Reconceptualising the Teaching and Learning of Common Fractions

Introduction

The MALATI fractions materials were designed according to the following basic principles:

- Learners are introduced to fractions using sharing situations in which the number of objects to be shared exceeds the number of 'friends' and leaves a remainder which can also be shared further (e.g. Empson, 1995; Murray, Human & Olivier, 1996).
- The introduction of fraction names and written symbols is delayed until learners have a stable conception of fractions. Written, higher order symbolization is not the result of natural learning, and learners struggle to construct meaning for such representations of fractions in the absence of instruction which builds on their own informal knowledge (Mack, 1995).
- Learners are encouraged to create their own representations of fractions; pre-partitioned manipulatives and geometric shapes do not facilitate the development of the necessary reasoning skills and may lead to limiting constructions (Kamii and Clark, 1995).
- Learners are exposed to a wide variety of different fractions at an early stage (not only halves and quarters) and to a variety of meanings of fractions, not only the fraction as part-of-a-whole where the whole is single discrete object, but also for example the fraction as part of a collection of objects, the fraction as a ratio, and the fraction as an operator.
- Learners can and should make sense of operations with fractions in a problem context before being expected to make sense of them out of context (Piel and Green, 1994).
- The materials repeatedly pose problems with similar structures to provide learners repeated opportunities to make sense of particular structures. Fractions are taught continuously throughout the year, once or twice a week rather than in a concentrated 'block' of time.
- A supporting classroom culture is required in which learning takes place via problem solving, discussion and challenge and in which errors and misconceptions are identified and resolved through interaction and reflection. Teachers do not demonstrate solution strategies, but expect learners to construct and share their own strategies and thus to gradually develop more powerful strategies.

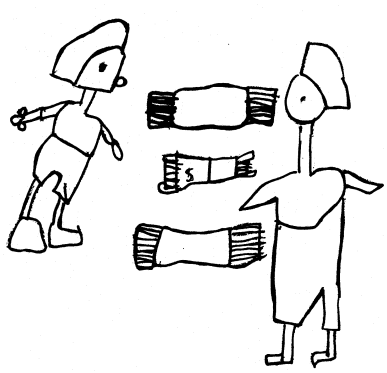
In this document, we describe how we arrived at these basic principles using two pre-test research studies that we conducted in 1997 before we began designing the materials. The first study conducted with Grade 1 learners showed that very young learners can make sense of fractions problems when they are presented in contexts that are accessible to them. The second study, conducted with Grade 4 and 6 learners, showed that by the time they reach the Intermediate Phase, learners have acquired serious limiting constructions when it comes to common fractions. Our material is designed to build on young children's ability to *make sense* of meaningful situations and to prevent these limiting constructions.

This document is illustrated with examples of learners' work from the pre-test studies, and also examples from our materials showing how we attempted to build on their informal knowledge and prevent and/or challenge limiting constructions. We also refer to research in other countries where appropriate.

Introducing common fractions

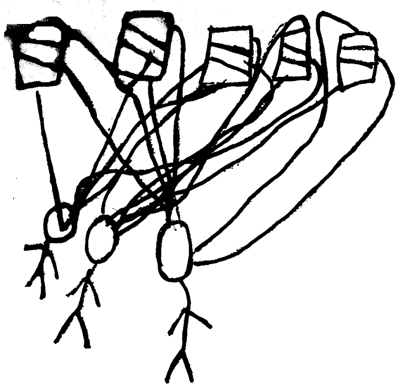
Research has found that effective problems for the introduction of the fractions concept are *sharing problems* in which there is a remainder which can be divided. These sharing situations elicit the informal knowledge that the children bring to the learning situation and can be used successfully for introducing fractions (Mack, 1990; Empson, 1995; Murray, Olivier & Human, 1996). Our own pre-test study with Grade 1 learners (De Beer & Newstead, 1998) shows that very young children have the ability to make sense of such fraction problems, even if their lack of social knowledge prevents them from producing the correct fraction *name* or *symbol*. For example, some learners called $1\frac{1}{2}$ 'two pieces'. Here are two examples from our pre-test study:

