LEARNING THROUGH PROBLEM SOLVING*

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Introduction

By 1988, our group had completed several studies on young students' understanding of particular concepts before, during and after instruction. These experiences led us to conduct two small scale and several informal teaching experiments based on the idea that the teacher should pose problems to students for which they do not yet have a routine solution method available, and that learning would take place while the students were grappling with the problems. The outcomes of these experiments helped us to develop the following *tentative* model for learning and teaching mathematics:

> *Learning occurs when students grapple with problems for which they have no routine methods. Problems therefore come before the teaching of the solution method. The teacher should not interfere with the students while they are trying to solve the problem, but students are encouraged to compare their methods with each other, discuss the problem, etc.*

In the years since 1988, we have come to realise that this naïve description actually represents an enormously complex *series* of learning situations. *Some* of the issues that we were confronted with to research further and resolve as well as we could, were:

The role of the teacher. To what extent should the teacher be part of the problem solving process? Keeping in mind that the problem solving process involves mathematical as well as social processes, do different processes demand different types of support and intervention?

The classroom culture. Although the classroom culture includes the didactical contract between teacher and students (their mutual expectations and obligations), it also includes the ways in which the learning situations are physically set up and the rules under which they operate. What did we have to learn about this?

Interaction patterns among students. These interaction patterns depend on the role which the teacher has assumed, the classroom culture and the way in which the teacher sets up learning situations, reflecting her beliefs about how mathematics is learnt. To what extent do different interaction patterns influence learning, and might there be different kinds of interaction patterns for different learning situations (i.e. different kinds of tasks)?

The kind of problem posed. Mathematical tasks or activities come in a variety of guises: investigations, projects, traditional story sums, real-life problems, abstract problems, puzzles, etc. Were all of these suitable for learning through solving problems, or were some more suitable than others?

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The mathematical structure of the problem. Would it be a good idea if Grade 1 students worked at addition-type problems for a long time, so as to establish strong understandings of the operation and its solution methods? Would it matter if Grade 1 students never met a proportional sharing problem?

Sustained learning. It might be possible to achieve single, successful learning episodes, or even the satisfactory development of a group of students over a period of weeks, but would such a programme maintain students' mathematical development over a number of years?

The type of response elicited from the student. Should the teacher accept verbal answers and explanations, or should she insist on written explanations, and if so, for what purpose and in what format?

Teacher awareness, understanding and co-operation. We knew from experience that many teachers of the lower elementary grades were to some extent aware that young children invented their own methods and, provided they received *sufficient* information and some continued support, achieved significant success. We were concerned about large-scale implementations which necessarily implied brief training sessions with large groups of teachers.

Informing the larger community. Would we be able to communicate well enough with the larger community (parents and other members of the public, pre-school, elementary and high school teachers and lecturers at local colleges and universities, and remedial teachers and educational psychologists) so that these different groupings could understand and identify with our basic principles and appreciate the quality of the mathematics that students were doing?

Theoretical base

Our present theoretical base can be described as follows:

Contrary to an empiricist view of teaching as the transmission of knowledge and learning as the absorption of knowledge, research indicates that students construct their own mathematical knowledge irrespective of how they are taught. Cobb, Yackel and Wood (1992) state: "… we contend that students must necessarily construct their mathematical ways of knowing in any instructional setting whatsoever, including that of traditional direct instruction," and "The central issue is not whether students are constructing, but the nature or quality of those constructions" (p. 28).

A problem-centred learning approach to mathematics teaching (e.g. Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti & Perlwitz, 1991; Olivier, Murray & Human, 1990) is based on the acceptance that students construct their own knowledge and therefore attempts to establish individual and social procedures to monitor and improve the nature and quality of those constructions. We hold the view that the construction of mathematical knowledge is firstly an *individual* and secondly a *social* activity, described as follows by Ernest (1991):

a) “The basis of mathematical knowledge is linguistic knowledge, conventions and rules, and language is a social construction.
b) Interpersonal social processes are required to turn an *individual's subjective* mathematical knowledge, after publication, into *accepted objective* mathematical knowledge.
c) Objectivity itself will be understood to be social.” (p. 42, our italics.)

