FACTORS INFLUENCING STUDENTS' GENERALISATION THINKING PROCESSES

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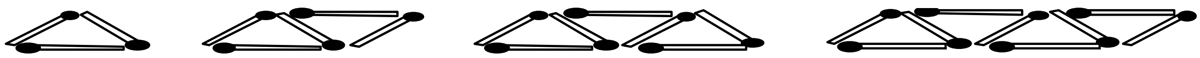
In this study we presented students with generalisation activities in which we varied the representation along several dimensions, namely the type of function, the nature of the numbers, the format of tables, and the structure of pictures. Our results show that varying these dimensions has little effect on children's thinking – as in our previous study, few children tried to find a functional relationship between the variables, but persisted with using the recursive relationship between function values, making many logical errors in the process.

INTRODUCTION

Number patterns, the relationship between variables and generalisation are considered important components of algebra curricula reform in many countries. These curricula often use generalised number patterns as an introduction to algebra. However, there is insufficient research that deals with the cognitive difficulties students encounter and the feasibility of such an approach. Much of the available research on students' thinking processes in generalisation reports on students' strategies in abstracting number patterns and formulating general relationships between the variables in the situation (e.g. Garcia-Cruz and Martinon, 1997; MacGregor and Stacey, 1993; Orton and Orton, 1994; Taplin, 1995).

In a previous study (Linchevski, Olivier, Sasman & Liebenberg, 1998) we presented grade 7 students with problems like the following:

(C3): Matches are used to build pictures like this:



Picture 1   Picture 2   Picture 3   Picture 4

The table shows how many matches are used for the different pictures. Complete the table.

Picture number	1	2	3	4	5		20		100		n
Number of matches	3	5	7	9							

Few students managed to construct a function rule to find function values. Rather, they focussed on recursion (e.g. $f(n + 1) = f(n) + 2$ in problem C_3 above), which led to many mistakes as they tried to find a manageable method to calculate larger function values. The most common, nearly universal mistake was to use the proportionality property that if $n_2 = k \times n_1$, then $f(n_2) = k \times f(n_1)$. For example, in problem C_3 above, from f(5)[1] = 11 they deduced that $f(20) = 4 \times 11 = 44$. Although this property applies only to functions of the type $f(n) = an$, students erroneously applied it to any function. It is possible that our use of "seductive numbers" in a sequence like n = 5, 20 and 100 *stimulated* the error (we regarded these numbers as seductive from a multiplicative point of view).

[1] Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for ease of communication.

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We found that most students' generalisations and justification methods were invalid, because they are not aware of the role of the database in the process of generalisation and validation. For example, in problem C3 above, they did not, and seemed unable, to verify their generalisations against the given data pairs (1 ; 3), (2 ; 5), (3 ; 7), (4 ; 9).

We also found that students worked nearly exclusively in the number context and did not use the structure of the pictures at all.

Based on the above we viewed the following as questions for further research:

- whether the use of non-seductive numbers will *prevent* students from making the multiplication error, also when they encounter seductive numbers in other problems
- whether the visual impact of the table, as for example shown in problem C_3 above, also contributed to the persistence of the proportional multiplication error
- whether pictorial representations in which the function rule is "transparent" will help students to use the structure of the pictures to more easily find function rules.

In this paper we report on some first findings on these three questions.

RESEARCH SETTING

The activities

We designed a series of eight generalisation activities in which we varied the representation of the activities. Four activities were formulated in terms of numbers only (in the form of a table of values), and four were formulated in terms of pictures only (in the form of a drawing of the situation). Each numerical representation had a corresponding pictorial representation. Two of the functions were linear functions of the form f(n) = an + b, and two functions were simple quadratic functions

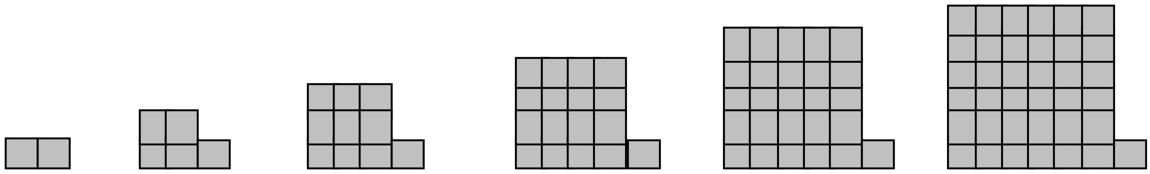
The numerical tables of values were presented in different formats: "continuous" (e.g. I_T below, in which input values for which the corresponding function values had to be calculated were included) and "non-continuous" (e.g. II_T, where the input values were not given, but were presented verbally by the interviewer). The tables were presented in both vertical and horizontal format.

