Slightly adapted version of Plenary presented at the *SOSI Geometry Imperfect Conference,*
2-4 October 1996, UNISA, Pretoria.

The Future of Secondary School Geometry

Michael de Villiers
Mathematics Education, University of Durban-Westville

Introduction

Recently in a *Mathematical Digest* (Jul '96, no. 104:26) published by the Mathematics Department at UCT someone wrote the following:

> "*... South Africa is the habitat of an endangered species, for Euclidean Geometry has disappeared from the syllabuses of most other countries* ..."

Such a statement is rather common amongst many mathematicians and mathematics educators in South Africa. However, geometry is alive and well, and experiencing an exciting rebirth in many countries; not only at school level, but at university level as well. The geometry done at school level in many countries may of course be not as highly formalised as in our senior secondary phase, and their interpretation of geometry usually goes well beyond a narrow Euclidean view. There is great danger if policy makers in mathematics education in South Africa are unaware of these dramatic new developments.

Some developments in contemporary geometry

The only geometry most South Africans know is the Euclidean geometry they learnt at school. Furthermore, there appears to be a belief that the old Greeks and other civilizations before them had discovered all the geometry there is to know. Very few realize that many exciting new results in Euclidean geometry were discovered in the nineteenth and twentieth centuries, for example, the theorems of Morley, Miquel, Feuerbach, Steiner, etc.

Apart from that, the previous century saw the development of the non-Euclidean geometries of Lobachevsky-Bolyai and Riemann. The counter-intuitive axioms for these two geometries completely revolutionized mathematicians' understanding of the nature of axioms. Whereas many had previously believed that axioms were "*self-evident truths*", they now realized that they were simply "*necessary starting points*" for mathematical systems. From believing that mathematics dealt with "*absolute truths*" in relation to the real world, they for the first time realized that mathematics only dealt with "*propositional truths*" which may or may not have applications in the real world, and in fact, that applicability was not a necessary criteria for mathematics.

In Table 1 two examples from the respective non-Euclidean geometries of Lobachevsky-Bolyai and Riemann are given. Respective models are the so-called Poincare' disk and the geometry of the sphere.

Lobachevsky-Bolyai	**Riemann**
(Playfair) Axiom: Through a point P not on a line l at least two lines parallel to l can be drawn.	(Playfair) Axiom: Through a point P not on a line l no lines parallel to l can be drawn.
Theorem: The angle sum of a triangle is less than 180 degrees and its area is proportional to the "defect" of its angle sum.	Theorem: The angle sum of a triangle is more than 180 degrees and its area is proportional to the "excess" of its angle sum.

Table 1

The previous century also saw the axiomatic development of projective geometry whose origins can be traced way back to Pappus (350 AD) and Desargues (1639). A major breakthrough was the discovery and independent proof of the principle of *duality* by Poncelet, Plucker and Gergonne in 1826. Two theorems or configurations are called *dual* if the one may be obtained from the other by replacing each concept and operator by its dual concept and operator. In projective geometry we find the following duality:

vertices (points)	-	sides (lines)
inscribed in a circle	-	circumscribed around a circle
collinear	-	concurrent

This duality is strikingly reflected by the projective theorems of Pascal (1623 - 1662) and Brianchon (1785 - 1864) as follows:

Pascal's theorem

If a hexagon is *inscribed* in a circle, then the three *points of intersection of the opposite sides* are *collinear* (lie in a straight line) (Figure 1a).

Brianchon's theorem

If a hexagon is *circumscribed* around a circle, then the three *lines* (the diagonals) *connecting opposite vertices* are *concurrent* (meet in the same point) (Figure 1b).

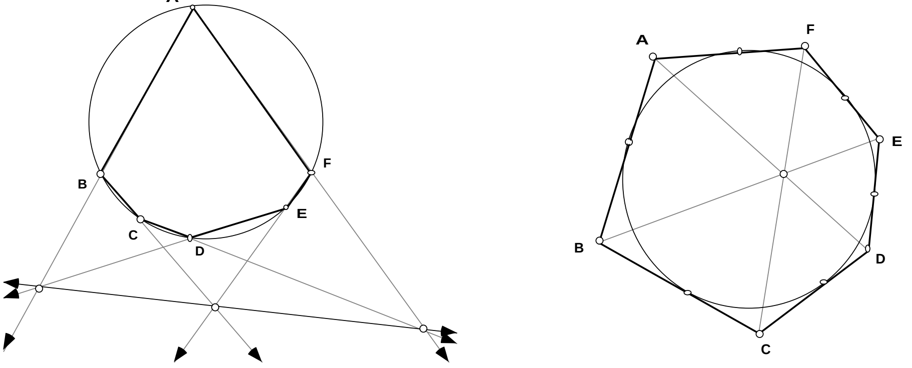


Figure 1: Pascal's & Brianchon's theorems

Although the initial axiomatic treatment of projective geometry was purely synthetic, gradual incorporation of analytical methods occurred in the latter part of the previous century. Most notably was Klein's famous *Erlangen*-program (1872) which described geometry as the study of those geometric properties which remain *invariant* (unchanged) under the various groups of transformations. In short, geometry could be classified according to this view as follows:

- *isometries* - transformations of plane figures which preserve all distances and angles (congruency)
- *similarities* - transformations of plane figures where shape (similarity) is preserved
- *affinities* - transformations of plane figures where parallelism is preserved
- *projectivities* - transformations of plane figures which preserve the collinearity of points and the concurrency of lines
- *topologies* - transformations of plane figures which preserve closure and orientability

Since time immemorial, one- and two-dimensional geometric patterns have been used by human beings to adorn their dwellings, clothes and implements. Figure 2a for example shows a Moorish tiling from the Alambra in the south of Spain. The Dutch artist Maurits Escher used tessellations extensively in the production of his artwork in the period 1937-1971. (See Figure 2b for an Escher-like tessellation, and also, Schattschneider, 1990)). Interestingly, the study of border patterns and tessellations (tilings) has received unprecedented interest by mathematicians in the twentieth century. Nevertheless, in the seventies a housewife Marjorie Rice discovered four new convex pentagons that tessellate, although mathematicians had thought at that stage that the list of tessellating pentagons was complete (see Schattschneider, 1981). Most recently, Grunbaum & Shepherd (1986) produced a systematic investigation of symmetry in tilings and tessellations which to some degree equals Euclides' *Elements.*

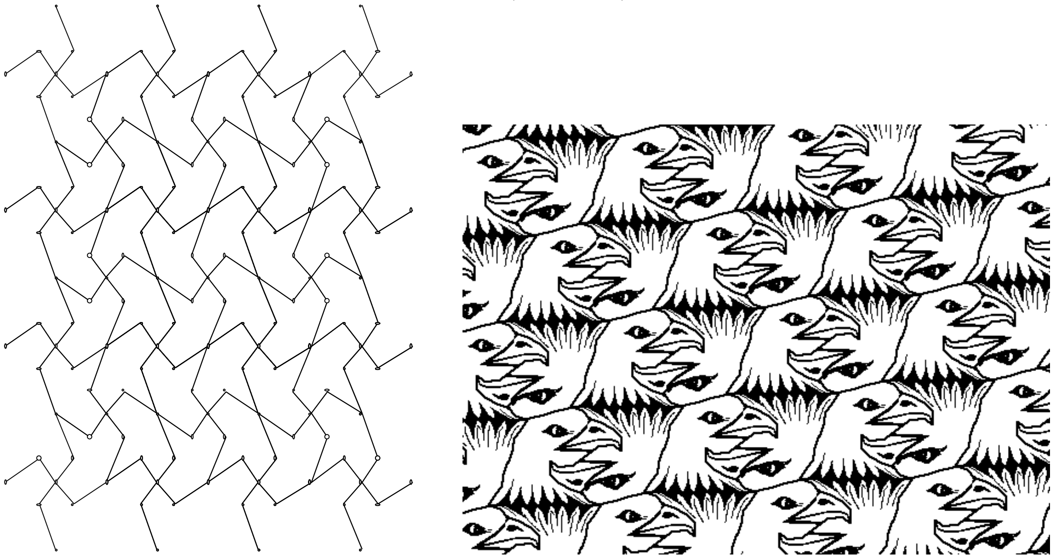
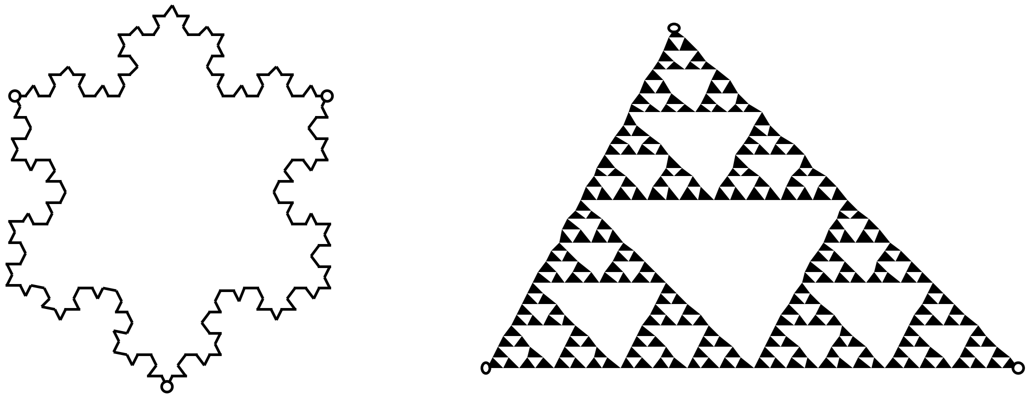


Figure 2: Examples of tessellations

One of the important concepts in the classification of border patterns and tessellations is that of *symmetry* . Using this concept, border patterns can be classified into seven different types and tessellations into seventeen different types. An obvious property of any tiling is that of a repetition of the pattern. If a tiling has translational symmetry in two independent directions it is called *periodic*. Although most common tilings are periodic, only about twenty years ago, the British mathematician Penrose discovered a surprising set of two quadrilaterals that tile non-periodically (eg. see Benade, 1995). In fact, it is still an open question whether or not a single tile exists with which one can only tile non-periodically.

Another interesting development in recent years has been *fractal* geometry, which is the study of geometrical objects with fractional dimensions. For example, a cloud is a good example of a fractal. Although it is not really quite three-dimensional, it is certainly not two-dimensional; one could therefore say that its dimensions lie somewhere between twee and three. In fact, many real world objects such as coastlines, fern leaves, mountain ranges, trees, crystals, etc. have fractal properties. Fractal image compression is also used today in a variety of multimedia and other image-based computer applications. An important property of fractals is that of *self-similarity* which loosely means that any small subset of the figure is similar to the larger figure. Two examples of fractals are given in Figure 3 where this property is clearly illustrated.