Social interaction serves at least the following purposes in problem-centred classrooms:

- Social interaction creates opportunities for students to *talk* about their thinking, and this talk encourages reflection. “From the constructivist point of view, there can be no doubt that reflective ability is a major source of knowledge on all levels of mathematics … To verbalise what one is doing ensures that one is examining it. And it is precisely during such examination of mental operating that insufficiencies, contradictions, or irrelevancies are likely to be spotted.” Also, “… leading students to discuss *their* view of a problem and their own tentative approaches, raises their self-confidence and provides opportunities for them to reflect and to devise new and perhaps more viable conceptual strategies” (Von Glasersfeld, 1991, p. xviii, xix).

- Through classroom social interaction, the teacher and students construct a consensual domain (Richards, 1991) of taken-to-be-shared mathematical knowledge that both makes possible communication about mathematics and serves to constrain individual students' mathematical activity. In the course of their individual construction of knowledge, students actively participate in the classroom community's negotiation and institutionalisation of mathematical knowledge (Cobb et al, 1991).

Whereas a traditional, transmission-type approach necessarily leads to subjective knowledge which is largely reconstructed objective knowledge, our version of a problem-centred learning approach reflects the belief that subjective knowledge (even if only in young children) should be experienced by the students as personal *constructions* and not re-constructed objective knowledge. (When we aim at children creating their own knowledge, as opposed to reconstructing existing objective knowledge, we do not imply that children are actually creating knowledge that does not already exist as objective knowledge; we do state that the children in this approach assume that they are creating *their* knowledge as new.)

We therefore regard problem-solving as the *vehicle* for learning.

It is necessary to distinguish sharply between learning *to* solve problems and learning *through* solving problems. Davis (1992) describes the process of learning through solving problems as follows: “Instead of starting with ‘mathematical’ ideas, and then ‘applying’ them, we would start with *problems or tasks*, and as a result of working on these problems the children would be left with a residue of ‘mathematics’ – we would argue that mathematics is what you have left over *after* you have worked on problems. We reject the notion of ‘applying’ mathematics, because of the suggestion that you *start* with mathematics and then look around for ways to use it.” (p. 237). Also: “According to Dewey (1929), these relationships and understandings are what is left after the problem has been resolved" (Hiebert et al, 1996, p. 15).

However, no matter how well-designed a problem or sequence of problems is, the amount and quality of the learning which takes place depend on the classroom culture and on students' and teachers' expectations. "Tasks are inherently neither problematic nor routine. Whether they become problematic depends on how teachers and students treat them." (Ibid, p. 16). Neither do we imply that learning *to* solve problems is not important in its own right, nor that *routine* problems should never be posed. This is discussed again later.

Implementation

In 1988, one of the local Departments of Education made available to us eight schools of their choice as experimental sites, and identified a control school for each of the experimental schools. We conducted a two-day workshop for the Grade 1, 2 and 3 teachers of the experimental schools at the end of 1988, and these teachers started implementing a problem-centred approach at the beginning of the school year in 1989. A small team of three researchers and three subject advisors from the Department of Education supported the process by brief visits to classrooms and short follow-up meetings.

All the students involved in the project wrote two sets of tests in the course of a school year for evaluation purposes. It was already clear after the first test in August (six months after inception) that there was a marked improvement in both the understanding of word problems and in computational skills (Malan, 1989).

The Department of Education accordingly requested us to provide in-service training for the teachers of another sixteen schools. In the years that followed, all the elementary schools of the Department from Grade 1 to Grade 6 eventually became involved. The approach also spread to four other Departments of Education. By 1993, the lower elementary grades of more than a thousand schools from five Departments of Education were involved. This very wide implementation was against our advice, because we support an organic model of growth with sufficient teacher support.

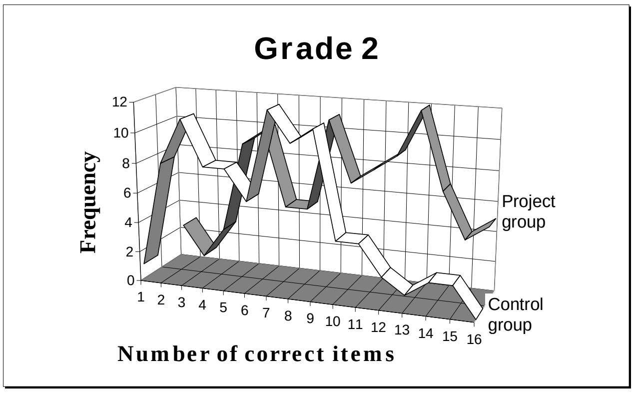
The problem-centred approach was also introduced to the elementary grade teachers of more than 50 special schools (schools for mentally and physically handicapped students). In a small number of these schools teachers have adopted and are using the approach with success.

