An Analysis of the Geometric Understanding of Grade 9 Pupils Using Fuys et al.'s Interpretation of the Van Hiele Theory

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The performance of Grade 9 pupils on a written geometry test was analysed using the van Hiele descriptors constructed by Fuys, Geddes, Lovett and Tischler (1988). On the basis of this analysis, three pupils were selected to be the subjects of case studies in which their geometric understanding was explored further. This paper reports on two of the three case studies, focusing on the problems encountered during the analysis and on the initial conclusions regarding the use of this particular interpretation of the van Hiele theory.

The Context

Dissatisfaction with the secondary school geometry curriculum and poor pupil performance in geometry have been the topic of many discussions in mathematics education over the past decades. During 1997 the Geometry Working Group at Malati has been attempting to reconceptualise the teaching and learning of geometry. It is felt that if this is to be done and if changes to the geometry curriculum are to be proposed, a means to understand the geometric thinking of learners is required. The Group has found the van Hiele model of thinking (see below), and particularly the interpretation of this theory proposed by Fuys, Geddes, Lovett and Tischler (1988), useful in providing a framework in which to work and for designing geometry activities which are to be trialed in 1998. For as Usiskin (1982) points out, the theory not only provides an explanation of why pupils have problems, but also suggests a remedy for these problems.

The Van Hiele Theory

The work of the Dutch mathematics educators, Pierre van Hiele and Dina van Hiele-Geldorf, focused on levels of thinking in geometry and the role of instruction in assisting pupils to move through the levels. In his doctoral thesis completed in 1957, Pierre van Hiele formulated the five levels of thinking in geometry and discussed the role of insight in the learning of geometry. Van Hiele reformulated the original five levels into three during the 1980's. Dina van Hiele-Geldorf's doctoral thesis, which was completed in 1957, focused on the role of instruction in the raising of a pupil's thought levels. The van Hiele's ideas have since been studied and used by mathematics educators elsewhere, particularly in the Netherlands, the former Soviet Union and the United States.

Fuys *et al* .(1988) were involved in a three year study during the early 1980's to determine whether the van Hiele model could be used to describe how students learn geometry. Part of this involved the development of their own working model of the levels. This working model, and the corresponding level descriptors, were based on English translations of the work of Dina van Hiele-Geldof and Pierre van Hiele and were examined by van Hiele himself, as well as other van Hiele researchers, Hoffer and Burger. Following the study, the researchers suggested some modifications to the original descriptors and provided guidelines to avoid misinterpretation of their model. These modifications and comments were taken into account in this study.

For the purposes of this study it was only necessary to concentrate on the first three van Hiele levels (these will be referred to as Recognition, Analysis and Informal Deduction respectively). Fuys *et al.* (1988) characterise these levels as follows:
Recognition: The pupil identifies, names and compares geometric figures on the basis of their appearance as a whole.
Analysis: The pupil analyses figures in terms of their properties, establishes the properties of a class of figures empirically, and uses the properties to solve problems.
Informal Deduction: The pupils understands the relations within and between figures, gives informal deductive arguments, and formulates and uses definitions.

Methodology

The Sample: This consisted of twenty eight English-speaking pupils from one Grade 9 class in a Malati project school. Grade 9 was chosen as the focus as it was felt that a study of the understanding of these learners would be of particular interest in determining whether they are likely to cope with formal geometry in Grade 10.

The test was administered in early October and the interviews were conducted three weeks later. The written test will still be administered to a wider sample.

The Instrument: The format of tests designed to assess the geometric understanding of school pupils and pre-service teachers varies: multiple choice tests, written tests which require that explanations are provided for answers, and clinical interviews have been used for this purpose. As the aim of this study was to assess the understanding of the pupils, it was decided to construct a written test in which pupils were requested to provided explanations for their answers (this was indicated in each question and stressed by the researcher when administering the test). The written tests would be followed up by interviews with some of the subjects.

The written test consisted of ten questions on triangles and quadrilaterals and was designed to be completed in fifty minutes. These topics were chosen because much of the literature discusses the theory of van Hiele in terms of polygons and this work could be used as a basis for the construction of the test. It was also felt that pupils in Grade 9 would be familiar with these topics.

The ideas for the test were obtained from the work of Burger and Shaughnessy (1984) and Fuys *et al.* (1988). The construction of the items and the adaptation of these items after piloting were done in consultation with colleagues at Malati. The items were designed to assess certain aspects of the first three levels of the Theory, for example, the ability to identify a figure from given properties, class inclusion and the ability to use minimum properties in describing a shape. An attempt was made to ensure that each aspect was assessed using at least two different classes of shapes.