Koch Snowflake Sierpinski Triangle

Figure 3: Examples of fractals

Recent years have also seen the development and expansion of Knot Theory and its increased application to biology, the use of Projective Geometry in the design of virtual reality programs, the application of Coding Theory to the design of CD players, an investigation of the geometry involved in robotics, etc. Even Soap Bubble Geometry is receiving new attention as illustrated by the special session given to it at the Burlington MathsFest in 1995. In 1986 Eugene Krause wrote a delightful little book on Taxicab Geometry, introducing a new kind of non-Euclidean geometry. Several international conferences on geometry have been held over the past decade. In fact, David Henderson from Cornell University, USA recently told the author that presently they have more post-graduate students in geometry or geometry related fields than in pure algebra.

Even Euclidean geometry is experiencing an exciting revival, in no small part due to the recent development of dynamic geometry software such as *Cabri* and *Sketchpad*. In fact, Philip Davies (1995) describes a possibly rosy future for research in triangle geometry. Recently Adrian Oldknow (1995, 1996) for example used *Sketchpad* to discover the hitherto unknown result that the Soddy center, incenter and Gergonne point of a triangle are collinear (amongst other interesting results). The Soddy center is named after the Nobel prizewinning chemist, Frederick Soddy, who published the following result in 1936: If three circles with centers at the three vertices of a triangle are drawn tangent to each other as shown in Figure 4 (each triangle has a unique set of three such circles), then there exists a fourth circle tangent to all three as shown. (The center of this circle is now known as the (inner) Soddy center S - there is also an outer one).

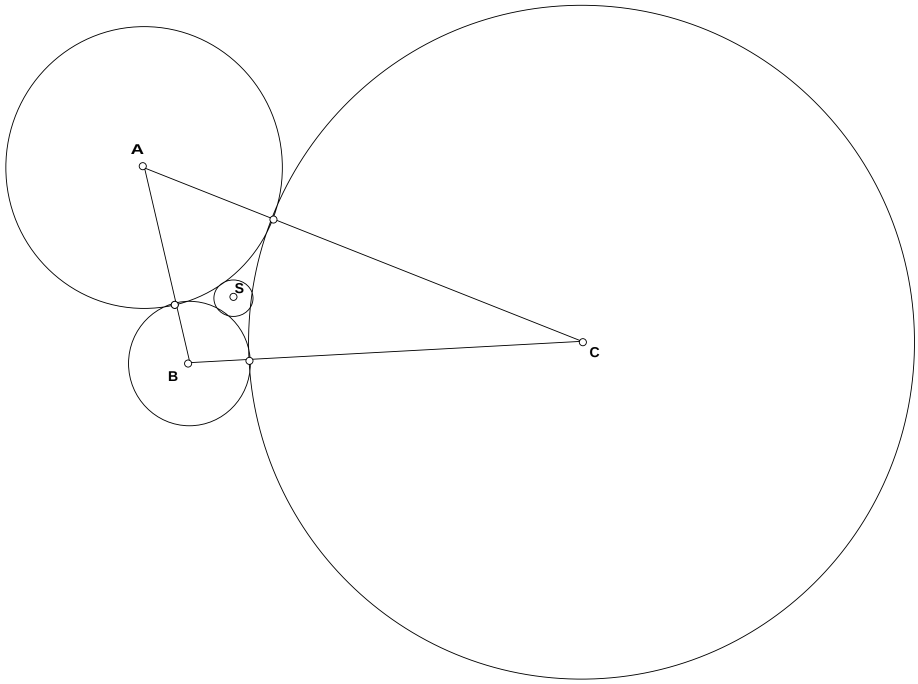


Figure 4: Soddy center

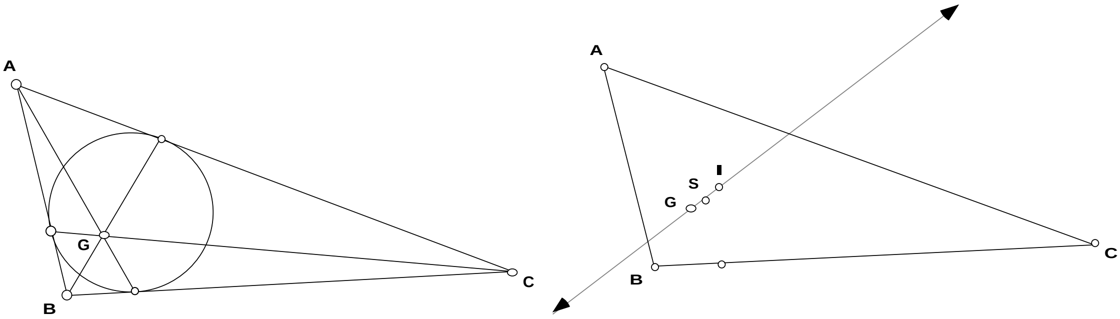


Figure 5: Gergonne point & Gergonne-Soddy-Incenter line

The Gergonne point G of a triangle is the point of concurrency of the three line segments from the vertices to the points of tangency of the incircle with the opposite sides (see Figure 5a). (The Gergonne point incidentally is just a special degenerate case of Brianchon's theorem). Then as shown in Figure 5b, we find the surprising result that the Gergonne point G, the Soddy center S and the incenter I are collinear. (The outer Soddy center also lies on this line).

The author also recently discovered two interesting generalizations of Van Aubel's theorem using dynamic geometry software. This theorem states that if squares are constructed on the sides of any quadrilateral then the line segments connecting the centers of the squares of opposite sides are always equal and perpendicular (see Yaglom, 1962 or Kelly, 1966). After some experimentation, the author managed to further generalize it for similar rectangles and rhombi on the sides as shown in Figures 6 & 7 (proofs are given in De Villiers, 1996 & 1997). In Figure 6, *EG* is always perpendicular to *FH*. Also *KM* is congruent to *LN* where *K*, *L*, *M* and *N* are the midpoints of the line segments joining adjacent vertices of the similar rectangles as shown.

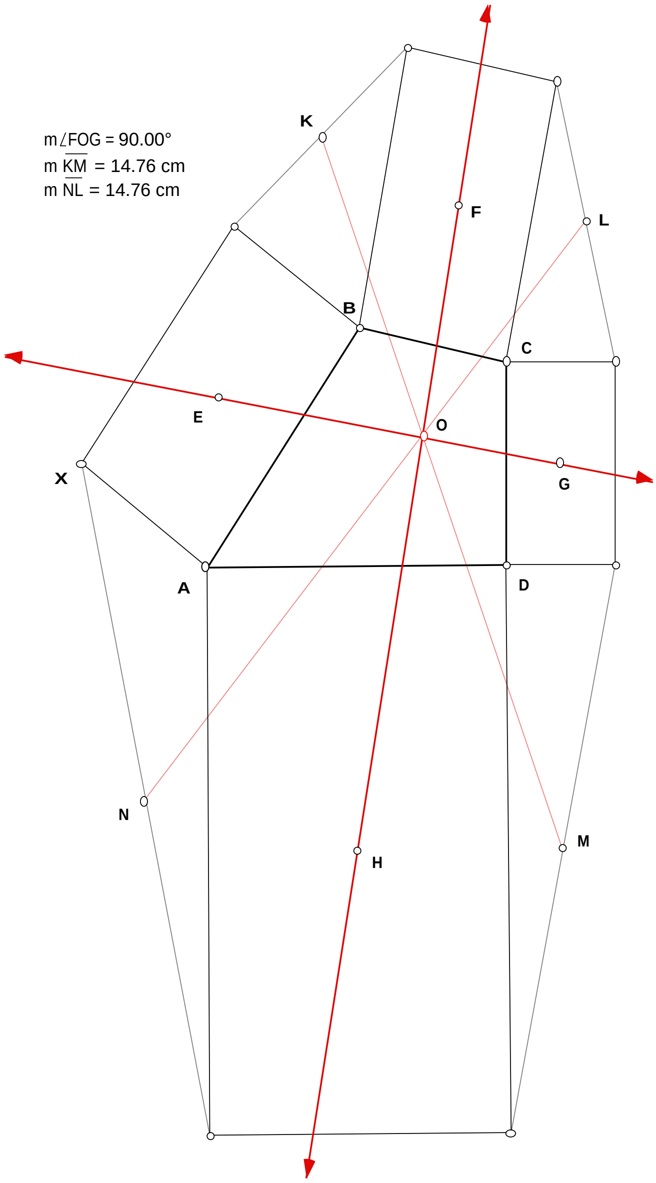


Figure 6: Van Aubel rectangle-generalisation

In Figure 7, *EG* is always congruent to *FH*. Also *KM* is perpendicular to *LN* where *K*, *L*, *M* and *N* are the midpoints of the linesegments joining adjacent vertices of the similar rhombi as shown. The "intersection" of these two results therefore provides Van Aubel's theorem.

Slightly adapted version of Plenary presented at the *SOSI Geometry Imperfect Conference*, 2-4 October 1996, UNISA, Pretoria.