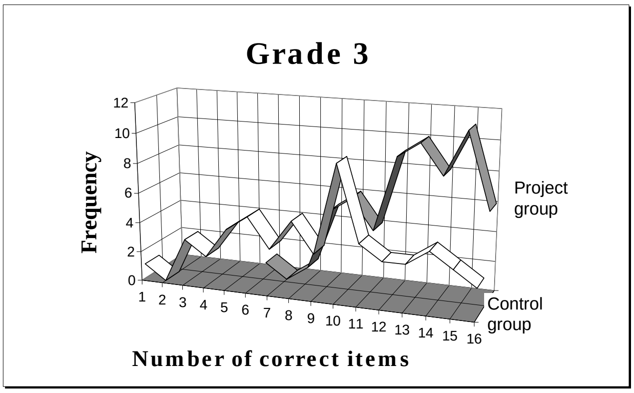
In the majority of schools involved, the implementation has petered out in the sixth and seventh grades, partly because in-service training received by the upper elementary teachers was very brief, and partly because the documentation they received was not comparable to the extensive and detailed teachers' guide, compiled by subject advisors, teachers of the first eight experimental schools and researchers, which the lower elementary teachers received.

It is important to make clear that the subject advisors and teachers made immense contributions to the development of the approach: We really *did* approach the original group of teachers with our naïve idea of how learning through problem solving works, but because we were not prescriptive, we could observe how different teachers had made sense of their workshop experiences by initiating different practices in their classrooms. It was our great fortune that we could observe these different practices and their effects on students' learning over time, enabling us to identify useful and potentially problematic practices.

Evaluation

All the Departments of Education involved in the project evaluated the approach independently by comparing second, third and fourth grade students' performance on written tests with the performance of students of the same school in the previous year on similar tests. Two Departments also compared experimental and control schools. The tests included straight calculations and word problems. Students at all grade levels in all the project schools showed marked improvement. When the frequencies of the number of correct answers for a particular test are plotted, the graphs typically show a consistent shift to the right for project students, i.e. more students did well than students in traditional teaching. We give two examples from different schools.





After six years the Cape Education Department commissioned a large-scale independent evaluation which also yielded positive results (Taylor, Glover, Kriel & Meyer, 1995), as did the James and Tumagole (1994) and Newstead (1996, 1997) evaluations.

De Wet's study (1994) found that fourth grade remedial students in a problem-centred approach progressed better than a matched group in a traditional approach (compare Thornton, Langrall & Jones, 1997).

Concerns from the larger community

The fears, accusations and arguments of the "California Math Wars" are very similar to those that arose a few years ago around the problem-centred approach in South Africa, giving rise to heated public debates in the letter columns of newspapers and magazines.

These fears and objections have at least two distinct sources (there are more).

Firstly, *different views* about what mathematics is and what is mathematically valuable; how children learn mathematics; and what our version of a problem-centred approach entails.

Secondly, very real concerns about the implementation of an innovative approach for which teachers *may not have the necessary mathematical and didactical knowledge and skills*, and for which there may be a shortage of appropriate materials.

Here are some examples of specific questions:

- How and when will students learn to perform the basic operations? (Implying that if the students do not learn the standard vertical algorithms, they have not learnt "the operations" for whole number arithmetic.)
- How will students learn their bonds (number facts) and multiplication tables if there is no drilling and memorisation?
- If students cannot do long division, how will they do algebraic division?
- What about the weaker student? Weak students cannot construct their own methods, they need to be shown.
- Communication plays an important part in this approach. What about students who receive instruction in a second language?

These problems will be discussed later.

Research results

The wide "forced" implementation of the approach within a relatively short time led to teachers receiving some quite divergent messages during and after their in-service training. Although this by itself is unfortunate, and caused uncertainty among teachers and parents, it did enable us as researchers to identify the crucial elements of the approach, and to research the reasons for less effective implementations. We have also been able to chart the *development* of young students' number concept and multiplication and division strategies, their *initial conceptions* of fractions and division, and social interaction patterns which lead to improved learning. These results have been described in nine PME papers from 1989 to 1996.

