# DEVELOPING AND STIMULATING GENERALISATION THINKING PROCESSES AND SKILLS

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In many countries pattern recognition and generalisation are considered fundamental to developing mathematical thinking and has thus become important components of mathematics curricula reform. South Africa's new curriculum plans (Curriculum 2005) also emphasises the importance of generalisation as is evident from the following specific outcomes for Mathematical Literacy, Mathematics and Mathematical Sciences:

- SO 2: Manipulate numbers and number patterns in different ways
- SO 9: Use mathematical language to communicate mathematical ideas, concepts, generalisations and thought processes
- SO 10: Use various logical processes to formulate, test and justify conjectures

In our work at the Mathematics Learning and Teaching Initiative (MALATI), a project aimed at informing curriculum development, we have developed learners' materials aimed at developing a broad notion of function. We build on the procedural view of function, namely *generalising certain input-output relations described or represented by situations, tables, graphs and mathematical rules***.** The activities address the aspects of function we find essential.

We found, in the current literature, that some research has been done on algebraic generalisation (e.g. Garcia-Cruz and Martinon, 1997; Taplin, 1995; Orton and Orton, 1994, 1996; MacGregor and Stacey, 1993). However, there is not enough in these reports to enable us to delve into the students' thinking in the *processes* of generalisation and validation. For example, do learners view their efforts at generalising as hypotheses? Do they realise the necessity to validate their strategies and answers, i.e. do they reflect on their strategies and answers? How do they become convinced that their generalisations are correct or wrong? How do they convince others? Moreover, in the current literature there is not enough to sufficiently answer the question as to how to design the learning activities and the necessary teaching approach in order to address children's difficulties and teachers' needs.

In order to address these difficulties we are engaging in ongoing research with grade 7 and grade 8 learners at our project schools. In the first round we presented the students with a series of generalisation problems in which we varied the representation of the problems. Some problems were formulated in terms of numbers only (in the form of a table of values), some were formulated in terms of pictures only (in the form of a drawing of the situation).

The questions were in each case basically the same, namely given the values of f(1)[1], f(2), f(3), f(4), f(5), and f(6), we asked students to find the values of f(7), f(8) and other further-lying output values and to explain and justify their answers and strategies. Some of the functions were linear functions of the form $f(n) = an + b$ and some functions were simple quadratic functions. Based on their responses the researcher asked questions to create cognitive conflict. All interviews were videotaped. In addition to the video protocols, written transcripts of the subjects' verbal responses as well as their paper-and-pencil activities were used in the analysis.

[1] Formal functional notation was not used in the actual problems or in communications with the students. It is merely used here for reporting on the students.

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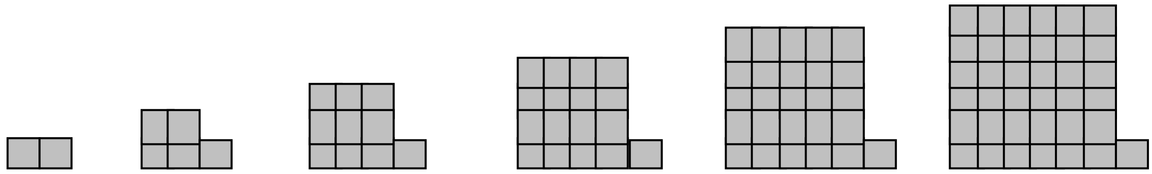
We supply below a selection of the activities.

**I**$_{\mathbf{P}}$ [2] [3]

Blocks are packed to form pictures that form a pattern as shown below:



Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

**I**$_{\mathbf{T}}$

Tiles are used to build pictures to form a pattern. The table below shows the number of tiles in a particular picture.

| Picture number | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | … | 20 | … | 60 | … | $n$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Number of tiles | 2 | 5 | 10 | 17 | 26 | 37 | | | | | | | | |

**II**$_{\mathbf{T}}$

Matches are used to build shapes to form a pattern. The table shows the number of matches used to build a particular shape.