The pictorial representations of the activities were chosen to be either "transparent", i.e. the function rule is embodied in the structure of the pictures (e.g. in I_P below), or "non-transparent", i.e. the function rule is not easily seen in the structure of the pictures (e.g. in III_P). As with tables, pictures were presented in both "continuous" and "non-continuous" format. All the drawings were presented to students in vertical format, but is here given horizontally due to space considerations.

The questions in each activity were basically the same, namely given the values of f(1), f(2), f(3), f(4), f(5) and f(6), we asked students to first find f(7) and f(8), and then the function values of certain further input values and to explain and justify their answers and strategies. These input values were both "seductive" (e.g. 20, 60) and "non-seductive" (e.g. 19, 59).

We supply below a selection of the activities. The subscript P indicates that the activity was presented in a pictorial representation and the subscript T indicates the problem was presented in the form of a table of values.

I$_P$: *Blocks are packed to form pictures that form a pattern as shown below:*

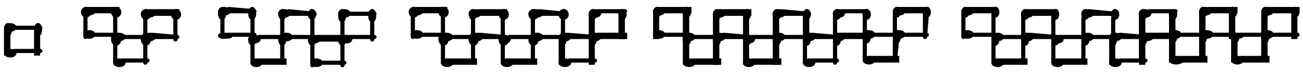


Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

I$_T$: *Tiles are used to build pictures to form a pattern. The table below shows the number of tiles in a particular picture.*

Picture number	1	2	3	4	5	6	7	8	…	20	…	60	…	*n*
Number of tiles	2	5	10	17	26	37								

II$_P$:*Matches are used to build shapes. A different number of matches is used to build each shape.*

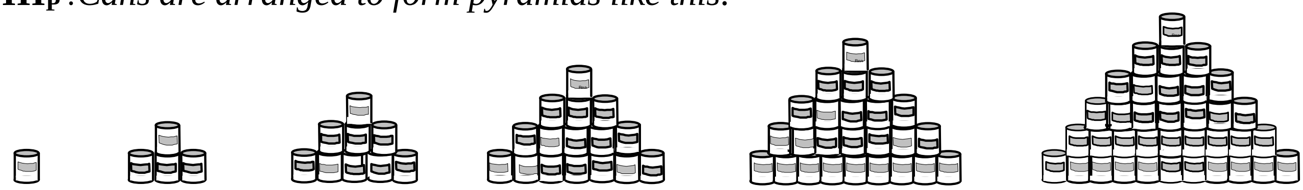


Shape1 Shape 2 Shape 3 Shape 4 Shape 5 Shape 6

II$_T$:*Matches are used to build shapes to form a pattern. The table shows the number of matches used to build a particular shape.*

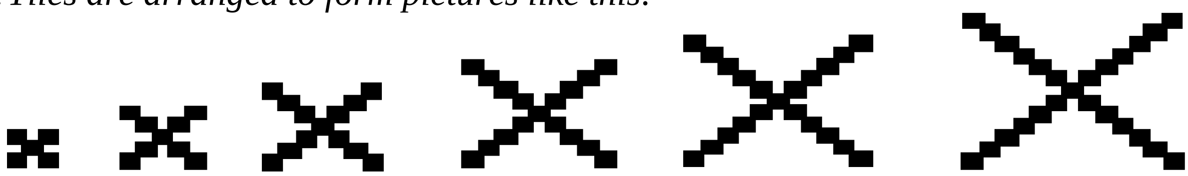
Shape number	Number of matches
1	4
2	12
3	20
4	28
5	36
6	44

III$_P$:*Cans are arranged to form pyramids like this:*



Pyramid1 Pyramid 2 Pyramid 3 Pyramid 4 Pyramid 5 Pyramid 6

IV$_P$:*Tiles are arranged to form pictures like this:*



Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

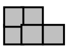
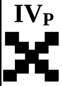
Methodology

We interviewed ten grade 8 students at one of our project schools in a historically disadvantaged area of Cape Town before they had received any instruction on patterns, sequences or algebra. The students were selected by the teacher so that they were representative of the grade 8 class. Each student was interviewed three times in 45-minute sessions. All interviews were videotaped and the tapes transcribed. The analysis will be used to design a teaching intervention to address the cognitive difficulties students have in the processes of generalisation.