Problem: Share 3 chocolate bars equally between 2 friends



Learner: "I will give one chocolate to each of them and break the one left into two pieces and give each a piece"
Interviewer: "How much chocolate does each friend get?"
Learner: "A big one and a small one"

Problem: Share 5 chocolate bars equally among 3 friends



(Starts sharing out pieces, but gets confused counting all the lines)
Learner: "Each one gets 5"
Interviewer: "What is each piece called?"
Learner: "A Bar One"

We distinguish between the representation (as shown clearly by the children's examples above) and the name (word) and the number symbol. While the physical representation is logico-mathematical knowledge, the name (e.g. half) and the number symbol ($\frac{1}{2}$) are social knowledge, to be introduced by the teacher. *The representation (concept) is therefore not dependent on the notation*. According to Mack (1995), the fraction *notation* causes problems and should be *delayed* until the concept is stable. The *language* is, however, *needed* as soon as children have solved problems involving, e.g. a half and a third, in order to distinguish between different 'pieces'. The following extracts from activities in the MALATI fractions materials demonstrate the order in which we introduce these various aspects of fractions.

From Worksheet 1 of Phase 1, demonstrating our introduction to fractions based on our knowledge that learners have the ability to make sense of equal sharing problems with remainders:

LISA SHARES CHOCOLATE

1. Lisa and Mary have 7 bars of chocolate that they want to share equally between the two of them so that nothing is left. Help them to do it.

From Worksheet 3 of Phase 1, introducing the fraction words:

GIVING NAMES

When we divide something into 2 equal parts, we call these parts halves

When we divide something into 3 equal parts, we call these parts thirds

When we divide something into 4 equal parts, we call these parts fourths or quarters

When we divide something into 5 equal parts, we call these fifths

1. What would you rather have, a third of a chocolate bar or a fifth of a chocolate bar? Why?

From Worksheet 7 of Phase 1, introducing the fraction symbols:

FOOD FOR THE NETBALL TEAM

A short way to write a half is $\frac{1}{2}$

A short way to write a seventh is $\frac{1}{7}$

A short way to write a twentieth is $\frac{1}{20}$

2. Two netball teams play a game. There are 14 children all together. The sports teacher wants to give each child $\frac{1}{2}$ an orange. How many oranges does she need?

This introduction to common fractions differs from many of the traditional ways and sequences of teaching fractions. In our pre-test study (Newstead & Murray, 1998), we found evidence that the traditional teaching of fractions resulted in misconceptions. For example, we found that many learners associated fractions with meaningless 'recipes' (often incorrect), or simply did not have any meaningful association for fractions. For example, here are three separate responses to the same problem:

Problem: What does $\frac{4}{5}$ mean?

$\frac{4}{5}$ Noemer
Lynstuk
Teller

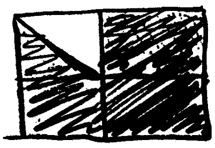
it mean 4 is a Numarotor
and 5 is a Decimal Fractions

5 5 5 5

$\frac{4}{5}$ mean a number that has been shaded

In the past, many teachers have introduced learners to common fractions using pictures of pre-partitioned shapes, or actual manipulatives. The Grade 4 and 6 pre-test study (Newstead & Murray, 1998) pointed to the dangers of such an approach. Not only can it lead to a very limited interpretation of fractions, but the idea of equal partitioning is often lost. Both of these consequences are illustrated in these examples of learners' responses:

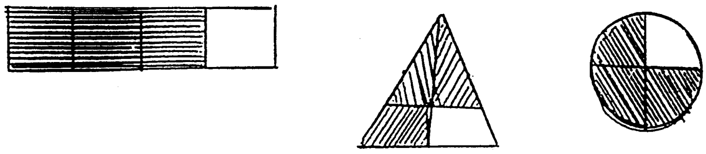
Problem: What does $\frac{4}{5}$ mean?



Problem: Describe or show $\frac{3}{4}$ in 3 different ways.

Square
Rectangle
Triangle

Problem: Describe or show $\frac{3}{4}$ in 3 different ways.