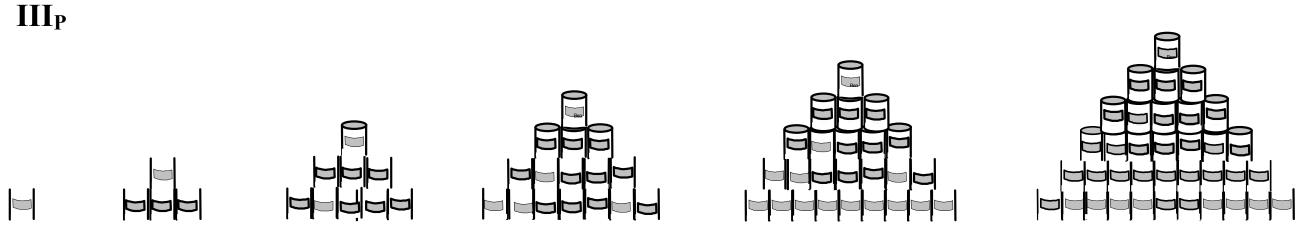
| Shape number | Number of matches |
|---|---|
| 1 | 4 |
| 2 | 12 |
| 3 | 20 |
| 4 | 28 |
| 5 | 36 |
| 6 | 44 |

**II**$_{\mathbf{P}}$

Matches are used to build shapes. A different number of matches is used to build each shape.

Shape 1 Shape 2 Shape 3 Shape 4 Shape 5 Shape 6

**III**$_{\mathbf{P}}$



Pyramid 1 Pyramid 2 Pyramid 3 Pyramid 4 Pyramid 5 Pyramid 6

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[2] The subscript P indicates that the problem was presented in a spatial context in the form of a pictorial representation of the situation and the subscript T indicates the problem was presented in a numerical context in the form of a table of values.

[3] All the drawings were presented to students in vertical format, but is here given horizontally due to space considerations.

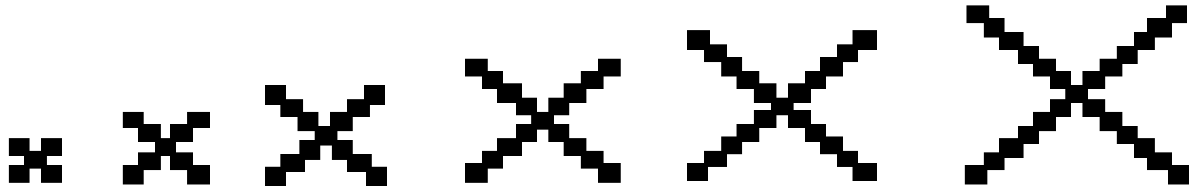
**$\mathrm{III_T}$**

Peter uses blocks to build pictures that form a pattern. The table shows the number of blocks he needs to build a particular picture.

| Picture number | Number of blocks |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 3 | 9 |
| 4 | 16 |
| 5 | 25 |
| 6 | 36 |

**$\mathrm{IV_P}$**

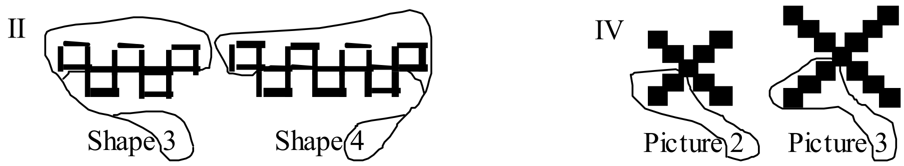
Tiles are arranged to form pictures like this:



Picture 1 Picture 2 Picture 3 Picture 4 Picture 5 Picture 6

Most students had no difficulty finding f (7) and f (8) in any of the activities – they either found and used the function rule correctly, or used recursion correctly for these nearby values. However, in trying to find a manageable strategy for finding further-lying function values, children invented a variety of different strategies. While using recursion was successful in extending number patterns to nearby values, students find it tedious for finding larger function values. They then mostly attempted to adapt their recursion strategy in some way, but made many logical errors in the process. Our biggest concern is not so much the fact that students make many errors, but that they do not feel the need, or do not have the know-how, to verify their methods or answers against the given data. We also found that students worked nearly exclusively in the number context and not with the pictures.

