LAYING THE FOUNDATION FOR ALGEBRA: DEVELOPING AN UNDERSTANDING OF STRUCTURE

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Abstract

In this workshop we will provide our motivation for an approach to developing a better understanding of structural notions in the number system as one of the routes to understanding algebraic structures. Firstly we analyse the notion of a structural view in both numerical and algebraic contexts. We will then present an overview of the approach as well as tasks from the teaching sequence to illustrate this. Finally we will reflect on how the development of pupils' "structure" sense in numerical contexts can benefit them when dealing with algebraic structures.

Introduction

While many research studies show that there is clearly a relationship between pupils' difficulties in algebra and their lack of understanding of the structural notions in arithmetic, this "arithmetic connection" is not clear cut. Matz (1980) and Lins (1990) suggest that the transition from the arithmetical context to the algebraic context is not a direct one as argued by Booth (1988). According to Matz and Lins many of the obstacles in the algebraic context do not necessarily reflect difficulties in the numerical context, they probably reflect difficulties in interpreting the new context. This theory suggests that there are situations in which the correct knowledge from the numerical context will be transferred correctly to algebraic context and situations when it will be transferred incorrectly.

An important difference between arithmetic and algebra is that in arithmetic we can often bypass the conventions related to the algebraic structure. For example, if it had been agreed that every possible pair of brackets should be inserted in each arithmetic string, we could have avoided the need for a convention about the order of operations in most cases. In algebra, however, even simple equations cannot be handled without a convention about

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the order of operations. At the beginning of algebra the focus shifts to the structural properties of the expressions. Our pedagogical approach therefore focuses on numerical expressions whose **structures** have a possible input on the necessary and relevant algebraic expressions that the pupils will be dealing with at the beginning of algebra. Our decision to start in the context of numbers is based on the following:

1. The structure of the algebraic system is based on the properties of the number system;
2. The numerical context is a familiar context for the pupils;
3. It is a meaningful context through situations for the construction of schema;
4. It allows meaningful reflection and verification procedures through calculations.

Furthermore we take cognisance of recent research by Linchevski (1996) which shows that pupils' interpretation of mathematical structures in a numerical context is often related to the specific numerical combinations. For example, this research shows that the following three expressions 27 – 5 + 3; 167 – 20 + 10 + 30 and 50 – 10 + 10 + 10, while having the same structure, triggered different rates of detachment (adding all the numbers after the subtraction and then subtracting the answer from the first number). It was found that 50 – 10 + 10 + 10 triggered the highest rate of detachment since many pupils over-generalised the primitive model of multiplication as repeated addition. We have therefore developed tasks that present the pupils with this kind of conflict between their number sense and their structure sense in an attempt to address these cognitive obstacles.

On The Meaning and Understanding of Structure

According to Kieran (1988), structural knowledge means being able to identify "all equivalent forms of the expression". It is however not mere recognition that reflects structural knowledge since a pupil who for example recognises the following pairs of numerical expressions (2 + 3 and 3 + 2; 2 + 3 and 2 + 2 + 1) as being equivalent might not be able to understand the structural justification for their equivalence. These pupils will probably justify the equivalence of the expressions on the basis that they produce the same result and not on a structural justification that order does not matter or the decomposition of the numbers. Linchevski and Vinner (1990) further argue that Kieran's definition of structural knowledge should also include the ability to discriminate between forms relevant to the task - generally one or two - and all the others.

Viewing an expression structurally really depends on what we choose to be the main focus in the structure. The focus may sometimes be only the numbers in the expressions. In other situations we may choose to focus only on the operations or even both. For example, if we consider the expressions $a + b$ and $b + a$, the operation is fixed while the numbers change. In another situation we may consider expressions with the same structure, $a * b$ and $b * a$, but here the numbers and the operations are changed to see if the commutative property is generalised over all the operations. Viewing an expression structurally also implies having the ability to "see" both it's "surface" structure as well as its "hidden" structure. For example, when looking at the following expression $a \times b + c \times d + e \times f$, many pupils see six numbers (it's "surface" structure), while some pupils see the expression as having three numbers (its "hidden" structure). Those pupils who are able to view the structure $a \times b + c \times d + e \times f$ as having three numbers are able to see $a \times b$ as a single object (x) and to see the structure as $x + y + z$ and addition as the dominant operation. The ability to see the hidden structures in complex algebraic structures and to relate the structure to its equivalent "simplified" form is one of the major obstacles that confront pupils when having to deal with these structures.

