 72? [*strategy 1 as above*]
David: I go my way, adding 3 and 3 and 3
Sipho: [*to David*] I see the method is right, but can you tell me what I have done wrong to get the wrong answer?

At this point Sipho confronts the two methods which is significant since he realises that there is a conflict. He is sure the recursion method gives the correct answer and realises that his multiplication answer gives an incorrect answer. He is interested in *why* the multiplication method is wrong. However, in the very next moment Sipho again succumbs to the multiplication error:

Interviewer: What about shape 100?
Sipho: I times because you know that I get 5 20’s .... I think I’ll times 63 by 5 to get it.
David: That’s the wrong way.
Sipho: If I didn’t times, I added 3,3,3, ... I would get the same answer.
Interviewer: Where do you see multiplication? [*Strategy 3*] I can understand where the 3 came from. I saw that it’s given here [*points at the table and the differences between the number of matches*]. Where did you get the multiplication? Can we check?
Sipho: Shape 3. I just go to shape 4.
Interviewer: If you want to get to shape 4 with your method what would you times?

Sipho: I would times the number of matches here [*points at shape 2*] by 2.
Interviewer: And what number of matches will you get?
Sipho: 18
Interviewer: And what is written here? [*points at f(4) in the given data base*]
Sipho: 15
Interviewer: So?
Sipho: Yes,eh

At the end of the interview we are left with the impression that Sipho is convinced that his multiplication method is incorrect, because the given database does not reflect the multiplication method.

In the third interview it appears as if the previous discussions with Sipho had not taken place. He still uses the incorrect multiplication method to obtain f(20). He is taken back to the given database to reflect on how he obtained f(5):

Sipho: Because shape 1 is 3 and shape 2 is 5 and the difference is 2. [*referring to C3*]

Sipho is now pushed to reflect on the given database and his method for obtaining f(20). He realises that if he uses the multiplication method on f(1) to obtain f(4), it will not be the same as the value for f(4) in the given database. This conflict leads him to use recursion to find f(20) = 41. Yet he reverts to the multiplication method to obtain the value for f(100). He is challenged by the interviewer:

Interviewer: It does not work for f(20) but you think it might work if you go from 20 to 100?
Sipho: Yes, because I think the number of matches in shape 20 is *now* right.

This remark sheds some light on Sipho's line of thought. He thinks since he *now* has the correct value of f(20) he can use it for f(100). For him, the problem was not the *method* but the wrong *value* of the take off point. It seems that Sipho is sure about the value of f(20) which was obtained by recursion. He is convinced that the multiplication method does not work for f(20), but nevertheless, from his perspective, it still works for f(100), provided that the value of f(20) is correct. He is now challenged to use the multiplication method on f(5) to obtain f(20). This yields a value of 44, which he knows is wrong because he obtained f(20) = 41 by recursion. He is puzzled:

Interviewer: Now, you think 5 times 41, 205, you say it's right for f(100).
Sipho: I think it's wrong.
Interviewer: Why
Sipho: Because I did the same thing when I multiplied. I tried to multiply the number of matches by 5 … I saw that I was wrong.
Interviewer: So how will you then do 100 [*shape number 100*]?
Sipho: I think I have to do it like this [*points at the list for f(20)*] but it will take a long time.
Sipho: [*Long pause*]…I'm trying to think if I can do another method to get the answer of eleven [*the answer for f (5) - he is trying to look for a functional rule*]. I'm trying to multiply the number of, number of shape 5.

Yet, albeit a slow development, there were successes: in the final written test six out of the ten students avoided making the multiplication error.

DISCUSSION

Our study shows that our interviewees did not view their answers as hypotheses that should be validated. They were not aware of the role of the database in the process of generalisation and of validation.

Although we were aware that students frequently succumb to the proportional multiplication error, its persistence and obstinance to change surprised us. On the one hand students easily convinced themselves that when a value for f(n) they obtained using recursion differed from the value they obtained using their multiplication method (our validation strategy 1 described above), the result obtained by multiplication was *incorrect*. On the other hand, the knowledge that their multiplication method produced incorrect answers did not *prevent* them from making the mistake again (and again). Indeed, when they were asked for the value of f(m) for m > n in the same problem, all the students again resorted to the multiplication method and were sure that their answers are correct. Our efforts to create cognitive conflict by leading students to apply the multiplication method to different take-off points and getting different values for f(n) (our strategy 2 above), or drawing their attention to the fact that the multiplication method is not applicable in the domain given in the table (our strategy 3), did not easily eliminate the error. Most students continued using the multiplication method throughout the interviews. Six out of the ten students eventually avoided the error in the fourth session.

One could argue that our choice of numbers *triggered* the proportional multiplication error, i.e. that our use of "seductive numbers" like n = 5, 20 and 100 *stimulated* the error. One could also argue that that if we used non-seductive numbers like n = 17, 27 and 83 children would not use the erroneous multiplication method. However, we believe that our evidence shows that children, in their quest for a manageable short method, *create* "seductive numbers" themselves. For example, Thandi was busy using a laborious recursive strategy on her way to calculate f(100) – she continued to f(50) and then suddenly stopped and calculated f(100) as $2 \times f(50)$, probably because she immediately recognised the multiplicative relationship between 50 and 100. Nevertheless, it remains a question for further research to establish whether an approach with non-seductive numbers will *prevent* children from making the multiplication error, also when they encounter seductive numbers in other problems.

REFERENCES

Garcia-Cruz, J. A. and Martinon, A. (1997). *Actions and invariant in schemata in linear generalising problems*. In E. Pehkonen (Ed.), Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education, Vol. 2, 289-296. Lahthi, Finland.

MacGregor, M. and Stacey, K. (1993). *Seeing a pattern and writing a rule.* In I. Hirabayashi, N. Nohda, K. Shigematsu and F. Lin (Eds), Proceedings of the 17th Conference of the International Group for the Psychology of Mathematics Education, Vol. 1, 181-188. Tsukuba, Japan.

Orton, J. and Orton, A.(1994). *Students' perception and use of pattern and generalization*. In J.P. da Ponte and J.F. Matos (Eds.), Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3, 407-414. Lisbon, Portugal.

Taplin, M. (1995). Spacial patterning: *A pilot study of pattern formation and generalisation.* In L. Meira and D. Carraher (Eds.), Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education, Vol. 3, 42-49. Recife, Brazil.

Drouhard, J.P. (1997). Triple Approach. A theoretical frame to interpret student's activity in Algebra. In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education,* Vol. 2, 225-232.

Booth, L.R.(1989). See the Pattern: Approaches to Algebra. *The Australian Mathematics Teacher,* Vol.45, no.2, pp. 12-13.

Western Cape Education Department, AMESA Western Cape Region Inservice Training Partnership for Junior Secondary Mathematics, Inservice Curriculum Material for Algebra, 1995, pp.1-51.

Western Cape Education Department, Junior Secondary Course, Syllabus for Mathematics, Standards 5 to 7, South Africa (For implementation 1996-1997).

Wong, M.P.H. (1997). Numbers versus letters in algebraic manipulation: which is more difficult? In E. Pehkonen (Ed.), *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education,* Vol.4, 285-289.