RESULTS AND ANALYSIS

Most students had no difficulty finding f (7) and f (8) in any of the activities – they either found and used the function rule correctly, or used recursion correctly for these nearby values. However, in trying to find a manageable strategy for finding further-lying function values, students invented a variety of different strategies, both correct and incorrect. These strategies and their frequency are summarised in Table 1.

Table 1: Number of students using each strategy per activity

Activity number and representation format	Recursion	Proportional multiplication	Decomposition of input value	Difference method	Extended recursion:	Function rule	Other
I_P  Transparent picture, continuous Seductive values (20, 60, *n*) Quadratic function ($f(n) = n^2 + 1$)	2		1		2 wrong variations	5	
I_T Horizontal continuous table Seductive values (20, 60, *n*) Quadratic function ($f(n) = n^2 + 1$)		4		1	1 wrong variation	3	1
II_P  Transparent picture, continuous Non-seductive values (19, 59, *n*) Linear function ($f(n) = 8n – 4$)	1		1	3	1 2 wrong variations	2	
II_T Vertical non-continuous table Seductive values (20, 60, *n*) Linear function ($f(n) = 8n – 4$)		4		3	1 2 wrong variations		
III_P  Non-transparent picture, non-continuous Non-seductive values (23, 79, *n*) Quadratic function ($f(n) = n^2$)	3		1	1	2 wrong variations	2	1
III_T Vertical non-continuous table Seductive values (29, 87, *n*) Quadratic function ($f(n) = n^2$)	1	1			1 wrong variation	6	1
IV_P  Transparent picture, non-continuous Seductive values (20, 60, *n*) Linear function ($f(n) = 4n + 1$)	1	2		2	2 2 wrong variations	1	
IV_T Horizontal continuous table Non-seductive values (23, 117, *n*) Linear function ($f(n) = 4n + 1$)	1		3	1	2 2 wrong variations	1	

The nature of the function

Finding function rules It is interesting to note from the data in Table 1 that more than half of the students found and used the function rules in activities I_P and II_T. These both represent simple quadratic functions. One could be tempted to conclude that students easily recognise such simple quadratic function rules. However, one immediately also notices the marked differences in students' responses for the same functions in the picture and the table contexts. In activity I_P the picture is transparent, but students find it much more difficult to recognise the same function rule from the equivalent table in activity I_T. In activity III on the other hand, students easily find the rule in the table, but not in the non-transparent picture.

It is clear that students found it much more difficult to formulate function rules for linear functions. From our interviews it seems that students try to construct simple multiplication (proportional) structures, but when it does not fit the database, they quickly give up and then invent all kinds of error-prone recursion strategies.

Recursion When students focus on recursion patterns, however, they find the constant difference between consecutive terms in linear functions much easier to handle than the changing (increasing) difference in quadratic functions, leading to many errors. We describe these strategies and errors in the following sections.

Seductive vs. non-seductive numbers

The proportional multiplication error In our earlier work with grade 7 students we found a persistence with the erroneous proportional multiplication error. Also in this study six of the ten students interviewed used it at least once in the series of activities. For example:

Interviewer:	How many tiles in Picure 20? (*in* IV_P)
Peter:	OK, I am using 5 (*meaning* $n = 5$*;* $f(5) = 21$ *in the picture)* to get to 20. So 21 times 4 is 84, because 5 times 4 is 20.

This erroneous strategy was used only with what we call "seductive numbers". When students could easily find the function rule the nature of the input values was immaterial, i.e. they did not make the multiplication error, even for seductive numbers.

Extended recursion A few students managed to adapt their focus on recursion to a manageable strategy for finding further-lying function values. This extended recursion method is symbolised by $f(n) = (n - k)d + f(k)$, where d is the common difference between consecutive terms. Here is an example:

Interviewer:	OK, Shape 59? (*How many matches in Shape 59 in* II_P*?*)
Hamid:	So first I subtract 19 (*he had previously calculated* $f(19) = 148$) by 59 and then you get your answer of 40 and then I times it by 8 (*the common difference between terms*) and then I get my answer and then I add it by 148, that is Shape 19's answer.