A long-term study is of value in at least three respects: It provides information on the long-term effects of specific interventions on students' learning, it makes possible the identification of unforeseen gains in conceptual development and understanding of the properties of numbers and operations, and it provides information on teacher development (Murray, Olivier & Human, 1995).

We now describe some issues which have emerged over the years. Some of these issues enabled us to clarify and refine our ideas on learning environments and the design of material which sustains development; other issues, notably establishing the link between arithmetic and algebra, are not yet resolved.

The social component of the learning environment

The role of the teacher. In the effort to help teachers make the paradigm shift from believing themselves to be the sources of knowledge and their main responsibility to transmit knowledge, towards accepting that students construct their own knowledge, we initially gave some teachers the impression that teachers should involve themselves as little as possible in the learning process. Later on, when some teacher groups received very sketchy in-service training, this impression was wide-spread and caused concern in the community.

It is necessary to clarify the role of the teacher in a problem-centred approach very thoroughly. We find Piaget's classification of mathematical knowledge as physical, social and logico-mathematical knowledge (Kamii, 1985) to be of great use in this respect. Teachers then realise that they have to provide the necessary information (social knowledge) for students to understand the problem, to communicate with each other and to capture their thoughts on paper in a generally acceptable (intelligible) way. They also have to show students how to use tools like measuring instruments and calculators, and they have to lay down (or negotiate with students) the social norms which govern interaction and general classroom behaviour.

It is only when student activity is *mainly* focused on the construction of logico-mathematical knowledge that the teacher should not interfere, except to monitor the social procedures and social needs.

The classroom culture. We suggest very tentatively that in the lower elementary grades at least, the classroom culture and the quality of students' interactions when solving problems have a greater influence on the students' mathematical constructions than the facilitatory skills of the teacher during discussions (Murray, Olivier & Human, 1993). We have not had the opportunity to research this further, but if the hypothesis holds, it has important implications for the general viability of an inquiry-based problem-solving approach. Many experiments and projects reported on internationally which use similar approaches leave the impression that the teacher involved has well-developed mathematical knowledge and didactical skills. Such reports serve as good examples of delicate and suitable teacher interventions, but most of our teachers, even the supposedly well-trained ones, do not possess similar skills.

Interaction patterns among students. Since this topic has received much attention during the past few years from researchers on a descriptive as well as a theoretical level, we mention only the contentious (but we perceive possibly *very* contentious) idea that we hold on this issue, and that is the need for grouping students into like-ability groups or allowing them to group themselves in such a way for tasks that are mainly focused on the construction of logico-mathematical knowledge (Murray, Olivier & Human, 1993). This must immediately be qualified as follows:

- The streaming or tracking of students into different lower-, middle- and upper-ability *classes* is not implied.
- Rigid groupings that are maintained for *all kinds* of mathematical tasks is not implied.

Using Piaget's classification of different kinds of mathematical knowledge as a guide, we propose that the acquisition of mainly *physical* and *social* knowledge is probably eased by co-operating with more knowledgeable peers or with the teacher (Vygotsky, 1978, p. 86). Logico-mathematical knowledge, however, needs to be constructed where the student's thinking space is not invaded by more advanced ideas, as happens when weaker or slower thinkers are physically near faster thinkers.

We therefore propose that for student interaction, the kind of mathematical *knowledge* that has to be constructed and the kind of *task* (e.g. routine/problematic/measuring/constructing) should guide possible groupings, as well as the physical facilities available. We have found that where students know what is expected of them, *they* are able to choose suitable partners.

The problem or task component of the learning environment

The kind of problem posed. Although in no way denigrating the role that investigations and projects can play in students' mathematical development, we base our learning trajectories on "problems" which present sufficiently clear *structures* for students to respond to. (Such problems need not have only one answer, but frequently attempt to elicit controversy as to the "best" answer.) Such "problems" may have realistic, abstract or imaginary contexts, although we tend to avoid contrived situations. This is later discussed more fully. In the discussion that follows, "problems" therefore mean *problems* and not, for example, projects or investigations.