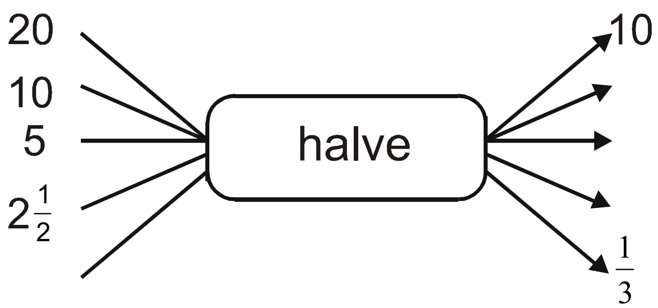


Kamii and Clark (1995) agree that children should be encouraged to generate their *own* diagrams to represent a fraction (for example, see Teacher Notes from Worksheet 1, Phase 1). These may closely resemble the diagrams that can be found in textbooks, but represent children's own understandings rather than someone else's thinking. Prepartitioned *apparatus/material* should be *delayed* as such material can lead to the misconceptions that we have illustrated.

As mentioned above, in order to ensure a meaningful concept of fractions, we suggest introducing fractions through sharing problems with a remainder, and we also include in our materials number concept development activities for fractions, for example snakes for counting forwards and backwards in fractions. Similar oral activities can also be given to the learners. Here is an example of a number concept activity from Worksheet 19 of Phase 1:

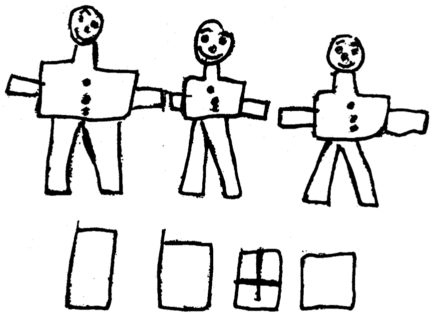
SPIDERS

Complete these flow diagrams:



In our Grade 1 pre-test study (De Beer & Newstead, 1998), we found that many learners were able to use only halves and quarters, even though in some cases they were aware that the remaining chocolate bar had not been shared equally. This may result from everyday experience in which any piece smaller than a whole is referred to as a 'half' or a 'quarter'. If only halves and quarters are used, children can come to think that any 'piece' is a half or a quarter (Murray, Olivier & Human, 1996; De Beer & Newstead, 1998). For example:

Problem: Share 4 chocolate bars equally among 3 friends



Learner:	"Each one gets one and a half"
Interviewer:	"Show me"
Learner:	"This one gets one, this one gets one, this one gets one..."
Interviewer:	"And then?"
Learner:	"Break it in half, they must break it in two sides."

In the MALATI materials, learners are introduced to fractions other than just halves and quarters very early in their work with fractions. For example, already in Worksheet 2 of Phase 1, other fractions are introduced:

EQUAL SHARING

1. Five friends want to share 11 chocolate bars equally. How must they do it?
2. Five friends want to share 21 chocolate bars equally. How must they do it?

Because of the introduction to fractions by sharing problems with remainders, learners are exposed to improper fractions and mixed numbers from the beginning. According to Kamii and Clark (1995), this is important so that they will think about parts and wholes at the same time. Such improper fractions were traditionally considered to be more difficult that 'proper' fractions, and were therefore delayed.

Different meanings of fractions

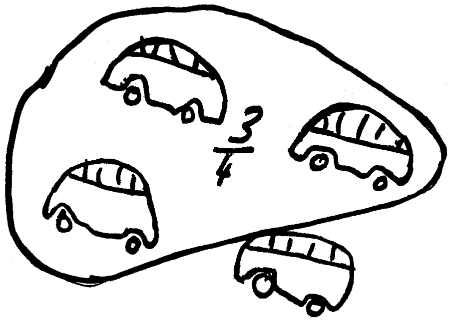
The MALATI materials also expose learners to several *different meanings* of fractions (e.g. Watson, Collis & Campbell, 1995).

One interpretation of a fraction is as part of a whole where the whole is a single object. The following extract from Worksheet 3 of Phase 1 illustrates this meaning of a fraction.