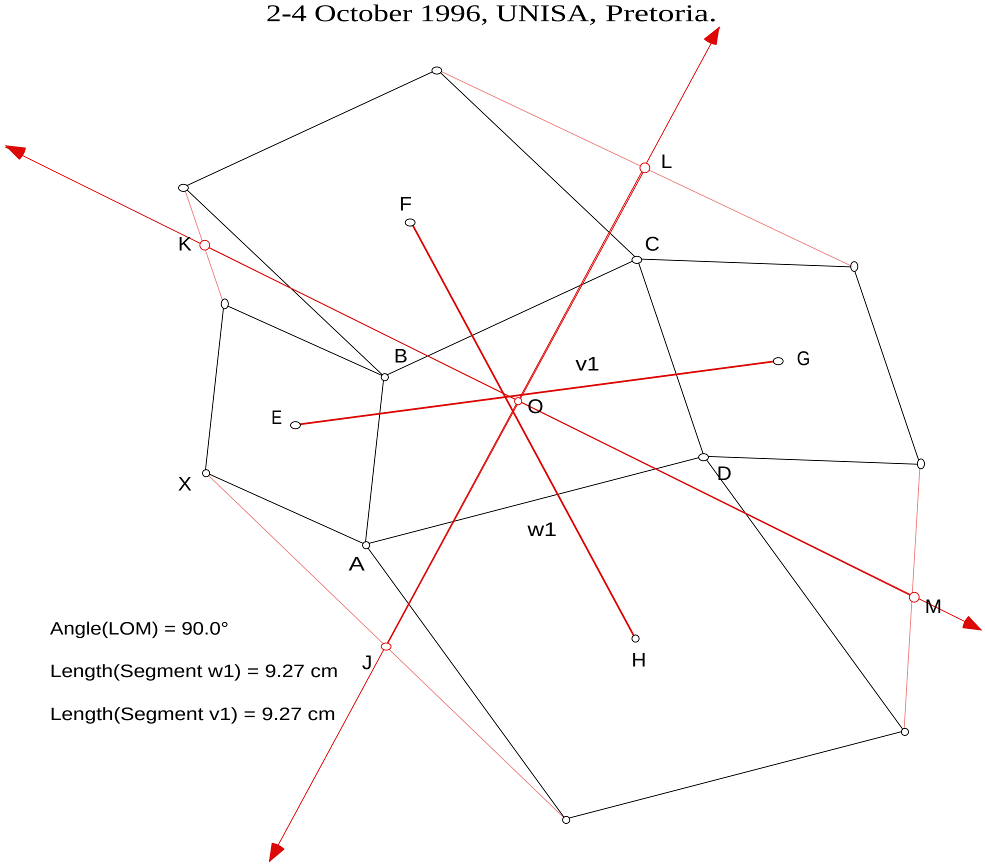


Figure 7: Van Aubel rhombus-generalisation

Just a brief perusal of some recent issues of mathematical journals like the *Mathematical Intelligencer*, *American Mathematical Monthly*, *The Mathematical Gazette*, *Mathematics Magazine*, *Mathematics & Informatics Quarterly*, etc. easily testify to the increased activity and interest in traditional Euclidean geometry involving triangles, quadrilaterals and circles. The mathematician Crelle once said: "*It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties*". Perhaps this applies even more widely to Euclidean geometry in general!

Some developments in geometry education

The Van Hiele theory

The Van Hiele theory originated in the respective doctoral dissertations of Dina van Hiele-Geldof and her husband Pierre van Hiele at the University of Utrecht, Netherlands in 1957. Dina unfortunately died shortly after the completion of her dissertation, and Pierre was the one who developed and disseminated the theory further in later publications.

While Pierre's dissertation mainly tried to explain why pupils experienced problems in geometry education (in this respect it was **explanatory** and **descriptive**), Dina's dissertation was about a teaching experiment and in that sense is more **prescriptive** regarding the ordering of geometry content and learning activities of pupils. The most obvious characteristic of the theory is the distinction of five discrete thought levels in respect to the development of pupils' understanding of geometry. Four important characteristics of the theory are summarised as follows by Usiskin (1982:4):

- **fixed order** - The order in which pupils progress through the thought levels is invariant. In other words, a pupil cannot be at level n without having passed through level n-*1*.
- **adjacency** - At each level of thought that which was intrinsic in the preceding level becomes extrinsic in the current level.
- **distinction** - Each level has its own linguistic symbols and own network of relationships connecting those symbols.
- **separation** - Two persons who reason at different levels cannot understand each other.

The main reason for the failure of the traditional geometry curriculum was attributed by the Van Hieles to the fact that the curriculum was presented at a higher level than those of the pupils; in other words they could not understand the teacher nor could the teacher understand why they could not understand! Although the Van Hiele theory distinguishes between five different levels of thought, we shall here only focus on the first four levels as they are the most pertinent ones for our secondary school geometry. The general characteristics of each level can be described as follows:

Level 1: Recognition

Pupils visually recognize figures by their global appearance. They recognize triangles, squars, parallelograms, and so forth by their shape, but they do not explicitly identify the properties of these figures.

Level 2: Analysis

Pupils start analysing the properties of figures and learn the appropriate technical terminology for describing them, but they they do not interrelate figures or properties of figures.

Level 3: Ordering

Pupils logically order the properties of figures by short chains of deductions and understand the interelationships between figures (eg. class inclusions).

Level 4: Deduction

Pupils start developing longer sequences of statements and begin to understand the significance of deduction, the role of axioms, theorems and proof.

The differences between the first three levels can be summarised as shown in Table 2 in terms of the objects and structure of thought at each level (adapted from Fuys et al, 1988:6).

	Level 1	Level 2	Level 3
Objects of thought	**Individual figures**	**Classes of figures**	**Definitions of classes of figures**
Structure of thought	**Visual recognition Naming Visual sorting**	**Recognizing properties as characteristics of classes**	**Noticing & formulating logical relationships between properties**
Examples	**• Parallelograms all go together because they "look the same" • Rectangles, squares and rhombi are not parms because they do "not look like one"**	**A parallelogram has: • 4 sides • opp. angles = • opp. sides = • opp. sides // • bisecting diagonals; etc. A rectangle is not a parm since a rectangle has 90° angles but a parm not.**	**• Opposite sides = imply opposite sides // • Opposite sides // imply opposite sides = • opposite angles = imply opp. sides = • bisecting diagonals imply half-turn symmetry**

Table 2

By using task-based interviews, Burger & Shaughnessy (1986) characterized pupils' thought levels at the first four levels more fully as follows:

Level 1

(1) Often use irrelevant visual properties to identify figures, to compare, to classify and to describe.

(2) Usually refer to visual prototypes of figures, and is easily misled by the orientation of figures.

(3) An inability to think of an infinite variation of a particular type of figure (eg in terms of orientation and shape).

(4) Inconsistent classifications of figures; for example, using non-common or irrelevant properties to sort figures.

(5) Incomplete descriptions (definitions) of figures by viewing necessary (often visual) conditions as sufficient conditions.

Level 2

(1) An explicit comparison of figures in terms of their underlying properties.

(2) Avoidance of class inclusions between different classes of figures, eg. squares and rectangles are considered to be disjoint.

(3) Sorting of figures only in terms of one property, for example, properties of sides, while other properties like symmetries, angles and diagonals are ignored.

(4) Exhibit an uneconomical use of the properties of figures to describe (define) them, instead of just using sufficient properties.

(5) An explicit rejection of definitions supplied by other people, eg. a teacher or textbook, in favour of their own personal definitions.

(6) An empirical approach to the establishment of the truth of a statement; eg. the use of observation and measurement on the basis of several sketches.

Level 3

(1) The formulation of economically, correct definitions for figures.

(2) An ability to transform incomplete definitions into complete definitions and a more spontaneous acceptance and use of definitions for new concepts.

(3) The acceptance of different equivalent defintions for the same concept.

(4) The hierarchical classification of figures, eg. quadrilaterals.

(5) The explicit use of the logical form "*if ... then*' in the formulation and handling of conjectures, as well as the implicit use of logical rules such as *modus ponens.*

(6) An uncertainty and lack of clarity regarding the respective functions of axioms, definitions and proof.

Level 4

(1) An understanding of the respective functions (roles) of axioms, definitions and proof.

(2) Spontaneous conjecturing and self-initiated efforts to deductively verify them.

Russian research on geometry education

Geometry has always formed an extremely prominent part of the Russian mathematics curriculum in the nineteenth and twentieth centuries. This proud tradition was no doubt influenced by (and instrumental in) the achievements of several famous Russian geometers (like Lobachevsky) in the past two centuries. Traditionally the Russian geometry curriculum consisted of two phases, namely, an *intuitive* phase for Grades 1 to 5 and a *systematisation* (deductive) phase from Grade 6 (12/13 year old).

In the late sixties Russian (Soviet) researchers undertook a comprehensive analysis of both the intuitive and the systematisation phases in order to try and find an answer to the disturbing question of why pupils who were making good progress in other school subjects, showed little progress in geometry. In their analysis, the Van Hiele theory played a major part. For example, it was found that that at the end of Grade 5 (before the resumption of the systematisation phase which requires at least Level 3 understanding) only 10-15% of the pupils were at Level 2.

The main reason for this was the insufficient attention to geometry in the primary school. For example, in the first five years, pupils were expected to become acquainted, via mainly Level 1 activities, with only about 12-15 geometrical objects (and associated terminology). In contrast, it was expected of pupils in the very first topic treated in the first month of Grade 6 to become acquainted not only with about 100 new objects and terminology, but it was also being dealt with at Level 3 understanding. (Or frequently, the teacher had to try and introduce new content at 3 different levels simultaneously). No wonder they described the period between Grades 1 and 5 as a "*prolonged period of geometric inactivity*"!

The Russians subsequently designed a very successful experimental geometry curriculum based on the Van Hiele theory. They found that an important factor was the continuous sequencing and development of concepts from Grade 1. As reported in Wirszup (1976: 75-96), the average pupil in Grade 8 of the experimental curriculum showed the same or better geometric understanding than their Grade 11 and 12 counterparts in the old curriculum.

The primary & middle school geometry curriculum

The parallels from the Russian experience to South Africa are obvious. We still have a geometry curriculum heavily loaded in the senior secondary school with formal geometry, and with relatively little content done informally in the primary school. (Eg. how much similarity or circle geometry is done in the primary school?) In fact, it is well known that on average, pupils' performance in matric (Grade 12) geometry is far worse than in algebra. Why?

The Van Hiele theory supplies an important explanation. For example, research by De Villiers & Njisane (1987) has shown that about 45% of black pupils in Grade 12 (Std 10) in KwaZulu had only mastered Level 2 or lower, whereas the examination assumed mastery at Level 3 and beyond! Similar low Van Hiele levels among secondary school pupils have been found by Malan (1986), Smith & De Villiers (1990) and Govender (1995). In particular, the transition from Level 1 to Level 2 poses specific problems to second language learners, since it involves the acquisition of the technical terminology by which the properties of figures need to be described and explored. This requires sufficient time which is not available in the presently overloaded secondary curriculum.

It seems clear that no amount of effort and fancy teaching methods at the secondary school will be successful, unless we embark on a major revision of the primary school geometry curriculum along Van Hiele lines. The future of secondary school geometry thus ultimately depends on primary school geometry!

In Japan for example pupils already start off in Grade 1 with extended tangram, as well as other planar and spatial, investigations (eg. see Nohda, 1992). This is followed up continuously in following years so that by Grade 5 (Std 3) they are already dealing formally with the concepts of congruence and similarity, concepts which are only introduced in Grades 8 and 9 (Stds 6 & 7) in South Africa. No wonder that in international comparative studies in recent years, Japanese school children have consistently outperformed school children from other countries.

Although the recent introduction of tessellations in our primary schools is to be greatly welcomed, many teachers and textbook authors do not appear to understand its relevance in relation to the Van Hiele theory. Although tessellations have great aesthetic attraction due to their intriguing and artistically pleasing patterns, the fundamental reason for introducing it in the primary school is that it provides an intuitive visual foundation (Van Hiele 1) for a variety of geometric content which can later be treated more formally in a deductive context.