<u>Analysis of the Written Test and Selection of Pupils for Case Studies:</u> After the reading of a range of literature in the field, the descriptors constructed by Fuys *et al.* (1988), and the modifications suggested by these researchers after their study, were selected for use in the analysis. This particular model was chosen, firstly, because it had been constructed in consultation with van Hiele and other mathematics educators working in the field and, secondly, because the descriptors were elaborated more clearly and in greater detail than in other sources.

An attempt was made to assign each pupil to a level for each of the ten items. In some cases the responses did not make any sense, for example, when lists of properties of a figure were given. These responses often did not correspond to the analysis on other items. In such a case, the possibility of memorisation was noted in the analysis.

Analysis of the responses on the written test using these particular descriptors proved more difficult than expected and some interesting features emerged as a result. In cases where there appeared to be certain trends in a pupil's understanding, it was decided to conduct a semi-structured interview with the pupil in order to explore these features in greater detail and to obtain clarification on the initial analysis.

The work of three pupils, who will be called Kim, Nancy and Robert in this paper, were selected as being of particular interest. The analysis of each pupil's written test was used as a basis to prepare additional questions on the existing items and further activities (using the sources mentioned above). Each interview was audiotaped and lasted approximately thirty minutes. The interviews with Kim and Robert yielded a number of particularly interesting aspects of this model, and have been selected for discussion below.

Kim

<u>Reason for Selection for the Interview</u>

This pupil was classified as being on the Recognition level for four of the ten items. For example, shapes were grouped together because they had the "same shape", and the "corners" of a parallelogram were mentioned.

It appeared that orientation and position in space of figures were of concern to her. When required to draw four **different** triangles she provided two which were similar but had different orientation and when providing instructions for the identification of the parallelograms from a given collection of figures, she wrote, "They must have space between each other and not be near one another".

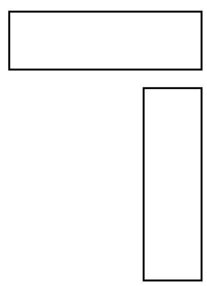
The listing of properties in those items designed to assess Analysis and Informal Deduction level thinking did not make sense and Kim used inappropriate vocabulary, for example, she named an acute angled triangle, as a "corresponding angle". Such a response suggested that this vocabulary does not have any meaning to her but that she has simply memorised the words.

The objective of the interview was to study Kim's handling of figures in different orientations by encouraging her to draw more figures of different types and to identify given figures in different orientations. More questions designed to assess thinking on the Analysis level would be provided to establish whether her use of vocabulary and properties was owing to memory rather than understanding.

<u>The Interview</u>

<u>Language Levels:</u> During the interview Kim admitted that the only figures she knew were the rectangle, square and triangle and she seemed unable to respond to my initial questions about right angles or diagonals. Initially it felt to me as if we were speaking different languages. I had to adapt the level of my language accordingly and could only speak about the shapes with which she was familiar. This conversation seemed to confirm van Hiele's suggestion that people need to be speaking the same language in order to understand one another (1986), and supported the findings of Fuys *et al.*(1988) and Mayberry (1983) regarding language use on different levels.

<u>Orientation:</u> During the interview I was able to confirm my initial analysis that Kim distinguished between shapes on the basis of their orientation and relative position in space. She claimed that the two triangles in the written test "pointed in different ways". When asked to draw three different rectangles on dotty paper she could only provide the following congruent rectangles:



This observation was confirmed in her grouping of angles. She also demonstrated how the page could be turned so that angles would be "opening the same way".

When asked to describe parallel lines to a friend over the telephone, Kim drew a pair of parallel lines and the following dialogue took place:

Kim: *They shouldn't be close together, they don't have to be the same size. (pause) They shouldn't, they must, must have space between them, on the same level.*

Interviewer*: Can you explain what you mean by "the same level"?*

Kim: *They don't have to be the same size, but they must be next to each other.*

These findings shed light on two aspects of the Fuys *et al.* model. Firstly, Kim was able to recognise figures such as squares and triangles in different orientations and to recognise a rectangle and a square within an irregular polygon. Fuys *et al.* (1988), Mayberry (1983) and Burger and Shaughnessy (1986) classify this as Recognition level thinking.

What appears to be absent from the descriptors used in the Fuys et al model is any indication of how the different orientations are viewed by the subject. Kim, for instance, could recognise shapes in different orientations, but the above examples indicate that she regarded the figures as different because the orientation varied. Would this also be categorised as Recognition, or is there perhaps a level below Recognition to which this type of reasoning would belong? Clements, Sarama, and Swaminathan (1997) have also proposed the existence of such a pre-representational level, but this is based on a different kind of thinking, namely, that pupils on this level cannot actually distinguish between different geometric figures, for example, squares and triangles.

