STUDENTS' AWARENESS OF THE DISTRIBUTIVE PROPERTY *

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In this paper we briefly compare the role of the distributive property in arithmetic calculation and in algebraic manipulation. We formulate a model describing different levels of awareness of the distributive property. We describe a teaching experiment to facilitate students' awareness of the distributive property and discuss the outcomes of this experiment.

Introduction

Students' difficulties in learning the basic ideas of early algebra has been well documented. Kieran (1989) emphasises that an important aspect of this difficulty is students' difficulty to recognise and use *structure*. Structure includes the "surface" structure (e.g. that the expression $3(x + 2)$ means that the value of x is added to 2 and the result is then multiplied by 3) and the "systemic" structure (the equivalent forms of an expression according to the properties of operations, e.g. that $3(x + 2)$ can be expressed as $(x + 2) \times 3$ or as $3x + 6$).

Kieran sees algebra as the formulation and manipulation of general statements about numbers, and hence hypothesises that children's prior experience with the structure of numerical expressions in elementary school should have an important effect on their ability to make sense of algebra. Booth expresses the same view:

> ...a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic. (Booth,1989, p. 58)

From this perspective Booth formulates two tasks for research:

- To examine students' recognition and use of structure and how this recognition may develop.

- To devise new learning activities and environments to assist students in this development.

Our study shares the same assumptions about the importance of understanding numerical structure as a prerequisite for understanding algebraic structure. This paper reports on a five year study which addressed Booth's two research tasks, as it relates to the development of students' awareness of the distributive property of operations.

The relationship between arithmetic calculation and algebraic manipulation

The essential nature of any non-counting computational algorithm is that it is a set of rules for breaking down or transforming a calculation into a set of easier calculations of which the person already knows the answers (Olivier 1992). This process of changing the task to an equivalent but easier task involves three distinguishable sub-processes, illustrated here for a procedure to calculate 8×23:

1. *Transformation of the number(s) to more convenient numbers*, e.g. $23 = 20 + 3$.
2. *Transformation of the given computational task to a series of easier tasks*, e.g. the instruction $8 \times (20 + 3)$ is transformed to the equivalent task $8 \times 20 + 8 \times 3$.
   The ability to transform the task to an equivalent task depends on the student's awareness of certain properties of operations, here the distributive property of multiplication over addition.
3. *Computation*, e.g. $8 \times 20 = 160$; $8 \times 3 = 24$; $160 + 24 = 184$.
   These final calculations are performed at the "automatic" level, i.e. without much thinking and requires knowledge of basic number facts.

The process of calculation does not necessarily follow this sequence. The way you initially decompose the numbers depends on your anticipation, "looking ahead" to the transformation you intend in phase 2 and the basic facts you are going to use in the final phase of the calculation. Such a computational strategy therefore includes a realisation of the significance or usefulness of the transformations.

The same task transformation underlies "algebraic manipulation": Finding the product of 8 and $2x + 3$ involves the transformation of the product $8(2x + 3)$ to the sum $8 \times 2x + 8 \times 3$. Human (1988) illustrates this similarity as follows:

Arithmetic		**Algebra**
$8 \times 23 = 8 \times (20 + 3)$	*Transformation of number(s)*	$8(2x + 3)$
$= 8 \times 20 + 8 \times 3$	*Transformation of task*	$= 8 \times 2x + 8 \times 3$
$= 160 + 24 = 184$	*Calculation*	

In both the arithmetic and the algebraic case, the transformation from a product to a sum is determined by the distributive property. Explicit awareness of this principle is essential for students when they embark on a first course in elementary algebra. Equally important for these students is understanding that the similarity is confined to the sub-process of task transformation, i.e. that algebra is not "calculation with letters".

Awareness of the distributive property in the elementary school

The traditional standard algorithm for long multiplication involves the same transformations as described above. However, it is probably non-controversial to say that the algorithm has traditionally been taught instrumentally (Skemp, 1989), that the transformations were very much "hidden", and that students in traditional classrooms were not explicitly aware of the logic or the significance of the underlying transformations.

On the other hand, in reform classrooms in South Africa no specific strategies are prescribed and strategies are not "taught". Students are free to construct their own computational strategies. In the process they invent and use a variety of strategies (see e.g. Murray, Olivier & Human, 1994). This is a typical strategy used by young children to calculate 8×23:

$$8 \times 10 = 80$$
$$8 \times 10 = 80$$
$$8 \times 3 \ = \ 24$$
$$80 + 80 + 24 = 184$$

Strategies such as these that children in reform classrooms use in solving word problems generally indicate that they have a sound intuitive grasp of the properties of operations that underlie their strategies. Vergnaud (1988) refers to students' intuitive use of properties of operations as "theorems-in-action": Theorems-in-action are physical or mental actions performed by a student, often in a disguised or subtle way, that provide behavioural evidence of implicit knowledge of a more formal property or method or "theorem" of mathematics.