For example, in a triangular tessellation pattern such as shown in Figure 8, one could ask pupils the following questions:

(1) identify and colour in parallel lines

(2) what can you say about angles A, B, C , D and E and why?

(3) what can you say about angles A, 1, 2, 3 and 4 and why?

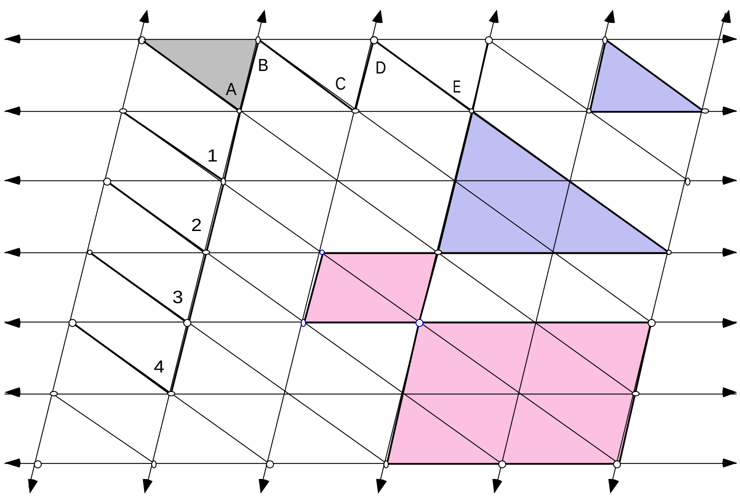


Figure 8: Visualisation

In such an activity pupils will realize that angles A, B, C, D and E are all equal since a halfturn of the grey triangle around the midpoint of the side AB maps angle A onto angle B, etc. In this way, pupils can be introduced for the first time to the concept of "*saws*" or "*zig-zags*" (alternate angles). Similarly, pupils should realize that angles A, 1, 2, 3 and 4 are all equal since a translation of the grey triangle in the direction of angles 1, 2, 3 and 4 consecutively maps angle A onto each of these angles. In this way, pupils can be introduced for the first time to the concept of "*ladders*" (corresponding angles). Pupils should further be encouraged to find different saws and ladders in the same and other tessellation patterns to improve their visualisation ability.

Since each tile has to be identical and can be made to fit onto each other exactly by means of translations, rotations or reflections pupils can easily be introduced to the concept of congruency. Pupils can also be asked to look for different shapes in such tessellation patterns, eg. parallelograms, trapezia and hexagons. They could also be encouraged to look for larger figures with the *same shape*, thus intuitively introducing them to the concept of *similarity* (as shown in Figure 8 by the shaded similar triangles and parallelograms).

Tessellations also provide a suitable context for the analysis of the properties of geometric figures (Van Hiele 2), as well as their logical explanation (Van Hiele 3). For example, after pupils have constructed a triangular tessellation pattern as shown in Figure 9, one could ask them questions like the following:

(1) What can you say about angles A and B in relation to D and E? Why? What can you therefore conclude from this?

(2) What can you say about angles F and G in relation to angles H and I? Why? What can you therefore conclude from this?

(3) What can you say about line segment JK in relation to line segment LM? Why? What can you therefore conclude from this?

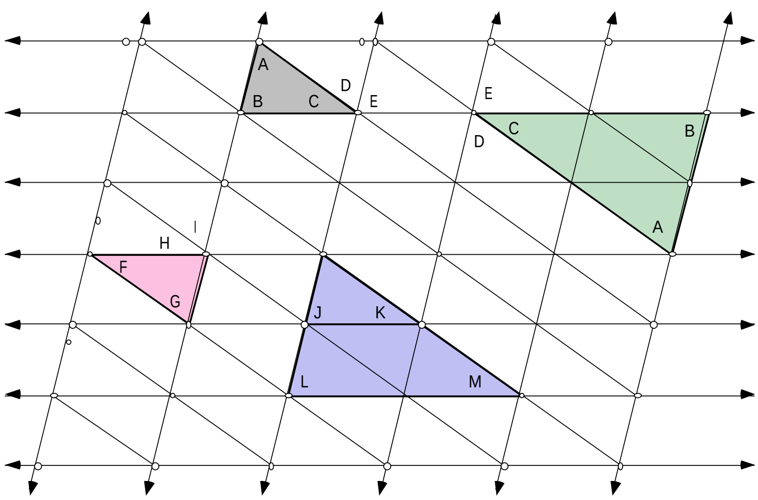


Figure 9: Analysing

In the first case, pupils can again see that angle A = angle D due to a saw being formed. Also angle B = angle E due to a ladder. It is then easy for them to observe that since the three angle lie on a straight line, that the sum of the angles of triangle ABC must be equal to a straight line. They can also observe that this is true at any vertex, as well as for any size triangle or orientation, thus enabling generalization. In the second case, the exterior angle theorem is introduced and in the third case, the midpoint theorem. Such analyses are clearly just a short step away from the standard geometric explanations (proofs); all they now need is some formalisation. In Figure 10 the three levels are illustrated for the discovery and explanation that the opposite angles of a parallelogram are equal.

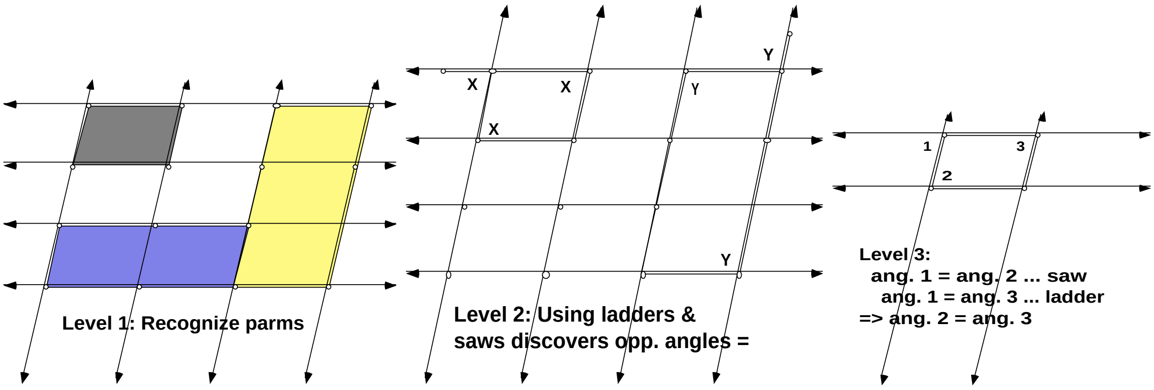


Figure 10: Three levels

Another important aspect of the Van Hiele theory is that it emphasizes that informal activities at Levels 1 and 2 should provide appropriate "*conceptual substructures*" for the formal activities at the next level. I've often observed teachers and student teachers who let pupils measure and add the angles of a triangle for them to discover that they add up to 180°. From a Van Hiele perspective this is totally inappropriate as it does not provide a suitable conceptual substructure in which the formal proof is implicitly embedded. In comparison, the earlier described tessellation activity clearly provides such a substructure. Similarly, the activity of measuring the base angles of an isosceles triangle is conceptually inappropriate, but folding it around its axis of symmetry lays the foundation for a formal proof later. The same applies to the investigation of the properties of the quadrilaterals. For example, it is conceptually inappropriate to measure the opposite angles of a parallelogram to let pupils discover that they are equal. It is far better to let them give the parallelogram a half-turn to find that opposite angles (and sides) map onto each other, as this generally applies to all parallelograms and contains the conceptual seeds for a formal proof.

Recently I had a conversation with a teacher who quickly dismissed a fellow teacher's introduction to tessellations who first let his pupils pack out little card board tiles. This teacher felt that it produced untidy patterns, was ineffective and time consuming, and that one should just start by providing pupils with ready-made square or triangular grids and show them how they can then easily draw neat tessellation patterns (see Figure 11). Although such grids are a useful and effective way of drawing neat patterns, it is conceptually extremely important for pupils to first have had prior experience of physically packing out tiles, ie. rotating, translating, reflecting the tiles by hand. The problem is that it is possible to draw tessellation patterns on such grids without any clear understanding of the underlying isometries which create them, which in turn are conceptually important for analysing the geometric properties embedded in the pattern.

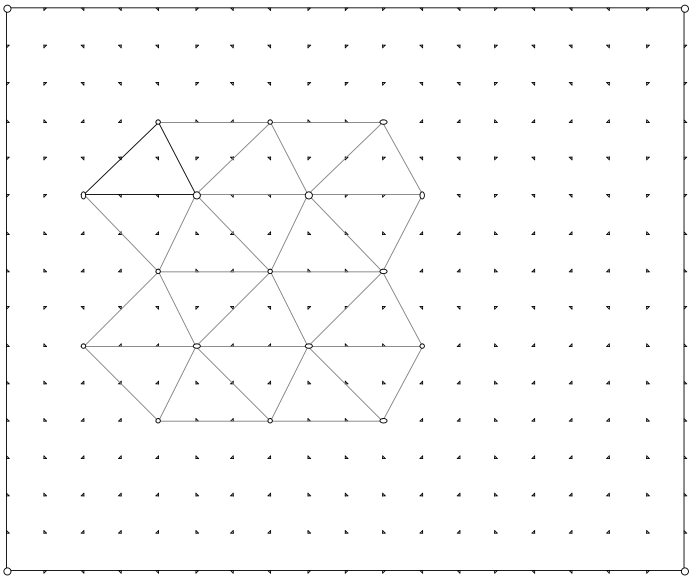


Figure 11: Using grids

Process versus product teaching in geometry

The distinction between "*processes*" and "*products*" in mathematics education is a relatively old one. With a product is meant here the end-result of some mathematical activity which preceded it. As far back as 1978, the Syllabus Proposals of MASA regarding the South African Mathematics Project, stated:

> "*The intrinsic value of mathematics is not only contained in the PRODUCTS of mathematical activity (i.e. polished concepts, definitions, structures and axiomatic systems, but also and especially in the PROCESSES of MATHEMATICAL ACTIVITY leading to such products, e.g. generalization, recognition of pattern, defining, axiomatising. The draft syllabi are intended to reflect an increased emphasis on genuine mathematical activity as opposed to the mere assimilation of the finished products of such activity. This emphasis is particularly reflected in the various sections on geometry.*" - MASA (1978:3)

Regrettably these good intentions, except for a few schools, were hardly implemented on a large scale in South African schools. Most teachers and textbook authors simply continued providing pupils with ready-made content that they merely had to assimilate and regurgitate in tests and exams.