GIVING NAMES

What would you rather have, a third of a chocolate bar or a fifth of a chocolate bar? Why?

However, fractions can also be part of a whole where the whole is a collection of objects. In our pre-test study (Newstead & Murray, 1998), we were concerned that learners' conceptions of fractions appeared to be mostly limited to the above-mentioned understanding of fractions. Very few learners illustrated $\frac{4}{5}$ or $\frac{3}{4}$ as part of a collection of objects. Here is a rare example of such a representation:



The learners in our study had difficulty solving a simple problem requiring this interpretation of a fraction, namely “Mother has 10 smarties. She says you can have $\frac{3}{5}$ of the smarties. How many smarties will you get?”.

In the MALATI materials, learners are also exposed to the fraction as part of a whole where the whole is a collections of objects, as illustrated by Worksheet 27 of Phase 1:

READING

1. John's book has 88 pages. He says: ”I have read more than half of the book. I am on page 41." Is it true? Explain.

However, fractions have several other meanings. Other meanings of a fraction include the fraction as an operator, illustrated here by an extract from Worksheet 11 of Phase 2:

FRACTIONS WITH CALCULATORS

1) Jane baked 60 biscuits. She gave $\frac{2}{5}$ of the biscuits to the bazaar.
   (a) How many biscuits did she give to the bazaar?
   (b) Now press the following on your calculator: 60 ÷ 5 =. What fraction of 60 have you now worked out?
   (c) Keeping your answer on the screen of the calculator, now press × 2 = . What fraction of 60 have you now worked out? What is your answer?

A fraction can also represent a ratio or a relationship, as illustrated by the following extract from Worksheet 31 of Phase 1:

GIFTS OF BILTONG

1. Karl’s brother Shaun was given some biltong. Shaun is much younger than Karl, so for every strip of biltong Shaun was given, Karl was given three.

   How many strips of biltong did Karl get if Shaun got 7?

A fraction can also represent a unit of measurement, as illustrated by the following extract from Worksheet 11 of Phase 1:

WIRE ANIMALS AND CARS

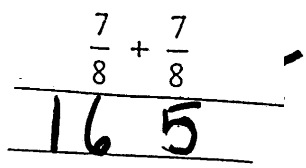
The children are making different animals and cars from wire.
A car needs $2\frac{1}{2}$ metres of wire.
An animal needs $1\frac{1}{2}$ metres of wire.

1. The children have 20 metres of wire.

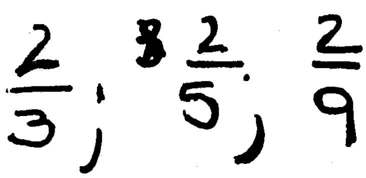
   (a) How many cars can they make from 20 metres of wire?
   (b) How many animals can they make from 20 metres of wire?

Another meaning of a fraction - the fraction as a rational number - is a difficult concept. During Phase 1 of the MALATI materials, learners are given the opportunity to reflect on and compare the sizes of fractions. The material has been designed to challenge learners' intuitive view of the fraction as *two* whole numbers which results in them adding fractions by simply adding the numerators and the denominators (Mack, 1990; D'Ambrosio & Mewborn, 1994; Mack, 1995). There was evidence of such misconceptions in the Grade 4 and 6 pre-test study (Newstead & Murray, 1998). 43% of the Grade 6 learners responded to this item as follows:

Other Grade 6 and Grade 4 learners attempted to carry out whole-number procedures with fractions.



Grade 4 and 6 learners responded to the item "Put these fractions in order from smallest to biggest: $\frac{2}{5}$, $\frac{2}{3}$, $\frac{2}{9}$" by ordering the whole numbers 2;2;2;3;5;9 or by examining only the denominators:



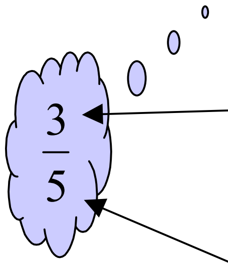
In order to prevent such misconceptions, formal introduction of the fraction as **a number** in which each part has a specific meaning is delayed until Worksheet 1 of Phase 2:

FRACTIONS

A fraction looks like two numbers but is actually one number!