Secondly, it appears that Kim was starting to recognise some properties of the familiar shapes, although she had not yet learned the required the vocabulary for these properties and the language used was still on a 'visual level'. Fuys *et al.* (1988) do claim in their model that pupils on the Recognition level can name geometric figures using "standard or non-standard names appropriately". Van Hiele (1986) himself refers to the period between Recognition and Analysis during which some properties are observed as Period 1. This analysis of Kim's understanding also seems to confirm Clements *et al*'s claim that although children at the Recognition level rely primarily on visual matching to distinguish shapes, they are also capable of recognising components and simple properties of familiar shapes (1997).

The above interpretations of the theory would suggest that Kim is in transition from Recognition to Analysis on familiar shapes. Burger and Shaughnessy (1986), however, suggest that a pupil is on the Analysis level if s/he explicitly uses the properties of a figure, even if the properties are imprecise. It appears, therefore, that more clarification is required on the use of vocabulary in classifying thinking according to the van Hiele levels.

<u>Measurement:</u> In providing a description of a rectangle, Kim provided measurements for the 'height' and the 'width' and stressed that "the height must be more than the width". When describing a rectangular prism, she requested permission to measure parts of the box. The use

of measurement in describing the properties does not appear to be dealt with explicitly in the descriptors used in this study. Fuys *et al.* (1988) do indicate that a pupil on the Analysis level is able to generalise properties for a class of figures, but at what stage of the transition or at what level is the measurement aspect no longer necessary?

<u>Memorisation:</u> From my discussion with Kim I was able to conclude that she had resorted to memorisation when answering questions to assess the Analysis and Informal Deduction levels. It was clear from the analysis mentioned above that she had not yet progressed to the Analysis level on the few shapes that were familiar to her, and it is highly unlikely that her responses on items using other figures could have made sense to her. When questioned about her use of the terms “equilateral” and “isosceles” she confessed that she had just heard these names and did not know what they meant. Her drawings confirmed this lack of understanding. It is interesting to note at this point that a number of researchers recognise the need to identify instances of rote learning in pupil responses but that little detail is provided on how this can be done.

On the basis of the interview I was able to confirm my initial analysis of Kim’s written work that, according to the Fuys et al model, she was on the Recognition level for rectangles and squares and had begun the transition to Analysis. Her responses on other geometric topics could be attributed to memorisation. What is interesting to consider in Kim’s case, is when and how she would begin the process of transition on the other figures with which she was clearly unfamiliar.

Robert

<u>Reason for Selection for the Interview</u>

In the written test, Robert mentioned the properties of familiar figures, such as triangles, squares and rectangles, although his vocabulary was still relatively informal. He identified rectangles, for instance, as those in which “all are equal in length and breadth”. When required to group unfamiliar shapes, however, his thinking could be classified as being on the Recognition level. He grouped two polygons together because “they will fit in together”, and others because “no matter how you change them or turn them they will still be the same”.

There was also an indication that some responses may have been due to memorisation: when asked to draw four different triangles, he gave an explanation , using congruency, which made little sense. He did appear to be using one step informal deduction, but this also made little sense, thus indicating the possibility of memorisation.

When required to identify rectangles from a given set of figures, Robert did include the squares in his classification. Class inclusion was not, however, evident when he was required to compare equilateral and isosceles triangles.

In order to study these aspects further, Robert was provided with more unfamiliar shapes with which to work and was question on the use of words such as ‘congruence’.

The Interview

Identification of Properties and Vocabulary Usage: The interview confirmed my analysis that Robert could identify properties in familiar shapes, such as triangles and rectangles. His use of vocabulary, however, was still informal: He identified “height” as “from this point to that point” (indicating on a diagram), and added that this indicated “how long it is”. He also referred to the “sides”, “breadth” and “height” in a rectangular prism. I thus encountered a similar problem to that experienced when analysing Kim’s thinking: Although Robert’s use of the properties of shapes suggested Analysis level thinking, his use of vocabulary indicated that he might still be in transition to this level.

When required to sort unfamiliar figures, Robert resorted drawings in which he was shifting points and dividing up shapes so that they looked the same. It appeared that he was looking for common figures and identifying these on the basis of their shape as a whole, for example, by dividing figures into triangles. This would confirm other findings on the oscillating nature of change between levels. Fuys *et al.* (1988) and Burger and Shaughnessy (1986) noted that pupils who they identified as being in transition between Recognition and Analysis tended to lapse to the visual level on unfamiliar shapes.