Traditional geometry education of this kind can be compared to a cooking and bakery class where the teacher only shows pupils cakes (or even worse, only pictures of cakes) without showing them what goes into the cake and how it is made. In addition, they're not even allowed to try their own hand at baking!

The distinction between some of the main processes and products of formal geometry can be summarised as shown in Table 3. Most formal products often require a number of prior processes, some of which have been indicated. The process of proving also has its own product, namely a proof, which should be distinguished from the theorem, definition or axiom to which it refers.

Product	Process
Axioms	**Axiomatizing** • **proving**
Definitions	**Defining** • **experimenting** • **proving**
Algorithms	**Algorithm construction & verification**
Theorems	**Theorem finding & formulating** • **Experimenting** • **Refuting** • **Pattern finding** • **Generalizing** • **Specializing** • **Visualising** • **Proving**
Classifications	**Classifying**

Table 3

Due to limitations of space, we shall here mainly focus on the handling of definitions at Van Hiele Level 3. The direct teaching of geometry definitions with no emphasis on the underlying process of defining has often been criticised by mathematicians and mathematics educators alike. For example, already in 1908 Benchara Blandford wrote (quoted in Griffiths & Howson, 1974: 216-217):

> "*To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To*

do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational."

The well-known mathematician Hans Freudenthal (1973:417-418) also strongly criticized the traditional practice of the direct provision of geometry definitions as follows:

"*... most definitions are not preconceived but the finishing touch of the organizing activity. The child should not be deprived of this privilege ... Good geometry instruction can mean much - learning to organize a subject matter and learning what is organizing, learning to conceptualize and what is conceptualizing, learning to define and what is a definition. It means leading pupils to understand why some organization, some concept, some definition is better than another. Traditional instruction is different. Rather than giving the child the opportunity to organize spatial experiences, the subject matter is offered as a preorganized structure. All concepts, definitions, and deductions are preconceived by the teacher, who knows what is its use in every detail - or rather by the textbook author who has carefully built all his secrets into the structure.*"

From our preceding discussion of the Van Hiele theory it should be clear that understanding of formal definitions only develop at Level 3, and that the direct provision of formal definitions to pupils at lower levels would be doomed to failure. In fact, if we take the constructivist theory of learning seriously (namely that knowledge simply cannot be transferred directly from one person to another, and that meaningful knowledge needs to be actively (re)-constructed by the learner), we should even at Level 3 engage pupils in the activity of defining and allow them to choose their own definitions at each level. This implies allowing the following kinds of meaningful definitions at each level:

Van Hiele 1

Visual definitions, eg. a rectangle is a quadrilateral with all angles 90° and two long and two short sides.

Van Hiele 2

Uneconomical definitions, eg. a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.

Van Hiele 3

Correct, economical definitions, eg. a rectangle is a quadrilateral with two axes of symmetry though opposite sides.

As can be seen from the two examples at Van Hiele Levels 1 & 2, pupils' spontaneous definitions would also tend to be *partitional*, in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van Hiele theory, definitions at Level 3 are typically *hierarchical,* which means they allow for the inclusion of the squares among the rectangles, and would not be understood by pupils at lower levels.

The presentation of formal definitions in textbooks is often preceded by an activity whereby pupils have to compare in tabular form various properties of the quadrilaterals, eg. to see that a square, rectangle and rhombus have all the properties of a parallelogram. The purpose clearly is to prepare them for the formal definitions later on which are *hierarchical.* (In other words, the given definitions provide for the inclusion of special cases, eg. a parallelogram is defined so as to include squares, rhombi and rectangles). However, research reported in De Villiers (1994) show that many pupils, even after doing tabular comparisons and other activities, if given the opportunity, still prefer to define quadrilaterals in *partitions.* (In other words, they would for example still prefer to define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal).

A constructivist approach would not directly present pupils with read-made definitions, but allow them to formulate their own definitions irrespective of whether they are partitional or hierarchical. By then discussing and comparing in class the relative advantages and disadvantages of these two different ways of classifying and defining quadrilaterals (both of which are mathematically correct), pupils may be led to realize that there are certain advantages in accepting a hierarchical classification. For example, if pupils are asked to compare the following two definitions for the parallellograms, they immediately realize that the former is much more **economical** than the latter:

hierarchical: A parallelogram is a quadrilateral with both pairs of opposite sides parallel.

partitional: A parallelogram is a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

Clearly in general, partitional definitions are longer since they have to include additional properties to ensure the exclusion of special cases. Another advantage of a hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases. For example, if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is also true for rectangles, rhombi and squares. If however, we classified and defined them partitionally, we would have to prove separately in each case, for parallelograms, rectangles, rhombi and squares, that their diagonals bisect each other. Clearly this is very uneconomical. It seems clear that unless the role and function of a hierarchical classification is meaningfully discussed in class as described in De Villiers (1994), many pupils will have difficulty in understanding why their intuitive, partitional definitions are not used.

The USEME experiment

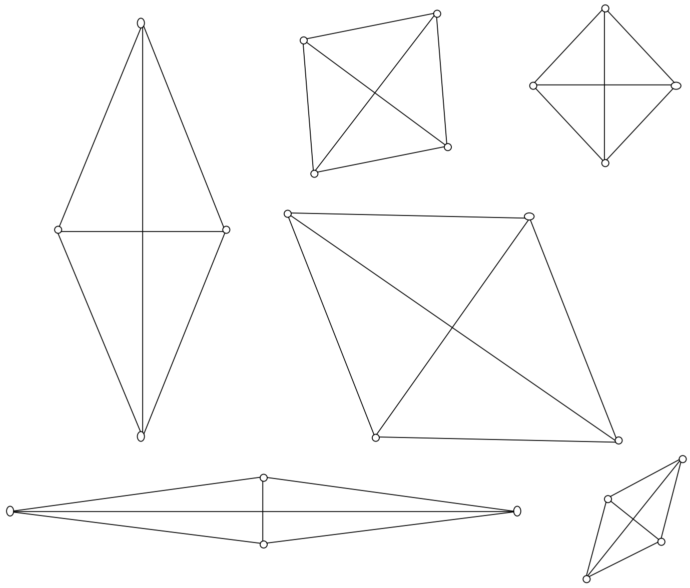
Is it possible to devise teaching stategies for the teaching of the processes of defining and axiomatising at Van Hiele levels 3 and 4? This in fact was the focus of the University of Stellenbosch Experiment with Mathematics Education (USEME) conducted with a control group in 1977 and an experimental group in 1978 (see Human & Nel et al, 1989a). The experiment was aimed at the Grade 10 (Std 8) level and involved 19 schools in the Cape Province. Whereas the traditional approach focusses overridingly on developing the ability of making deductive proofs (especially for riders), the experimental approach was aimed mainly at:

- developing the ability to construct formal, economical definitions for geometrical concepts
- developing understanding of the nature and role of axioms, definitions and proof

The following is an example of one of the first exercises in defining used in the experimental approach (see Human & Nel et al, 1989b:21).

EXERCISE

1(a) Make a list of all the common properties of the figures below. Look at the angles, sides and diagonals and measure if necessary.



(b) What are these types of quadrilaterals called?

(c) How would you explain in words, *without making a sketch*, what these quadrilaterals are to someone not yet acquainted with them?

The spontaneous tendency of almost all the pupils in (c) was to make a list of all the properties discovered in (a); thus giving a correct, but uneconomical description (definition) of the rhombi (thus suggesting Level 2 understanding). This led to the next two exercises which was intended to lead them to shorten their descriptions (definitions), for example:

2. A letter is addressed as follows:

   Mr. JH Nel
   "Nelstevrede"
   9 Venter Avenue
   PO Box 48639
   Stellenbosch
   7600

(a) The address is unnecessarily long. Give a shortened version of the above address so that the letter would still arrive at Mr. Nel. (Post in Stellenbosch is delivered in post boxes as well as to street addresses.)

(b) Are there other shortened versions of the above address whereby the letter would still reach Mr. Nel? Give as many shortened versions as you can. Everyone must be as short as possible.

3(a) Construct three different rhombi on your own.

(b) Look again at the verbal description of rhombi you gave in 1(c). Is your description perhaps unnecessarily long? If so, give a shorter description of rhombi which nevertheless would still definitely give you a rhombus if you constructed a figure according to the information contained in your (shorter) description: ensure therefore that it will have all the properties of a rhombus, even if all these properties are not mentioned in (your) shorter description.

(c) Give three different short verbal descriptions of rhombi.

(d) Try to construct a quadrilateral which is not a rhombus, but complies to the conditions of your first (shorter) descriptions in (b). If you can achieve that, your description is not an accurate description of the rhombi! Check your other two shorter descriptions of the rhombi in the same manner.

Clearly here pupils were led to shorten their descriptions (definitions) of rhombi by leaving out some of its properties. For example, in 3(a) pupils found that one does not need to use all the properties to construct a rhombus. One could for example obtain one by constructing all sides equal. In (b) and (c) pupils typically came up with different shorter versions, some of which were *incomplete* (particularly if they're encouraged to make it as short as possible by promising a prize!), for example: "*A rhombus is a quadrilateral with perpendicular diagonals*". This provided opportunity to provide a counter-example and a discussion of the need to contain enough (sufficient) information in one's descriptions (definitions) to ensure that somebody else knows exactly what figure one is talking about.

With some encouragement, pupils came up with several different possibilities. Also note at this stage that they were not expected to **logically** check their definitions, but by accurate **construction** and **measurement** (in other words a typical Level 2 activity). For example, pupils were expected to construct figures as shown in Figure 12 to evaluate definitions like the following:

(1) A rhombus is a quadrilateral with all sides equal.

(2) A rhombus is a quadrilateral with perpendicular, bisecting diagonals.

(3) A rhombus is a quadrilateral with bisecting diagonals.

(4) A rhombus is a quadrilateral with one pair of opposite sides parallel and one pair of adjacent sides equal.

(5) A rhombus is a quadrilateral with perpendicular diagonals and one pair of adjacent sides equal.

(6) A rhombus is a quadrilateral with both pairs of opposite sides parallel and one pair of adjacent sides equal.