For example, the fraction $\frac{3}{5}$ is a number somewhere between a half and one whole.

Imagine I could have $\frac{3}{5}$ of a chocolate bar...



The number on top of the line is called the *numerator*.

This would mean that I could have **THREE**...

...of the **FIVE** equal parts into which the chocolate bar is divided.

The number below the line (showing the number of equal parts into which the whole has been divided) is called the *denominator*.

What does $1\frac{3}{8}$ mean?

Introducing equivalent fractions and operations with fractions

The examples shown above of learners attempting to add fractions illustrate that Grade 4 and Grade 6 learners (Newstead & Murray, 1998) do not make sense of procedures for operations with fractions unless they have a stable concept of what a fraction is. Therefore, throughout the materials, different realistic problem situations are presented out of which the four basic operations arise. However, learners are not expected to carry out *formal operations with fractions* until they have a stable concept of a fraction. Other researchers (e.g. Sáenz-Ludlow, 1995) also mention the need for learners to conceptualize fractions as quantities before they are introduced to conventional symbolic algorithms.

This was confirmed in our Grade 4 and 6 pre-test study (Newstead & Murray, 1998), in which we found evidence of half-remembered procedures for generating equivalent fractions and adding fractions. We could ascribe this to the way in which these procedures have been taught in the classroom, without sufficient understanding of the meaning and purpose of equivalent fractions. For example, some Grade 6 learners knew how to find the common denominator but did not understand the need for changing the numerators:

$$\frac{2}{3} + \frac{4}{5} \qquad \frac{6}{15}$$

In other cases, misconceptions can arise from the sequence in which the content has been taught. For example, perhaps this learner was being taught addition of fractions in the following traditional sequence:

1) Adding fractions of which the denominator is the same;
2) Adding fractions of which the one denominator is a multiple of the other denominator, in which **we simply choose the bigger denominator**; and
3) Adding fractions with different denominators.

If the learner was currently being exposed to Stage 2, that might explain the following response to the item $\frac{2}{3} + \frac{4}{5}$:

My Answer is $\frac{6}{5}$ because I make addit
and Denormerator it is Not Change

Research (e.g. Kamii & Clark, 1995) suggests that the traditional ways of teaching such methods do not foster understanding of equivalent fractions. The MALATI material aims to develop a sound understanding of the *idea* of equivalent fractions, as it is basic to addition and subtraction of fractions. We believe that it is important that children realise that it is always possible to *find* an equivalent fraction (the same fraction by another name), even if they can't immediately generate it. It is also important that children understand the *purpose* of equivalent fractions, i.e. realise that only like units can be added and equivalent fractions are part of the process of moving towards like units. In the MALATI materials, formal methods for generating *equivalent fractions* are delayed as long as possible. Rather, problems such as the following (from Worksheet 14 of Phase 1) are given to develop the concept of equivalent fractions. The learners first complete a 'fractions wall', showing pieces of chocolate cut into various fraction pieces, before answering questions like the following:

CHOCOLATE PIECES OF THE SAME SIZE

Here are some pieces of a chocolate bar:

$\frac{6}{18}$	$\frac{5}{10}$	$\frac{1}{3}$	$\frac{2}{4}$	$\frac{3}{6}$
$\frac{4}{12}$	$\frac{6}{12}$	$\frac{5}{15}$	$\frac{5}{6}$	$\frac{4}{5}$
$\frac{12}{18}$	$\frac{12}{15}$	$\frac{10}{12}$	$\frac{2}{3}$	$\frac{2}{6}$

First say which of the above pieces of chocolate do you think are the same size. Explain why you say so.

Problem solving approach

In the MALATI materials, the concepts of fractions and operations with fractions are developed through *posing challenging problems to be solved collaboratively*, not through demonstration, recipes or definitions. As mentioned above, research (for example, Mack, 1990; Streefland, 1991; Newstead & Murray, 1998) has shown that mechanistic teaching of such rules and recipes leads to poor understanding of fractions. As mentioned above, throughout the materials, different realistic problem situations are presented out of which the four basic operations arise. The emphasis on problem solving and on communication is also in line with Specific Outcomes 1, 9 and 10 of MLMMS and with the Critical Outcomes of Curriculum 2005.