Robert’s sorting of the three-dimensional figures posed an interesting question with regards to the use of Fuys *et al*’s descriptors for tasks of this type. He was able to identify the “circles” in the cylinders. Does this mean that he was working with the properties of the polyhedra, or was he still recognising shapes as whole objects?

Spatial Ability: Robert displayed the ability to visualise three-dimensional objects from two-dimensional representations and to draw the nets of polyhedra. It appeared that he was using mental rotation in solving problems in planar geometry. Saads and Davis (1997) have suggested a link between the van Hiele levels, spatial ability and language use in pre-service secondary teachers, but there is clearly a need for more research in this area.

Memorisation: Robert showed a great willingness to share his geometric understanding with the interviewer and this provided a useful opportunity to explore his thinking and use of concepts such as congruency and area. He seemed to link congruency to individual triangles and used the terminology SAS and RHS to explain. He used the acronym "FUN" to remember to identify corresponding, co-interior and alternate angles, and although he could use these angles in informal arguments, he could not explain the relationships between the pairs of angles.

Robert showed a great interest in formulae and produced these even when not required. He had problem, however, remembering the formulae and could not explain why they were being used: "I just know, I can't recall. Basically I can just remember. I can't tell people how, it just pops up in my mind".

This interaction seems to suggest that, according to the Fuys *et al.* model, Robert had memorised much of what he had learnt in Grade 8 and 9 geometry. The analysis here seems to confirm van Hiele's ideas on reduction of levels: He suggested that pupils whose thinking has reverted to a lower level "are able to figure out all sorts of things; they can name results of their calculations and various other data, but they do not really know what they have calculated or what the names mentioned really signify"(1986). It appears that Robert had not had sufficient experiences at the Recognition level to enable him to understand the concepts of congruency, area and parallelism which are developed at the Analysis level.

Informal Deduction: Robert did display the ability to use a few steps of Informal Deduction when solving problems, which, according the model of Fuys *et al.* would be classified as being on the Informal Deduction level. Owing to his poor understanding of certain concepts as mentioned above, however, his reasoning made little sense. This indicates that a pupil can appear to be working on a particular level, but that this can be the result of rote learning.

Class Inclusion: Class inclusion is a feature of the Informal Deduction Level in Fuys et al's model, but de Villiers has noted that van Hiele assigns this to the Analysis level and then later contradicts himself (1987). Robert's performance could shed some light on this problem, as well as on the intrinsic/extrinsic nature of the model: When asked to identify the rectangles in a given set of figures, Robert included the squares in his choice of figures. It appears, however, that this reasoning had not become explicit for, when challenged on the relationship between a rectangle and a square he used the properties to claim that they were different figures.

The interview with Robert enabled me to conclude that his thinking was actually on a lower level than originally suggested by the analysis of his written responses. The discussion indicated that he had memorised much of the terminology and had little understanding of the concepts being studied. It appeared that his reasoning using familiar shapes was on the Analysis level, but that this reverted to Recognition level thinking on unfamiliar shapes.

Conclusion

In general it was found that the detail provided in this model was useful in analysing pupil responses. Problems were encountered, however, when analysing the use of vocabulary by the subjects and when attempting to categorise the thinking of pupils on the Recognition level.

The study was limited in its scope as it had to be accommodated in a wider research project at Malati and it is not possible to conclude at this stage whether the features that emerged in the case studies were the result of the nature of the test items, the result of the particular interpretation used, or features of the van Hiele theory itself. It is felt, however, that a number of interesting features of the theory and its interpretations have emerged, for example, the possible existence of a level below Recognition, the oscillating nature of change between levels, the level on which class inclusion occurs, and the need to identify memorisation. Four aspects are of particular interest to Malati and to the wider research community and require further study: namely,

- The relationship between language use and the van Hiele levels.
- The relationship between the van Hiele levels and spatial ability.
- The possible use of the van Hiele levels in explaining and rectifying misconceptions: The case studies revealed that all three pupils had misconceptions regarding certain geometric concepts.

- The effect of prior learning on movement through the levels. It appeared that the pupils in the particular class involved in the study had had limited exposure to geometric figures and it could be useful to track their progress through the van Hiele levels as regards unfamiliar shapes.

As a maths educator who has only recently been exposed to research in the field, this study has certainly highlighted a number of aspects of research which will be valuable in preparing me for a Masters dissertation. These include the usefulness of conducting interviews in obtaining information on pupil understanding and the need to consider the different factors that could have influenced my results. This study has also suggested a number of aspects of the van Hiele theory that could be studied in greater depth in a dissertation.

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