Such problems include situations where the development of basic skills are directly addressed, not through drill or memorisation, but by turning them into problematic situations (Hiebert, et al, 1996) and at a meta-level, discussing ways of mastering these facts through relationships and patterns. (For example, how many different ways are there to make 7 + 8 easier to calculate?)

The mathematical structures of the problems. The mathematical structures of the problems posed are important for a variety of reasons. Problems cannot be posed simply because they are good or seem interesting – the choice of problems should be based on thorough content analysis and a good understanding of how students develop concepts and misconceptions.

Problems also have different functions, or are used for different purposes, for example:

- Some problems are more suitable than others for initially establishing an inquiry-type classroom culture
- Problems may be used to introduce students to a valid *problem area* (e.g. calculus), so that analysis of the problem and reflection on its structure may in this case be more important than solving the problem
- In general, when students solve problems, they should be provided with opportunities to actualise existing (but not yet explicit) knowledge and intuitions; to make inventions; to make sense and assign meanings; and to interact mathematically.

Our views on suitable problems and on learning sequences were originally based on the thinking of some Dutch researchers, and developed further during our classroom research and observations.

The Van Hiele levels. Pierre-Marie and Dina van Hiele suggest that students should first be immersed in activities involving new concepts which they engage in informally, using what ever insights or skills they have available. “The aim is to acquire a rich collection of intuitive notions in which the essential aspects of concepts and structures are pre-formed. This, then, is laying the basis for concept formation.” (Treffers, 1987, p. 248). The teacher gradually introduces more generally acceptable terminology and more rigorous reasoning processes as the students become able to give meaning to these. The Van Hieles describe the next level as having constituted the ground level concepts as abstract entities, *related* to other advanced entities (Van Hiele, 1973).

Progressive schematisation. Although the problem posed may remain similar over a period of time, the students’ solution strategies should develop towards more numerically-aware, more advanced strategies (Treffers, 1987, p. 200).

The Van Hiele levels and the idea of progressive schematisation *can and are* interpreted in different ways.

Paul Ernest mentions “accepted objective mathematical knowledge” (Ernest, 1991, p. 42). But to what extent does the “accepted, objective” knowledge influence the learning trajectories of young children through the first Van Hiele level and along the path of progressive schematisation? In other words, what is perceived to be the *desired* objective mathematical knowledge for young children? We contend that this question is in fact one of the crucial points the mathematics education community should be debating: Where do our learning trajectories lead? The endpoint of each trajectory depends on a value judgement, and the learning trajectory for each topic may have a different goal. For example, some well-documented learning trajectories close towards specific algorithms or notations (e.g. Treffers, 1987, pp. 200–209). Why?

The answer to this question lies in our perspectives on what mathematics is, what it is used for (the needs of society) and why children have to study mathematics. For example, a learning trajectory may end with a particular algorithm or conceptualisation. What was the aim of the trajectory – the (objectively substantiated) need for the particular algorithm, or the learning and development of general concepts or theorems (theorems in action) which occurred in the process? This depends on the topic, on the associated algorithms and techniques of the topic, and on the utilitarian value or societal value attached to these algorithms and techniques.

For example, should a learning trajectory for the addition of whole numbers end in the vertical addition algorithm? This is debatable. We believe, for example, that the *knowledge of properties of numbers and operations*, and the *flexible number sense*, shown by Orlando’s (Grade 5) solution method for 784 ÷ 16, are more to be valued than the application of a (socially acceptable) version of a long division algorithm:

$$\begin{array}{lllllll} 1600 \times 100 & \to & 1600 \div 2 & \to & 800 - 16 & \to & 784 \\ & & 100 & & 50 & & 49 \end{array}$$

answer: 49.

On the other hand, should a learning trajectory for calculating areas at Grade 6 or 7 level end with students' knowing (and understanding) the formula for the area of a circle? Most decidedly!

Learning trajectories can also start in different ways. It now seems to be generally accepted that the Van Hiele first level experiences need not be *concrete* and need not be *real-world problems* either, but may be any experiences or tasks which make sense to the particular group of students, and which they are able to identify with and problematise. We go further than this.

We believe that, where possible, these ground level experiences should be *authentic* in the sense that they represent situations of which the ideas or concepts we wish to develop are natural parts. We therefore find it useful to reflect on the situations or problems which initially gave rise to the development of the mathematical tool or concept (the genetic principle, e.g. Klein, 1924) and seldom make use of contrived situations, although we admit that in a way almost all school problems *are* contrived.