It seems that these students do not have the necessary know-how of *how* to use the structure of a picture to find a functional relationship. If one wants to find a function rule in a table, one necessarily takes some specific value of the independent variable (input number) and tries to construct a relationship between this input-output pair. In the case of pictures, few children seem to intentionally take a specific input number and try to see this number in the structure of the picture, as illustrated in the following diagram:




Of course, it further requires a rich number sense, e.g. in II to see a further relationship in the numbers (2 is one less than 3, and 3 is one less than 4) before one can formulate the function rule $[n + (n - 1)] \times 4$. In IV one must see the multiplication or equal addition structure before one can formulate the rule $4 \times n + 1$. A weak number sense will therefore also contribute to students' difficulties in using the structure of pictures to see the general in the particular required to formulate function rules.

**INTERVENTION**

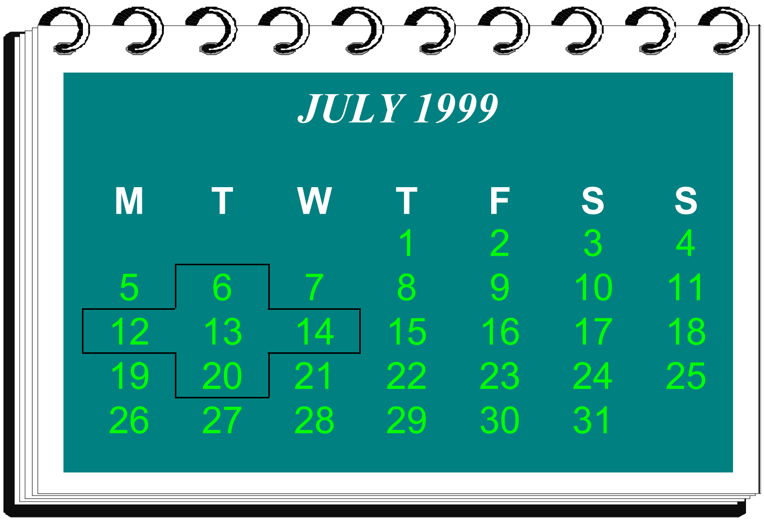
We have used the analysis of the research with the grade 7 and grade 8 pupils to design a teaching intervention aimed at addressing the cognitive difficulties children have in the processes of generalisation. In this intervention we focus children on some of the essential aspects of generalisation:

- predicting
- conjecturing (hypothesising)
  conjectures are essential because
  - they empower students by promoting ownership and inquiry
  - provide a means for students to construct mathematical knowledge
  - foster opportunities for students to make connections
- reasoning about the possible results of acting on one or another hypotheses and choosing one
- validating, testing the hypothesis, convincing others
- development of an awareness of the need to view any strategy as an hypothesis that should be validated against the given database
- skills of how to do it (children seem to lack simple strategic knowledge)
- presenting what is new but yet sufficiently familiar to evoke an effective response – large enough to challenge thought and small enough so that in addition to the confusion naturally connected to novel elements, there will be familiar things that jump at them from which helpful suggestions can be made
- developing an enquiring classroom culture
- attending to learners diversity

Below we supply two activities from the planned intervention:

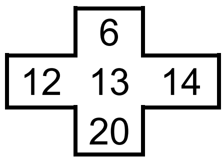
1. Six students attended a class party and ate a variety of foods. Something caused them to become ill. Joe ate pizza, hamburgers and sweets and became ill. Cindy ate hamburgers and sweets but not pizza. She became ill. Paul ate pizza but neither hamburgers nor sweets and felt fine. Thabo didn't eat anything and also felt fine. Jill ate pizza and sweets but no hamburgers and became ill. James ate hamburgers and sweets but stayed away from the pizza. He also got sick.
   Which food do you think caused the illness?

2.




1. Study the calendar page for the month of July 1999 above and describe as many number patterns as you can find.

2. Lester selected the following pattern of numbers from the calendar. He says that that *anywhere* in the calendar, for numbers arranged in this pattern, the sum of the outside four numbers will be four times the number in the middle. Do you think that Lester is correct?
   How can you check? Explain!




## OUTLINE OF THE WORKSHOP

During this workshop participants will

- engage with some of the problems themselves
- share our findings on children's strategies and difficulties
- engage with selected activities from the planned intervention
- discuss and give feedback on the effectiveness of these activities
- discuss the use of these materials for implementation in Curriculum 2005

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