Some students used this method also in the case of seductive numbers.

Students using this method often seemed to lose track of all the details. This was mostly because they worked verbally, and did not write down information or their strategy. For example, several students correctly calculated $(n - k) \times d$, but then did not add $f(k)$.

While the extended recursion method is correct for linear functions, many students also erroneously applied it to or adapted it for the quadratic functions. For example:

Interviewer:	Ok, and then Picture 20? (*How many matches in Picture 20 in* I_T*?*)
Harold:	I subtract 20 by 8 (*he had previously calculated* $f(8) = 65$) . . . I subtract 8 by 20, then I get 12 … with that 12 I times by 2 is equal to 24 … then I add 24 by 15, is equal to 39 then I add 39 to 65, is equal to 104.
Interviewer:	Just explain the 15 please
Harold:	That's the 15 I added by 50 ($f(7)$) to get 65 ($f(8)$).

Decomposition of input value The introduction of "non-seductive numbers" gave rise to other inappropriate strategies when students could not find a multiplicative relationship between the non-seductive numbers. For example:

Interviewer: OK, in Picture 117, how many tiles? *(How many tiles in Picture 117 in activity IV_T)*
Errol: *(Writes 30 = 121, 40 = 161, 50 = 201 . . . 100 = 401, 117 = 470)* Picture 117 is 470.
Interviewer: Can you explain to me how you got that?
Errol: Uhm, as I followed on Picture 100, I had to end up at 401 and I added Picture 17 to Picture 100 which gives Picture 117 *(He had previously calculated Picture 17 as 69)*

This method, generally symbolized by $f(n) = f(a) + f(b) + f(c)$ where $a + b + c = n$, was also used by other students. It seems that this strategy is born out of students' inability to find factors for numbers such as 19, 23, 59, 117.

The difference method The erroneous difference method, symbolised by $f(n) = n \times d$ was invoked with both "seductive" and "non-seductive" numbers. For example:

Interviewer: Ok, how many in Picture 23? (*How many tiles in Picture 23 in IV_T?*)
Linda: (*works on calculator)* . . . 92
Interviewer: Just explain please?
Linda: It will take too long to add 4 every time (*she previously found a constant difference of 4 between the terms of the sequence*). So I just said 23 times 4.

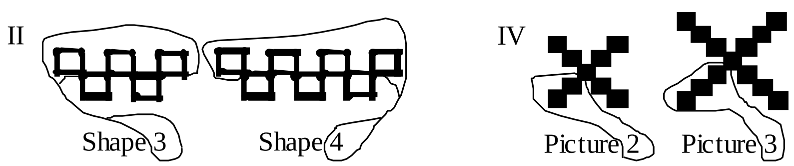
The visual impact of tables

From Table 1 it is clear that the visual presentation of the numbers in a table format for the function did not impact on the errors students made. The table in activity I_T was horizontal and "continuous" whereas the table in II_T was vertical and "non-continuous". Four students made the proportional multiplication error in both these examples. One student committed the difference method error in I_T whilst 3 students committed the error in II_T. The way we presented the questions as "continuous" or "non-continuous" in the picture activities also did not effect students' strategies. This can probably be explained by the fact that when the input numbers were not presented in writing, students made their own "continuous" "tables", so the visual distraction remained.

"Transparent" vs. "non-transparent" pictures

In I_P five of the ten students successfully recognised the function rule from the structure of the transparent picture. In II_P most students recognised that 2 squares (8 matches) were being added but then converted to numerical mode, constructing their own "table" of values, e.g. " … 1 = 4, 2 = 12, 3 = 20, etc.". Only two students described the function rule from the structure of the pictures, namely as $(n + n – 1) \times 4$ and $n \times 4 + (n – 1) \times 4$ respectively. Only one student used the structure of the picture in IV_P to identify the function rule. No child could recognise the function rule of using the non-transparent picture as the database in III_P. Two students found the function rule once they reverted to the number context.