Students' use of theorems-in-action in their calculations led us to hypothesise that students in reform classrooms are explicitly aware of the distributive property. However, this is not substantiated by research. In interviews with grade 5 students (Vermeulen, 1991) only 2 of 16 students (13%) responded positively to the question: "Will $37 \times (13 + 26)$ produce the same answer as $37 \times 13 + 37 \times 26$?". This was later followed by a questionnaire to grade 5 and 6 students which included the following question:

Will the following produce the same answers? Supply a reason for your answer.

1. *37×876 and $37 \times 344 + 37 \times 532$*
2. *58×356 and $58 \times 300 + 58 \times 50 + 58 \times 6$*

159 of the 240 students (66%) responded positively to both questions and could state a meaningful reason. We attributed the difference in the results of the interviews and the questionnaire to the use of brackets during the interviews. These results motivated the present teaching experiment to attempt to explicate elementary school students' intuitive knowledge of the distributive property.

Based on our observations and our theoretical framework, we designed the following model for levels of awareness of the distributive property:

Level 1: The spontaneous utilisation of the distributive property: When elementary school students who are free to construct their own computational strategies in arithmetic spontaneously use the distributive property.

Level 2: The recognition of the distributive property: When a student responds positively to questions like "Will 37×52 and $37 \times 30 + 37 \times 22$ produce the same answer?", and can supply a meaningful reason (e.g. 52 is broken down into $30 + 22$).

Level 3: The intentional utilisation of the property: The student intentionally investigates whether utilisation of the property will produce an equivalent number expression which is easier than the original one. Such a student will, for example, replace the expression $27 \times 13 + 27 \times 7$ by the equivalent expression 27×20 in order to simplify the calculation.

Level 4: The generalisation of the property: This entails two different aspects:

- The recognition that the distributive property applies to *all* real numbers, including very large numbers, decimal and common fractions, and (for older students) also to numbers represented by letter symbols.

- The ability to articulate the distributive property.

Level 5: The explanation of the property: Level 5 awareness is demonstrated when a student who has demonstrated level 2 awareness, can supply a meaningful *reason* for questions like: "*Why* do we get the same answer when we multiply both 30 and 22 by 37, as when we multiply 52 by 37?" The distributive property is of course an axiom of the real number system, and therefore cannot be proved. However, students' understanding of the property is based on their earlier concrete experiences. Level 5 awareness will thus be demonstrated if a student can substantiate the distributive property by referring to or connecting it to those earlier experiences.

Designing learning activities to explicate the distributive property

During 1994 we executed a teaching experiment which took the form of developmental research, as described by Gravemeijer (1994). We attempted to design learning activities through which we hoped to explicate grade 6 students' knowledge of the distributive property. By comparing pre-test and post-test results and studying student responses in class, we continuously tried to improve these activities such that they would enable students to explicate their theorems-in-action and lead to a higher level of awareness.

Our frame of reference is very much constructivist. Our intention in designing these activities was therefore that the activities should lead to cognitive conflict and that accommodation should take place. The challenge in designing the activities was evidently how to create such a disequilibrium that would facilitate successful reflection or reflective abstraction by the students (Inhelder, 1974).

It should be stressed that here it is not about forming new cognitive structures (in the Piagetian sense of the word), but rather to make students aware of knowledge which they do possess, but of which they are not explicitly aware. As such it could be argued that an attempt was to be made to transform "unknown" knowledge structures into "known" structures, that is cognitive structures of which students are explicitly aware, and which can be integrated into their existing cognitive structures.

We now briefly describe and discuss some examples of these initial learning activities:

Concrete problems: Students solved concrete problems, such as: "9 children each receive an ice cream of R2.83 and a chocolate of R2.17. What is the total cost?".

Students typically used one of two methods, namely either first adding 2.83 and 2.17 and then multiplying it by 9, or multiplying each of 2.83 and 2.17 by 9 and then adding the products. This represents a concrete manifestation of the distributive property. Through comparing and discussing their different computational strategies, we hoped that it would address their implicit knowledge of the distributive property.

Decontextualised problems: Students were asked to calculate the answer of a problem such as 15×36. Many students would break down the 36 into $30 + 6$, and then multiply 15 with both 30 and 6, hence using the distributive property. When students were challenged as to *why* he or she does it in such a way, and whether it is *correct* to do it like that, most students gave answers like: "It works out correctly, so surely it is correct?".

In our opinion, based on our observations of student reactions, decontextualised problems, as well as concrete problems, representing slightly different manifestations of the distributive property, did not succeed in bringing learners in touch with their theorems-in-action of this property. We were convinced that is was mainly because in both cases we were not able to create a disequilibrium (in Piaget's sense of the word), thereby not succeeding in forcing students to reflect.

Two equivalent number expressions: "Predict (do not calculate) whether the following two number expressions will produce the same answer. Give a reason for your answer".

$$25 \times 135 \quad \text{and} \quad 25 \times 98 + 25 \times 37$$

A variation of this question was to first predict, and then calculate the answer of each expression using a calculator.