For example, in our earlier research on young children's understanding of directed numbers and operations with directed numbers, the children *themselves* used patterns, analogies with the natural numbers and logical reasoning to make sense of this novel environment, and showed confusion when they were confronted with contexts like debt to give meaning to directed numbers, even though they understood the idea of debt itself (Murray, 1984; Malan, 1987; Hugo, 1987). Not debt, but mathematics, created the need for directed numbers. Likewise, when our students learn to count and calculate, they do so in contexts which suggest the original situations where the need for counting and calculating arose. There is therefore limited use of *pre-structured* counting materials, and the problem-solving situations cover a deliberate mix of problem structures.

Furthermore, we have ample evidence that if the Van Hiele ground level experiences are aimed at developing understandings of specific cases, or mathematically stream-lined situations, *limiting constructions* form very quickly. Studying special or easy cases first does not make the development of concepts and skills easier; it merely hampers understanding.

Limiting constructions. This is a phrase used by D'Ambrosio and Mewborn (1994) to denote the type of misconceptions which arise through limited exposure to a concept or through experiences of a particular (limited) kind. For example, the idea that "multiplication makes bigger" is viable in whole number arithmetic, but severely hampers students when they have to perform operations involving fractions.

There were some teaching practices in our traditional lower elementary classrooms that we knew about, but did not address in our initial in-service training sessions, because we were not yet aware of the severe impact they would have in the long run.

Teachers did not pose a wide variety of problem types (for example, they tended to favour division problems of the sharing type and neglected grouping problems). They tended to "block" the four basic operations by starting off the school year with three months' addition, followed by three months' subtraction, then multiplication and then division. When teaching addition, they first dealt with the special cases where no "carrying" of a ten from the units to the tens is necessary, and no "borrowing" of a ten is necessary. Finally, many of their number concept development activities highlighted only the tens-structure of the decimal number system.

What happened was that students developed strong, stable methods for addition based on decimal decomposition of the numbers involved. For example, for 27 + 35, the most popular methods constructed were

$20 + 30 \rightarrow 50 + 7 \rightarrow 57 + 5 \rightarrow 62$
or $20 + 30 = 50;\ 7 + 5 = 12;\ 50 + 12 = 62$

In analogy with this, the following division method might then be constructed to calculate 79 ÷ 13:

$70 \div 10 = 7$
$9 \div 3 = 3$
$7 + 3 = 10$

i.e. decimal decomposition, and "combining" tens-parts with tens-parts and units-parts with units-parts.

We therefore realised that a learning trajectory for whole number arithmetic required that problem types be mixed, not blocked, that the special ("easy") cases of adding and subtracting are not presented first, and that number concept activities and problem situations emphasise *multiples and factors, and not only decimal decomposition*. These practices had immediate and long-term positive effects on students' number sense, estimation abilities and especially on their construction of powerful multiplication and division strategies (Murray, Olivier and Human, 1994).

We have also reported on the limiting constructions about common fractions that third graders had built up as a result of teaching as opposed to first graders in the same school (Murray, Olivier & Human, 1996). Similarly, we observe that ninth graders who have studied linear functions and linear graphs for 18 months to the exclusion of all other functions, find even the *idea* of any other type of graph difficult to accept.

The role of time in the development of concepts. It has been possible for us to trace the development of groups of students and in some cases of individual students over periods of time of various lengths, i.e. a single lesson (of about 35 minutes), a set of three lessons, several weeks, several months and for one school, six years. Our data show clearly that many (if not the majority) of students who seem to be mathematically weaker or slower than others can and do construct powerful mathematical concepts and generalisations provided the integrity of their thinking is preserved (i.e. somebody doesn't decide they need help and start demonstrating methods to them), the tasks they are presented with remain challenging and are not made easier, and the inquiry nature of the mathematics classroom is maintained. Our description of the development of division strategies in a third-grade classroom clearly illustrates this (Murray, Olivier & Human, 1992).

Lower elementary grade teachers have found our model of number development (Murray & Olivier, 1989) of much practical use because it sensitised them to the fact that at different points in time students are at different levels of conceptual development and should not be forced to function at levels of abstraction which they have not (yet) reached, but which they will reach, given time.