It seems that these students do not have the necessary know-how of *how* to use the structure of a picture to find a functional relationship. If one wants to find a function rule in a table, one necessarily takes some specific value of the independent variable (input number) and tries to construct a relationship between this input-output pair. In the case of pictures, few students seem to intentionally take a specific input number and try to see this number in the structure of the picture, as illustrated in the following diagram:



Of course, it further requires a rich number sense, e.g. in II to see a further relationship in the numbers (2 is one less than 3, and 3 is one less than 4) before one can formulate the function rule $[n + (n - 1)] \times 4$. In IV one must see the multiplication or equal addition structure before one can formulate the rule $4 \times n + 1$.

Most students could see and use the structure of the pictures in a *recursive* way, e.g. in II students used the structure that 2 squares (8 matches) are added each time, and in IV they used the structure that 4 tiles are added to each successive picture. However, this did not help them to find the function rule, and students mostly then constructed a table of these values and then used the numbers in the table inductively. Of course, one could use the extended recursion method to use this recursive structure to formulate the function rules as $4 + 8(n - 1)$ and $5 + 4(n - 1)$ respectively.

Verification of strategies

Consider the following protocol:

Interviewer: Ok, Shape 19? (*How many matches in Shape 19 in II_P?*)

Peter: *(Peter successfully found f(7) and f(8) by counting the number of squares and then multiplying by 4 to get the number of matches. Now he starts making a systematic table of the number of squares in each Shape, using a recursive pattern:*

7	8	9	10	11	12	13	14	15	16
13	15	17	19	21	23	25	27	29	31

He then stops and goes back to looking at the pictures again.)

OK, I realised if I do this it is a bit of a hassle, so I looked at the pattern (*in his database*) and I figured the difference (*between f(n) and n*)

I took here (*pointing at f(5)*) . . . the difference between 9 (*f(5)*) and 5 (*n*) is 4 and by number 6 it is 5 . . . yes (*he checks again*) . . . 5. And by number 7 it is 6 and by number 8 it is 7. So I just tried it out. So I said to myself OK it is right and it will take too long to do it like this (*referring to his table of values*). So Shape 19 is 19 + 18, is 37, so 37 blocks times 4 gives you . . . 148 (*using the calculator*).

Clearly, Peter has constructed an efficient rule, which we can symbolise as $[n + (n - 1)] \times 4$, based on a sound analysis of patterns in the given and his extended database, and he verified that his pattern holds against the database several times. He was convinced and he could use the method with assurance. However, this style of working stands in stark contrast to most students' approach to such generalisations. While the students who used a function rule necessarily deduced the rule from the database, the other strategies reported in this paper are mostly not based on the database – students did not find the methods in the database, nor did they check it against the database. This applies to correct as well as to incorrect strategies. Students seem not to realise the need to validate their generalisations, and seem not to have the know-how of how to validate a generalisation against the database.

DISCUSSION

As in our previous study, students worked nearly exclusively in the number context and not with the pictures, favoured recursion methods, had difficulty in finding function rules and made many errors, including the proportional multiplication error. There is, however, one marked difference, namely the variety of strategies used by the students in the present study in comparison to the previous study. Of course this could be attributable to the differences in the subjects, who are at a different grade level, and from a different socio-economic background. We would argue, however, that the difference is mainly attributable to the introduction of non-seductive numbers in our activities.

It is for this reason (the greater variety of strategies), that we plan to extensively use non-seductive numbers in our planned intervention. However, as is evident from the examples in this paper, the use of non-seductive numbers will probably not prevent the ubiquitous proportional multiplication error when students encounter seductive numbers, nor will it prevent the other erroneous strategies reported here. For that we believe we should address two more fundamental issues, viz.

First: The development of awareness of the need to view any strategy as an hypothesis that should be validated against the database, and a focus on skills of how to do it. When one looks at the variety of strategies used by students, one can probably safely say that they have the ability and flexibility to find many patterns and relationships between numbers. The problem, however, is that most students are finding *random* relationships between the numbers without reference to the given database. We are struck not so much by the frequency and persistence of students' errors, but by their lack of an essential aspect of "mathematical culture", namely to view any strategy as an hypotheses that should be justified or verified against the given database. It seems that students lack simple *strategic knowledge*, e.g. to test a hypothesis against special cases.

Second: A more explicit study of the properties of different function types, and a comparison of such properties to become aware which properties apply to which function types.

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