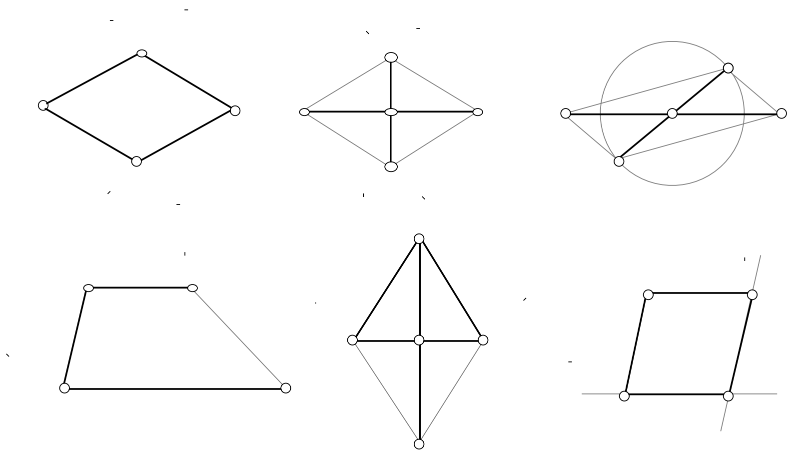


Figure 12: Construction & measurement

Psychologically, constructions like these are extremely important for the transition from Level 2 to Level 3. It helps to develop an understanding of the difference between a *premisse* and *conclusion* and their *causal* relationship; in other words, of the logical structure of an "*if-then*" statement. Logically each of the above statements can be rewritten in this form. For example, the last statement could be rewritten as: "**If** a quadrilateral has both pairs of opposite sides parallel and one pair of adjacent sides equal, **then** it is a rhombus (ie. has all sides equal, perpendicular bisecting diagonals, etc)". Smith (1940) reported marked improvement in pupils' understanding of "*if-then*" statements by letting them make constructions to evaluate geometric statements as follows:

"*Pupils saw that when they did certain things in making a figure, certain other things resulted. They learned to feel the difference in category between the relationships they* ***put*** *into a figure - the things over which they had control - and the relationships which* ***resulted*** *without any action on their part. Finally the difference in these two categories was associated with the difference between the* ***given*** *conditions and* ***conclusion****, between the if-part and the then-part of a sentence.*"

After some experimental exploration of different alternative definitions for the rhombi as described above, the pupils were then led into a deductive phase where starting from one definition they had to logically check whether all the other properties could be derived from it (as theorems). The same exercises were then repeated for the parallelograms. Eventually, it was explained to pupils that it would be confusing if everyone used different definitions for the rhombi and parallelograms, and it was agreed to henceforth use one definition only for each concept. (Note that the role and function of a hierarchical classification for the quadrilaterals was not adequately addressed at the time of the USEME experiment, and was one of the reasons for the subsequent study reported in De Villiers (1994)).

A common misconception among pupils (and even some of their teachers and textbook authors) is that axioms are *self-evident* truths, instead of necessary starting points for a mathematical system. An important objective of the USEME project was to let pupils understand the **necessity** of definitions and axioms by providing them with the experience that not all propositions within a formal system can be proved without getting a circularity, and that one consequently had to accept certain propositions as starting points (Van Hiele Level 4). Instead of presenting a finished axiomatic system to pupils, they were first engaged in the process of systematization as follows (see Human & Nel et al, 1989b: 43). (Note: Although pupils at this point knew the properties of parallel lines from informal exploration, they had not been given a formal definition for parallel lines nor logically derived any of the properties. They had also earlier been introduced to proof as a means of explanation of several interesting riders).

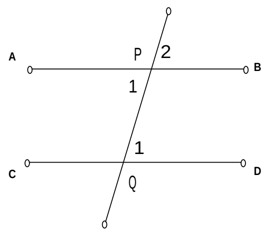
EXERCISE

1. Try to prove that if two parallel lines are cut by a transversal, then alternate angles are equal. You may make use of our other assumptions about parallel lines (corresponding angles equal, co-interior angles supplementary), as well as the theorem that when two straight lines intersect, vertically opposite angles are equal.
2. In your proof in no. 1 you made use of certain assumptions. Now try to prove these assumptions too.

3. Once again, in your proofs in no. 2, you made use of assumptions. Now make an attempt to prove these assumptions as well and to carry on in this way until you have proved all your assumptions.

In attempting to answer questions 1, 2 and 3, pupils inevitably argued circularly. The following is an example:

1.



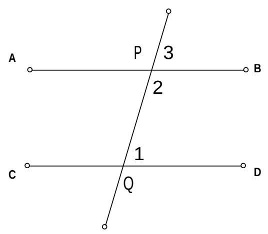
$\angle Q_1 = \angle P_2$ (corresponding angles, $AB//CD$)

$\angle P_1 = \angle P_2$ (directly opposite angles)

$\therefore \angle Q_1 = \angle P_1$

Alternate angles are therefore equal.

2.



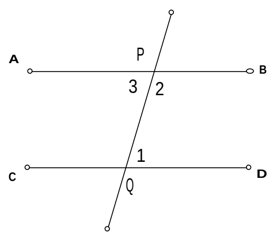
$\angle Q_1 + \angle P_2 = 180°$ (co-interior angles, $AB//CD$)

$\angle P_3 + \angle P_2 = 180°$ (QP extended forms straight line)

$\therefore \angle Q_1 = \angle P_3$

Corresponding angles are therefore equal.

3.



$\angle P_3 + \angle P_2 = 180°$ (APB is straight line)

$\angle P_3 = \angle Q_1$ (alternate angles, $AB//CD$)

$\therefore \angle Q_1 + \angle P_2 = 180°$

The sum of the co-interior angles on the same side of the transversal are therefore 180°.

corresponding angles equal

↓

alternate angles equal

↓

sum of co-interior angles is 180

Figure 13: A circular argument

These series of proofs can be schematically represented as shown in Figure 13 and clearly illustrate the underlying *circular* argument. The problem is that no matter how much they try, they inevitably land up with some kind of circularity. Although many pupils did not at first recognize the problem, some subsequent exercises alerted them to the underlying problem and the realization that it is impossible to prove all mathematical statements or properties of mathematical objects without obtaining a circular argument. They then realized that one had to accept one of these properties as a statement *without proof* (ie. as a definition or axiom) to avoid a circularity.

Comparative research at the conclusion of the USEME experiment indicated that not only had the experimental groups gained substantially in their ability to define known and unknown geometric objects (economically correct), but that they had developed a deeper understanding of the nature of axioms, as well as an ability to recognize circular and other invalid arguments (see Human, Nel et al, 1989a).

Dynamic Geometry Software

The development of dynamic geometry software in recent years is certainly the most exciting development in geometry since Euclid. Besides rekindling interest in some basic research in geometry, it has revitalized the teaching of geometry in many countries where Euclidean geometry was in danger of being thrown into the trashcan of history. For example, someone recently made the claim at the International Congress on Mathematical Education (ICME) in Spain (July 1996) that dynamic geometry had saved the geometry curriculum in the United States.

As we have seen earlier, one of the main reasons for the poor performance of pupils in geometry can be found in terms of the Van Hiele theory. For example, many pupils have undeveloped visualisation skills which are an important prerequisite for success in geometry. Furthermore, pupils are introduced too early to formal geometry without allowing sufficient experimental exploration of the properties of figures and the gradual introduction of appropriate formal terminology.

In the past, many teachers have simply avoided the informal exploration of geometric relationships by construction and measurement with paper-and-pencil, since they are so time-consuming (and relatively inaccurate). (Of course, there are also those teachers who from an extreme formalist philosohical position, disregard any form of experimental work in mathematics). Another problem is that such constructed figures are "*static*", one either has to redraw the figure or be able to visualise how it might chance shape.

This however has now all changed with the development of sophisticated software packages for geometry. One of the first such "*state of the art*" packages to be produced was Cabri-Geometre, a French program that was first introduced to the international Mathematics Education community at a conference in Budapest in 1988. Since then other similar packages have been developed, for example, Geometer's Sketchpad by an American company and with assistance from the National Science Foundation and the Visual Geometry Project at Swarthmore College, USA.

These geometric software packages were designed with the specific intention of putting at the disposal of the pupil or student a micro-world type environment for the experimental exploration of elementary plane geometry. In the past one either had to draw the geometric configurations on a sheet of paper, thereby obtaining a more or less exact, but fixed representation, thus severely limiting exploration. In these software packages the geometric figures can be constructed through actions and in a language which are very close to those in use in the familiar "*paper-and-pencil*" universe. In contrast to paper-and-pencil construction, dynamic geometry is accurate and is it extremely quick and easy to carry out complex constructions, and to vary them afterwards.

Once created, these figures can be redrawn by "*grasping*" their basic elements directly on the screen and moving them, while keeping the properties which had been explicitly given to them. In this way one can "*continuously*" change a triangle, and for instance notice that its altitudes always stay concurrent during the transformation. The software therefore allows one to easily repeat experiments in many different orientations and thereby checking which geometric properties stay invariant. In fact, Cabri has a property checking facility (only Macintosh version) that can check whether certain properties (eg. parallelism, concurrency, collinearity, orthogonality, etc) are true in general, and if they're not, it can construct counter-examples.

Probably the most welcome facility of dynamic geometry is its potential to encourage (re-introduce) experimentation and the kind of pupil oriented "*research*" in geometry described by Luthuli (1996) and others. In such a research-type approach, students are inducted early into the art of problem posing and allowed sufficient opportunity for exploration, conjecturing, refuting, reformulating and explaining as outlined in Figure 14 (compare Chazan, 1990). Dynamic geometry software strongly encourages this kind of thinking as they are not only powerful means of verifying true conjectures, but also extremely valuable in constructing counter-examples for false conjectures.