In fact, our informal observation is that students with a weaker number sense show great ingenuity and understanding of the properties of whole numbers and their operations by their (the students') use of theorems in action to make a calculation easier. For example, Niel (Grade 3) calculates 470 × 7 as follows:

$$\begin{array}{lllllll} 10 \times 470 & \rightarrow & 4700 \div 2 & \rightarrow & 2350 + 940 & \rightarrow & 3290 \\ & & \quad 10 & & \quad 5 \qquad 2 & & \quad 7 \end{array}$$

Claire-Anne (Grade 5) calculates 27 × 35:

$$\begin{array}{l} 27 \times 10 = 270 \\ 27 \times 10 = 270 \\ 27 \times 10 = 270 \\ 27 \times 5 = 270 \div 2 = 135 \\ 270 + 270 + 270 + 135 = 945 \end{array}$$

(Both of these students are obviously already at what we describe as level 3 number concept, but it is important to realise that level 3 methods are not invented if students are not confronted with numbers big enough to create the need for these methods. However, the teacher cannot *force* or *demonstrate* such methods; students *produce* them when they are able to do so.)

Anticipatory transformations. The above two examples clearly illustrate the ability of students to transform a given task into equivalent sub-tasks that *they know they can manage*.

The essential nature of any non-counting computational algorithm is that it is a set of rules for transforming a calculation into a set of easier calculations *the answers of which are already known or can easily be obtained*. This process of changing the task to an *equivalent* but easier task involves three distinguishable sub-processes, illustrated here with reference to a procedure to calculate 17 × 28 (Murray, Olivier & Human, 1994):

- Transformation of the numbers to more convenient numbers, e.g.

  28 = 30 – 2

  The ability to transform numbers in this way depends on the student's *number concept development.*

- Transformation of the given computational task to a series of easier tasks, e.g.

  17 × 28 = 17 × 30 – 17 × 2

  The ability to transform the task to an equivalent task depends on the student's awareness of certain *properties of operations* or theorems-in-action (here the distributive property of multiplication over subtraction).

- Calculation of the parts, e.g.

  17 × 28 = 17 × 30 – 17 × 2
  = 510 – 34
  = 476

When a Grade 3 student solves the problem "Find half of 237" as follows, he has clearly chosen to decompose 237 into numbers that *he anticipates* he can halve:

237	100	100	
	15	15	
	3	3	
	½	½	answer: 118½

The anticipatory transformations may not always be appropriate for a variety of reasons. They may still prove to be too difficult, or upon reflection, uneconomical and tedious. In the following example Marianne (Grade 3) *underestimated* her abilities and adjusted her transformations to a more sophisticated level.

Trying to solve 338 ÷ 13, she starts off by subtracting thirteens, then writes down "this will take too long" and switches to multiplying and doubling:

130 × 10 →	130 + 130 →	260	+ 52 →	312	+ 26 →	338	
	10	20	4		2	26	

answer: 26

The type of response elicited from the student. Davis (1992) rightly states: "Mathematics sometimes employs written notations of various sorts, but these symbols are not the mathematics itself, any more than lines drawn on a map are actual rivers and highways" (p. 255). Yet we have found access to appropriate notations to be crucial to young students' mathematical development. Social interaction and effective communication are essential to the approach and access to appropriate notations to capture methods so as to share them with others is a part of this. The ability to capture thought on paper is essential for individual reflection and analysis. In traditional mathematics classrooms at the *elementary* level, students have had no appropriate tools at hand to express (or capture) their thinking in writing. What they had available was a number sentence to be used in a prescribed format, and computational algorithms to be presented in prescribed formats. In such a situation Davis' stricture holds completely.

If, however, we make available notational tools *which students can match* to their thinking processes, they find it an important aid to individual as well as social construction of knowledge (Skemp, 1989, p. 103).

In mathematics, recording has different functions. It eases communication and serves as a thinking aid for the individual. Written explications which aim at proving something or convincing others also *need to measure up to standards* which are not applicable when the individual is simply trying to solve the problem.