When dealing with algebraic structures, pupils eventually operate on a purely syntactical level. It is our view that in developing the pupils' understanding of structure in both numerical and algebraic contexts, we need to encourage them to engage in a syntactic as well as a semantic discussion for the justification of the equivalence of expressions.

A semantic justification focuses on the meaning of the numbers in an expression. For example, when considering the equivalence of the expressions $87 - (25 + 25)$ and $87 - 25 - 25$ the pupils need to be able to explain that since $25 + 25$, namely 50, is being subtracted from 87 in the first expression both 25's need to be subtracted in the second expression. A syntactic justification on the other hand deals only with the rules. In the previous example the rule is simply, that when we "remove" brackets and there is a negative sign in front of it, the sign of the numbers inside the brackets have to change for the numerical value of the expression to remain the same. Research studies (Murray, Olivier &Human 1991) have shown that pupils are able to carry out various calculation strategies before being taught the formal methods. These strategies involve not only sophisticated transformation of the numbers involved but the transformation of the task too. Consider the way in which the following problem is performed:

$196 \div 17$

$10 \times 17 = 170$

$196 - 170 = 26$

$1 \times 17 = 17$

$26 - 17 = 9$

So the answer is 11 remainder 9.

An analysis of this method shows that the pupil is able to interpret the structure of the problem at an advanced level. The pupil transforms 196 [170 + 26], and intuitively applies the right-hand side distributive law for division over addition

[(170 ÷17) + (26 ÷ 17], hence transforming the task. The pupil is at this stage obviously not aware of the "hidden" syntactical rules that he is actually using. It may be necessary therefore when we challenge pupils with the formal representations of structures that we revisit and reflect on their "primitive" calculation procedures that are loaded with semantic meaning.

Developing Structure Sense in A Numerical Context

It is necessary that pupils have a good sense for the operations on numbers before dealing with numerical expressions structurally.

In this approach we use the order of operations and the role of brackets as the main vehicle through which we focus on the structure of numerical expressions. In a more traditional approach the focus was only on calculations. The emphasis in this approach is on the structure of the expression while not abandoning the need to do the calculations as a means of verifying the equivalence of expressions. In fact, we begin with a context in which both the scientific calculator and "sequential calculator" is used to calculate the value of numerical strings. This is used as a means of highlighting the conflict, for example, in the structure $a + b \times c$. Having been confronted with the different structures, for example, $a + b \times c$ and $a \times b + c$, the pupils are now encouraged to formulate the rule for the order of operation. As a way of shifting the focus to the structure of the expressions we now ask the pupils to build equivalent expressions with three numbers *a*, *b* and *c* and the operations + and × that are to be used once only. Clearly now the pupils are dealing with a lot more than just the order of operations, the fundamental number laws which pupils are aware of intuitively at this stage need to be applied. Calculations are also given but the numerical versions for a specific structure, for example, $a + b \times c$ is intentionally chosen so that the numbers either:

1. go with the structure [$57 + 2 \times 5$]
2. "competes" or "goes against" the structure [$13 + 7 \times 15$]
3. are neutral [$20 + 5 \times 3$]

As mentioned earlier it is not the intention to focus only on the syntactical features of the structures but to provide the pupils with meaningful experiences in the semantics of the structures as well. We therefore believe that if more emphasis is placed on the structural features of numerical expressions it may enable pupils to calculate better and, for example, to handle simple algebraic structures like $6 + 9 \bullet n = 60$ better and to avoid the common mistakes like $15 \bullet n = 60$.

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