To the first type of questions, a number of students readily responded that the answers would be the same, because the 135 was "broken up". When asked why one may "break up" a number, and then multiply with each of the "broken up parts", the answer was once again to the nature of: "It works out correctly, so surely it is correct?".

Intentional application of the distributive property: Here we gave students exercises such as: Calculate without a calculator: $43.6 \times 9 + 43.6 \times 5 + 43.6 \times 6$. Sometimes we would add: "...using the easiest method".

This type of activity did not seem to be successful – even after numerous exercises of the first three types described above, only the minority of students would intentionally first add the 9, 5 and 6, and then multiply the sum by 43.6.

Moving towards a teaching strategy

During 1994, through our observation of students' reactions, and our own reflection, coupled with our constructivist framework of knowledge, we became convinced that a successful teaching strategy should be based on the following principles:

- A powerful cognitive conflict should be created.
- Organised groupwork should be utilised, facilitating better opportunities for students to reflect.
- Calculators should be actively integrated into the learning process, supplying necessary feedback as a source of confirmation or cognitive conflict.

A teaching strategy which accommodated these principles was designed and implemented. We now briefly describe the essential characteristics of the teaching strategy:

- Students solve about three concrete problems. Essentially, there are two approaches to do the calculation, as explained previously. Students are then required to discuss and compare their answers and *methods* with group members, and are led to recognise that there are essentially two methods of calculation.

- Students then calculate the answers of given pairs of equivalent number expressions, using calculators, similar to the example described above. They are required to compare the answers of the two number expressions. For most students it was a surprise to realise that the answers of the two number expressions were equal. They were then challenged to explain why the two answers were equal. In our opinion, based on our observation of the students' reactions, this created quite a powerful disequilibrium, and the subsequent effort to try explain this result, could lead to a relatively high level of reflection, especially when students worked in groups.

- To facilitate the process of reflection, students were referred to the previous assignment, that is, the concrete problems. They were challenged to look for and describe similarities between that situation and the current one. Hence, reflection is forced onto a higher level, since they have to reflect on two apparently unconnected manifestations of the distributive property. In this process of reflecting about their reflection of the two separate situations, it is hoped that they will get in touch with their theorems-in-action about the distributive property.

- More problems, stressing the same principle, are then tackled in groups in order to explicate students' awareness even further.

Evaluation of the teaching strategy

During 1995 we implemented the teaching strategy in nine schools, involving 645 students from grades 3 to 7. Students worked together in groups of about four, and each student received a set of work sheets. There were seven work sheets in total, each consisting of four parts.

In evaluating the teaching strategy, we followed a typical pre-test, treatment, post-test design. As described above, students in reform classrooms have a good intuitive grasp of the distributive property at level 1. Given that the objective of the study was to increase students' level of awareness of the distributive property, we designed questions to test level 2 and 3 awareness. We did not test for level 4 and 5 awareness. There were essentially two types of questions in both tests, determining level 2 and 3 awareness. Examples of the type of question are:

- **Level 2**: *Predict whether the following two number expressions will produce the same answer. Give a reason for your answer.*

    37×150 *and* $37 \times 95 + 37 \times 55$

    If a student responded positively and was able to supply a meaningful reason, he or she was categorised as displaying level 2 awareness.

- **Level 3**: *Calculate, without using a calculator, the value of the following. It is important to show your work:*

    $18.3 \times 13 + 18.3 \times 7$

    If a student first added 13 and 7, and then multiplied the sum (20) by 18.3, he or she was categorised as displaying level 3 awareness. A student could fall in both categories.

Results and discussion

The results indicated that grade 6 and 7 students experienced a substantial increase in their awareness of the distributive property, as the sample data below illustrates. The teaching strategy was therefore fairly successful.

Results for grade 6 and 7 students*

PRE-TEST				POST-TEST			
level 2 awareness		level 3 awareness		level 2 awareness		level 3 awareness	
N	%	N	%	N	%	N	%
100	54.3	21	11.4	136	73.9	82	44.6

* 184 students wrote the pre-test and the post-test

The outcomes for grade 3 and 4 students were, however, very inconsistent, leading us to conclude that this teaching strategy did not benefit them in acquiring a higher level of awareness of the distributive property. A possible explanation for this is that the younger students found the calculations in the test questions very complex – their thought processes were occupied in trying to manage the calculations and not towards reflection on these thought processes. Using calculators in the test situation may change this.

Also, and this applies equally to the grade 6 and 7 students, we observed during the teaching experiment that the level of classroom discourse (Cobb, 1987) was a critical factor in determining the success of the teaching strategy. It became evident that left on their own, groups seldomly reached the level of discussion necessary for in-depth reflection to occur. It is clear that the teacher has a vital role to play in steering the group discussions in the correct direction.

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