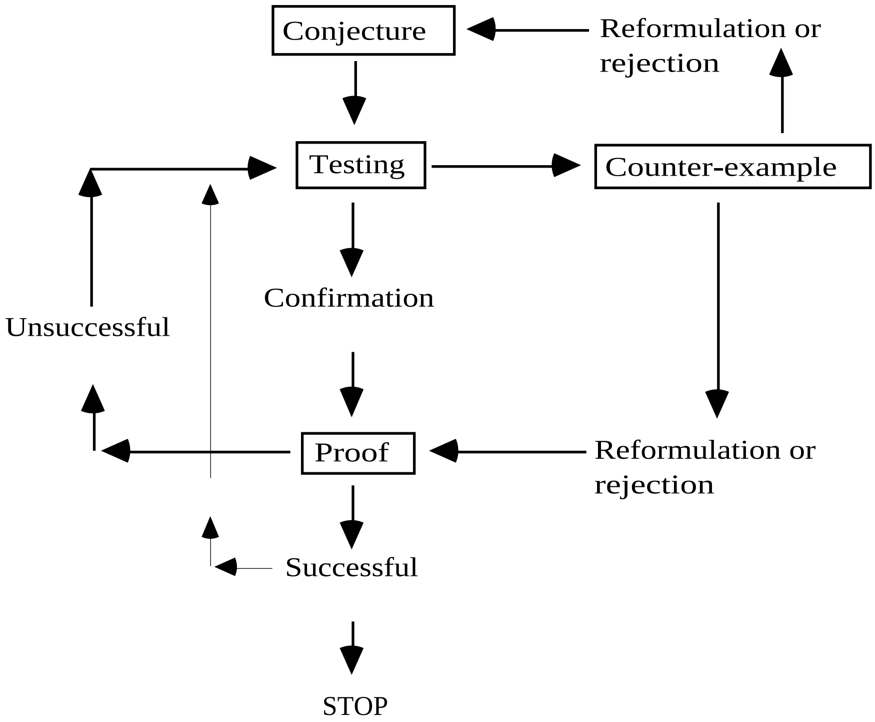


Figure 14: Pupil research in geometry

However, the development of dynamic geometry has also necessitated a radical change to the teaching of proof. Traditionally, the typical approach to geometry has always been to try and create doubts in the minds of pupils about the validity of their empirical observations, and thereby attempting to motivate a need for deductive proof. From experience, these strategies of attempting to raise doubts in order to create a need for proof are simply not successful when geometric conjectures have been thoroughly investigated through their continuous variation with dynamic software like *Cabri* or*Sketchpad*. When pupils are able to produce numerous corresponding configurations easily and rapidly, they then simply have no (or very little) need for further conviction/verification.

Although pupils may exhibit no further need for conviction in such situations, the author has found it relatively easy to solicit further curiosity by asking them **why** they think a particular result is true; i.e. to challenge them to try and *explain* it (also see De Villiers, 1990; 1991; Schumann & De Villiers, 1993). Pupils quickly admit that inductive verification merely confirms; it gives no satisfactory sense of *illumination*; i.e. an insight or understanding into how it is a consequence of other familiar results. Pupils therefore find it quite satisfactory to then view a deductive argument as an attempt at explanation, rather than verification.

Particularly effective appears to be to present pupils early on with results where the provision of explanations (proofs) enable surprising further generalizations (using proof as a means of discovery). Rather than one-sidedly focussing only on proof as a means of verification in geometry, it therefore appears that other functions of proof such as explanation and discovery should be effectively utilized to introduce proof as a meaningful activity to pupils.

The following is an example of a possible worksheet in this regard from De Villiers (1995a):

WORKSHEET

(a) Construct a *dynamic* kite using the properties of kites explored and discussed in our previous lessons.

(b) Check to ensure that you have a dynamic kite, i.e. does it always remain a kite no matter how you transform the figure? Compare your construction(s) with those of your neighbours - is it the same or different?

(c) Next construct the midpoints of the sides and connect the midpoints of adjacent sides to form an inscribed quadrilateral.

(d) What do you notice about the inscribed quadrilateral formed in this way? (Make some measurements to check your observation).

(e) State your conjecture.

(f) Grab any vertex of your kite and drag it to a new position. Does it confirm your conjecture? If not, can you modify your conjecture?

(g) Repeat the previous step a number of times.

(h) Is your conjecture also true when your kite is *concave*?

(i) Use the property checker of *Cabri* to check whether your conjecture is true in general.

(j) State your final conclusion. Compare with your neighbours - is it the same or different?

(k) Can you explain **why** it is true? (Try to explain it in terms of other well-known geometric results. *Hint*: construct the diagonals of your kite. What do you notice?)

(l) Compare your explanation(s) with those of your neighbours. Do you agree or disagree with their explanations? Why? Which explanations are the most satisfactory? Why?

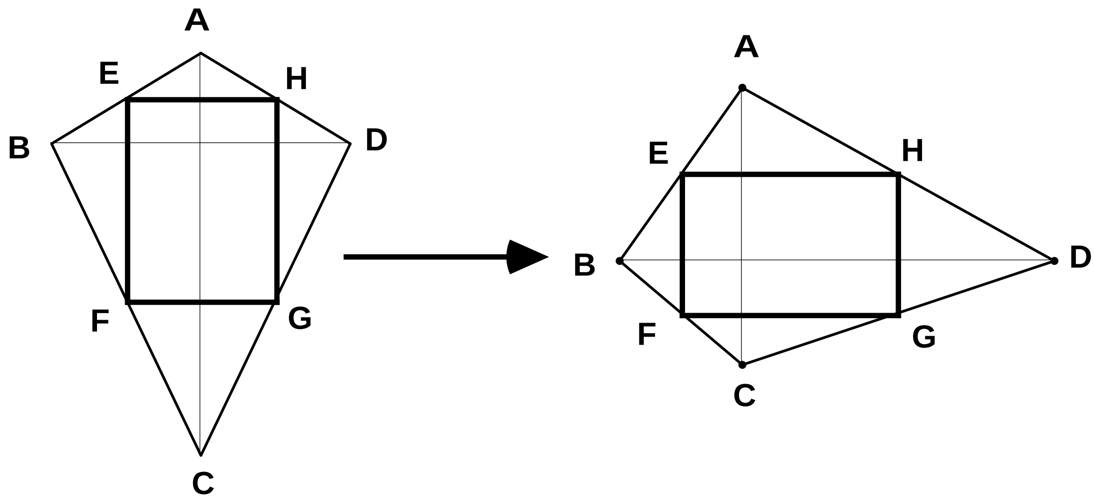


Figure 15: Explanation & discovery

Formulation

The line segments consecutively connecting the midpoints of the adjacent sides of a kite form a rectangle.

Deductive explanation

A deductive analysis shows that the inscribed quadrilateral is always a rectangle, because of the *perpendicularity* of the diagonals of a kite. For example, according to an earlier discussed property of triangles, we have EF//AC in triangle ABC and HG//AC in triangle ADC (see Figure 15a). Therefore EF//HG. Similarly, EH//BD//FG and therefore EFGH is a parallelogram. Since BD $\perp$ AC (property of kite) we also have for instance EF $\perp$ EH, but this implies that EFGH is a rectangle (a parallelogram with a right angle is a rectangle).

Looking back

Notice that the property of equal adjacent sides (or an axis of symmetry through one pair of opposite angles) was not used at all. In other words, we can immediately **generalize** the result to a *perpendicular quad* as shown in Figure 15b. (Note that it is also true for concave and crossed cases). This shows the value of understanding **why** something is true. Furthermore, note that the general result was not suggested by the purely empirical verification of the original conjecture. Even a systematic empirical investigation of various types of quadrilaterals would probably not have helped to discover the general case, since most people would probably have restricted their investigation to the more familiar quadrilaterals such as parallelograms, rectangles, rhombi, squares and rectangles. (Note that from the above explanation we can also immediately see that EFGH will always be a parallelogram in *any* quadrilateral. Check on *Cabri* or *Sketchpad* if you like!).

The teacher's language is particularly crucial in this introductory phase to proof. Instead of saying the usual: "*We cannot be sure that this result is true for all* *possible variations, and we therefore have to (deductively) prove it to make absolutely sure*", pupils (and students) find it much more meaningful if the teacher says: "*We now know this result to be true from our extensive experimental investigation. Let us however now see if we can EXPLAIN WHY it is true in terms of other well-known geometric results . In other words, how it is a logical consequence of these other results.*"

It is usually necessary to discuss in some detail what is meant by an "*explanation*". For example, the regular observation that the sun rises every morning clearly does not constitute an explanation; it only reconfirms the validity of the observation. To explain something, one therefore has to explain it in terms of something else, e.g. the rotation of the earth around the polar axis. Similarly, the regular observation that say the sum of the angles of a triangle is 180° does not constitute any explanation; in order to explain it, we need to show how (why) it is a logical consequence of some other results that we know.

Of course, proof has many other functions, e.g. verification, systematization, communication, discovery, intellectual challenge, etc. which also have to be communicated to pupils to make proof a meaningful activity for them. In fact, it seems meaningful to use a **spiral** approach as in De Villiers (1999) to introduce the various functions of proof more or less as given in Figure 16. It is important not to delay the first introduction to proof as a means of explanation unduly, as pupils might become accustomed to seeing geometry as just an accumulation of empirically discovered facts, and in which explanation plays no role. For example, even pupils at Van Hiele Level 1 could easily use symmetry to explain why certain results are true (e.g. why base angles of isosceles triangle are equal). Although the other functions can be introduced gradually as pupils progress through the levels from Level 1 to 3, the function of systematization should however be delayed until pupils have reached at least Van Hiele Level 3 or 4. (Examples of activities aimed at some different functions are given in De Villiers (1995b)). The function of communication is of course present all the time as the teacher needs to continuously negotiate with pupils the criteria for what constitutes an explanation, proof, etc.

Explanation → Discovery → Intellectual challenge → Verification → Systematization

Figure 16: Teaching functions of proof

The dynamic nature of geometric figures constructed in *Sketchpad* or *Cabri* may also make the acceptance of a hierachical classification of the quadrilaterals far less problematic than it is at the moment. For example, if pupils construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 17. In fact, it seems quite possible that pupils would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed.

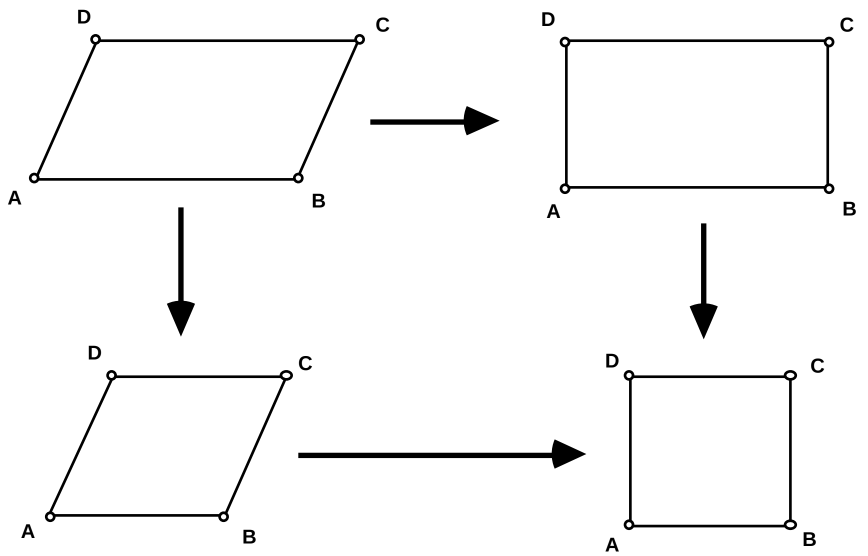


Figure 17: Dynamic transformation of parallelogram

The ability to quickly and efficiently transform geometric configurations with dynamic geometry software also allows one to effectively **model** real world situations and problems by dynamic scale drawings. It therefore becomes possible to give much more complicated real world problems to pupils to solve than is currently the case. Some examples are given in De Villiers (1994b). These software programs also have facilities for tracing the loci of certain objects, eg. points. This facility could easily be used, not only in many real world contexts, but also makes it feasible to introduce and study the conics as loci (in the classical Greek way - see Scher, 1995) instead of treating it purely algebraically as in the present syllabus.