For these reasons, the arrow notation, which can be used when the equal to sign would be incorrect, was introduced, and even in the lower grades students are required to record their thinking clearly and logically enough so that others can also follow it. It was found that recording skills take time to develop, and if teachers simply accept verbal explanations in the lower grades, some students experience serious problems from the fourth grade onwards, when they need a written record as a personal thinking aid and then do not know how to express themselves.

Answering the community

Community fears which are based on a lack of knowledge about the approach should of course be addressed by trying to provide the community with the necessary information. During the period when only a few schools were involved, information sessions with parents were very successful. As the number of schools involved increased, and teacher in-service training became sketchy, parents became increasingly badly-informed. A very small group of university lecturers in pure mathematics was (and remains) opposed to the approach. We understand their opposition to be rooted in an unawareness about the substantial international and local results available on the effects of the traditional teaching practices on students' thinking, the changed aims of mathematics education caused by the demands of a radically changed society and workplace, and the need for mathematics and mathematically related skills to be made accessible to the whole community.

Concerns that students who receive instruction through the medium of a second language are at a disadvantage in an approach where communication plays an important part are very valid. Yet the alternative seems to be a classroom where more time is spent on context-free mathematics and mainly teacher-directed explanations and examples. Would this be better?

In the environment of the handicapped with moderate to severe language problems, teachers have told us that it is necessary to stimulate children to listen and to try to communicate while they are doing mathematics. Simply trying to teach them arithmetic divorced from word problems does not equip them to handle everyday problems. Goodstein, himself deaf, strongly supports this view (Goodstein, 1992).

Also, one of the third-grade classrooms that we described (Murray, Olivier & Human, 1993), consisted of 32 students with ten different home languages. The majority of these students had a very poor command of English (the medium of instruction), yet the teacher was able to establish one of the best implementations of the problem-centred approach we have been able to document, and the students' mathematical development was very good.

However, the language issue remains serious and should be researched and debated further.

It is, however, very possible that a problem-centred approach to learning mathematics may not succeed if firstly, the problems posed are not chosen for their mathematical structures and the sequence in which they are posed is not well-planned (we have discussed this, with special reference to the prevention of limiting constructions).

Secondly, activities which are aimed at the development of *routine skills* should not be neglected. It is important, for example, that a flexible and wide-ranging knowledge of the multiplication tables be developed in the elementary grades, not by memorising the tables, which is an extremely inefficient way, but through encouraging students to use relationships and patterns. (Niel and Claire-Anne for example, showed flexible number knowledge.)

Thys (Grade 4) solved 711 ÷ 9 mentally, explaining as follows: "720 take away 9 is 711. So the answer is not 80, it is 79."

Schoenfeld (1994) states: "*Some* such skills are important for students, if only because not to be fluent at them means that one's clumsiness at them will get in the way when one needs to see past them" (p. 60). The following comment by Askey (1997) was obtained from the Internet: "Then NCTM tried Agenda for Action and later the Standards. Both of these were built on the idea that if you could solve problems, then you could do mathematics. You can, but at too low a level. All three are needed – problems, technique and structure."

To summarise, we think it unlikely that a problem-solving approach will be effective if:

- Progressive schematization is not encouraged, either through discussion or by the teacher posing the problem in such a way that more exact or more abstract responses are required even, and especially, for supposedly weaker students.
- The necessary important content is not covered.
- Useful mathematical techniques are not developed and sufficiently practised.
- Classes of problems do not achieve coherence (e.g. the function concept, algebra, statistics), so that the associated concepts and relationships cannot be constituted at an abstract level.

In conclusion

The problem-centred approach in the lower elementary grades is based on an over-simplified model called the three pillars:

- well-planned number concept activities, including activities which promote the building of patterns and relationships
- well-planned problems
- effective discussion

Neglect of any of these “pillars” shows in students’ behaviour or understanding, even only after a few months.

Hiebert et al (1997) identify five critical features of the very similar approach to the teaching and learning of mathematics they describe, and then go on to say: “The essential features are intertwined and work together to create classrooms for understanding. They define a system of instruction rather than a series of individual components. It makes little sense to introduce a few of the features and ignore the rest; their benefits come from working together as a coherent, integrated system.” (Hiebert et al, 1997, p. 172).

Initiating and sustaining mathematical development through posing problems that students have to work on has been found to be a successful way of learning mathematics, *but only if* the problems are well-designed and well-sequenced, and the classroom culture in its full complexity supports learning.

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