As described in the previous section, construction and measurement is extremely important from a learning psychological point of view (ie. in developing an understanding of the "*if-then*" nature of propositions and the inter-relationship between properties, and therefore in the transition from Van Hiele Level 2 to 3). Traditionally, many teachers have simply avoided paper-and-pencil constructions because it is tedious and rather inaccurate. The availibility of dynamic geometry however changes all of that since it is quick, efficient and accurate. With dynamic geometry, one does not have to redraw for example several different triangles, repeating the same constructions and checking each case individually, as the dragging facility of dynamic geometry allows one to continuously transform one triangle into another, but still maintaining the initial constructions one carried out on it.

It should also be pointed out that certain kinds of construction activities (on Sketchpad or otherwise) are completely inappropriate at Van Hiele Level 1. For example, at a recent PME conference someone commented that she was unpleasantly dismayed at how difficult young children found the task of constructing a "dynamic" square with Sketchpad. However, if the children were still at Van Hiele Level 1, then it is not surprising at all - how can they construct it if they do not yet know its properties (Level 2) and that some properties are sufficient and others not (i.e. the relationships between the properties - Level 3)?

In fact, at Van Hiele Level 1 it would appear to be far more appropriate to provide children with ready-made sketches of quadrilaterals which they can easily manipulate and first investigate visually. Next they could start using the measure features of the software to analyse the properties to enable them to reach Level 2. Only then would it be appropriate to ask them how they would construct such dynamic figures themselves, thus assisting the transition to Level 3.

Concluding comments

So what are some of the crucial changes necessary in secondary school geometry as we approach the year 2000? Basically the changes can be summed up as changes in content, process and teacher education. In terms of content there is a need to contemporize by including possible content such as fractals, graph theory, transformations, non-Euclidean geometry, etc. at various grades and at various levels of formality. In particular, the study of transformations could form a valuable golden thread through the entire curriculum, and in the high school show the powerful integration of algebra and geometry (see De Villiers, 1993).

However, even before any changes in the high school, many changes are necessary to our primary school geometry curriculum. Apart from content such as tessellations, vision- and 3D-geometry as described by Van Niekerk (1995, 1996) and Witterholt & Heinneman (1995) is absolutely essential for developing visualisation and spatial orientation skills, not only for formal geometry later on, but also for further study in woodwork, metalwork, architecture, art, computer graphics, engineering design, etc. More use could also be made of accurate scale drawings to solve complicated real world problems, and to develop an intuitive understanding of the process of modelling. These changes also have to be contextualised meaningfully in different contexts geographically, culturally, linguistically, etc.

However, perhaps even more important than changes in geometrical content, we need to focus far more on teaching and developing the process aspects of mathematics. It needs to be acknowledged that geometry content should not be presented in a ready-made form to pupils, but that they should actively (re)construct it in the class. In order to realize such a radical change in objectives, it is also necessary to change our evaluation procedures. Joubert (1980) and De Vries (1980) have for example developed several examples of how one could evaluate pupils' abilities to conjecture, define, axiomatize, classify, read critically, refute, etc. (For example see Joubert, 1988 & 1989).

Lastly, it is important to point out that none of the above would be realizable unless radical changes are made to teacher education programs around the country; both in pre-service and in-service. In particular, most high school teachers, even those with good qualifications, know hardly any more geometry than the pupils they have to teach. The reason is simple: most tertiary institutions (with the exception of UPE) do not teach any further geometry in their undergraduate courses. It is therefore important to seriously consider the (re)introduction of geometry in tertiary courses for secondary teachers, not only Euclidean, but different kinds of geometry (compare with Baart, 1992). However, the geometry education of primary school teachers also needs urgent attention. Burger (1992) for example has proposed an interesting geometry curriculum for primary mathematics teachers based on the Van Hiele model that could provide the basis for the development of a new college geometry curriculum.

References

Baart, L. (1992). Measuring the earth. **Pythagoras**, 28, April, 39-42.

Benade, G.(1995). Teelings van die vlak. **Pythagoras**, 37, Aug, 33-38.

Burger, W.F. (1992). A geometry curriculum for prospective elementary teachers based on the Van Hiele model of development. **Pythagoras**, 29, Aug, 9-17.

Chazan, D. (1990). Quasi-empirical views of mathematics and mathematics teaching. In Hanna, G. & Winchester, I. (Eds). **Creativity, thought and mathematical proof**. Toronto: OISE.

Davies, P.J. (1995). The rise, fall, and possible transfiguration of triangle geometry. **American Mathematical Monthly**, 102(3), 204-214.

De Villiers, M.D. (1990). The role and function of proof in mathematics. **Pythagoras**, 24, 17-24.

De Villiers, M.D. (1991). Pupils' needs for conviction and explanation within the context of geometry. **Pythagoras**, 26, 18-27.

De Villiers, M.D. & Njisane, R.M. (1987). The development of geometric thinking among black high school pupils in KwaZulu. **Proceedings of the 11th PME conference**, Montreal, Vol 3, 117-123.

De Villiers, M.D. (1993). Transformations: A golden thread in school mathematics. **Spectrum**, 31(4), Oct, 11-18.

De Villiers, M.D. (1994a). The role and function of a hierarchical classification of the quadrilaterals. **For the Learning of Mathematics**, 14(1), 11-18.

De Villiers, M.D. (1994b). The role of technology in mathematical modelling. **Pythagoras**, 35, Dec, 34-42.

De Villiers, M.D. (1995a). An alternative introduction to proof in dynamic geometry. **MicroMath**, Spring, 14-19.

De Villiers, M.D. (1995b). Why proof in dynamic geometry? **Proceedings of Justifying and Proving in School Mathematics Conference**, University of London, 6-8 Dec 1995, 155-173.

De Villiers, M.D. (1996). **Some Adventures in Euclidean geometry**. Durban: University of Durban-Westville.

De Villiers, M.D. (1997). The Role of Proof in Investigative, Computer-based Geometry: Some Personal Reflections. J. King & D. Schattschneider (eds). (1997). **Geometry Turned On**, Mathematical Association of America, 15-24.

De Villiers, M.D. (1999). **Rethinking Proof with Sketchpad**. Key Curriculum Press, USA.

De Vries, J.A. (1980). 'n Teoretiese en empiriese studie oor die onderrig van vierhoeke in standerd sewe. Ongepubliseerde M.Ed. verhandeling, Universiteit Stellenbosch.

Freudenthal, H. (1973). **Mathematics as an educational task**. Dordrecht: Reidel.

Fuys, D., Geddes, D. & Tischler, R. (1988). **The Van Hiele Model of Thinking in Geometry among Adolescents**. Monograph No. 3, NCTM.

Govender, M. (1995). Pupils' proof-writing achievement in circle geometry. Unpublished B.Ed. dissertation, University of Durban-Westville.

Griffiths, H.B. & Howson, A.G. (1974). **Mathematics: Society and Curricula**. Cambridge University Press.

Grunbaum, B. & Shepherd, G.C. (1986). **Tilings and patterns**. New York: Freeman & Co.

Human, P.G. & Nel, J.H. In medewerking met MD de Villiers, TP Dreyer & SFG Wessels. (1989a). **Alternatiewe aanbiedingstrategiee vir meetkunde-onderwys: n Teoretiese en empiriese studie.** ENWOUS verslag no. 2, Universiteit Stellenbosch. (Only in Afrikaans).

Human, P.G. & Nel, J.H. In co-operation with MD de Villiers, TP Dreyer & SFG Wessels. (1989b). **Appendix A: Curriculum Material**. RUMEUS Curriculum Material Series No. 11, University of Stellenbosch. (Ook in Afrikaans).

Joubert, G. J. (1980). Enkele vernuwingsmoontlikhhede in die evaluering van meetkunde in die Senior Sekondere Skoolfase. Ongepubliseerde M.Ed. verhandeling, Universiteit Stellenbosch.

Joubert, G.J. (1988). Evaluering van meetkunde. **Pythagoras**, 18, Nov, 7-13.

Joubert, G.J. (1989). Evaluering van meetkunde, No. 2. **Pythagoras**, 19, April, 13-15.

Kelly, P.J. (1966). Von Aubel's quadrilateral theorem. **Mathematics Magazine**, 49, 35-37.

Luthuli, D. (1996). Questions, reflection and problem posing as sources of inquiry in Euclidean geometry. **Pythagoras**, 40, Aug, 17-27.

Malkevitch, J. (Ed). (1990). **Geometry's future**. Arlington: COMAP.

Nohda, N. (1992). Geometry teaching in Japanese school mathematics. **Pythagoras**, 28, April, 18-25.

Oldknow, A. (1995). Computer adied research into triangle geometry. **The Mathematical Gazette**, 79(485), 263-274.

Oldknow, A. (1996). The Euler-Gergonne-Soddy triangle of a triangle. **The American Mathematical Monthly**, 103(4), 319-329.

Schattschneider, D. (1981). In praise of amateurs. In D. Klarner. **The mathematical Gardner**. Boston: Prindle, Weber & Schmidt.

Scher, D. (1995). **Exploring Conic Sections with the Geometer's Sketchpad**. Key Curriculum Press.

Schumann, H. & De Villiers, M.D. (1993). Continuous variation of geometric figures: interactive theorem finding and problems in proving. **Pythagoras**, 31, 9-20.

Smith, R. R. (1940). Three major difficulties in the learning of demonstrative geometry. **The Mathematics Teacher**, 33, 99-134, 150-178.

The Mathematical Association of South Africa. (1978). **South African Mathematics Project: Syllabus Proposals**. Pretoria: MASA. (Now Centrahil: AMESA).

Van Niekerk, R. (1995). From spatial orientation to spatial insight: A geometry curriculum for the primary school. **Pythagoras**, 36, 7-12.

Van Niekerk, R. (1996). "4-Kubers" in Africa. **Pythagoras**, 40, Aug, 28-33.

Witterholt, M. & Heinneman, H. (1995). Ruimtelike insig onder die loep. **Pythagoras**, 37, Aug, 20-24.

Yaglom, I.M. (1962). **Geometric Transformations I**. Washington: Mathematical Association of America.