In our pre-test study (Newstead & Murray, 1998), there was evidence that even Grade 4 learners could make sense of challenging problems in context. Here are three different responses to the question "Would you rather have $\frac{3}{5}$ or $\frac{3}{4}$ of a pizza? Why?":

if its a hawian

$\frac{3}{4}$ the pices of pizza are bigger

no

Because it is not a hole but me I want a Hole

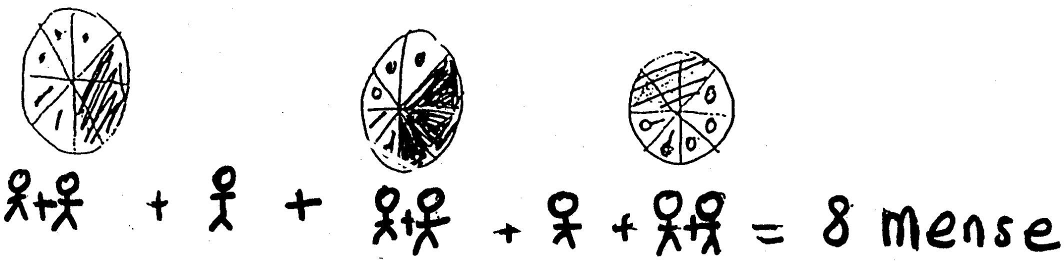
$\frac{3}{5}$, want ek eet nie baie nie,

(…because I don't eat a lot)

Problems that were traditionally perceived as *difficult* are also *not delayed in the MALATI materials*, e.g. division. We found evidence (Newstead & Murray, 1998) that learners can make sense of such problems. For example, we provide two examples from our pre-test study of learners making sense of traditionally 'difficult' problems by drawing. It is interesting to note that the first problem was solved with greater success (20%) by Grade 4 learners than its context-free equivalent ($2 \div \frac{1}{2}$), which was given to Grade 6 learners, of whom only 8% could solve it.

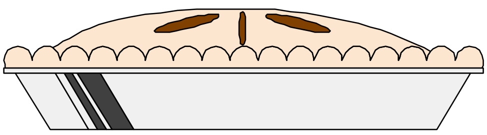
Problem: We need $\frac{1}{2}$ metre of material to make a scarf. How many scarves can we make if we have 2 metres of material?

Problem: Some friends go to a restaurant and order 3 pizzas. The waiter brings them the pizza, sliced into eighths. Each person gets $\frac{3}{8}$ of a pizza. How many people will get pizza?



We therefore include such problems in the MALATI materials, for example, Worksheet 9 of Phase 2:

MRS DAKU BAKES APPLE TARTS



Mrs. Daku bakes small apple tarts. She uses $\frac{3}{4}$ of an apple for one apple tart. She has 20 apples. How many tarts can she bake?

During our trialling of the fractions materials in primary schools, fractions were covered regularly (for example twice a week), throughout the entire school year. Learners were repeatedly exposed to similar problem types in order to facilitate the necessary concepts becoming stable. The approach as a whole did lead to an improvement in the learners' conceptions of fractions (Newstead & Olivier, 1999). We illustrate it using problems from Worksheet 1, Phase 1 and Worksheet 2, Phase 2 respectively:

FOOD FOR THE NETBALL TEAM

Two netball teams play a game. There are 14 children all together.
One of the parents brings a bag with 35 chocolate bars to share among the 14 players. How much chocolate bar does each player get?

CHOCOLATE BARS

Mrs Hermanus gives a prize to the group in her class that has behaved the best during the week. The prize is a box with 10 chocolate bars.
This week Ismail's group wins the prize. There are 4 people in Ismail's group. They all want the same amount of chocolate. How much chocolate does each child get?

Classroom culture

In our Grade 4 and 6 pre-test study (Newstead & Murray, 1998), there was evidence that learners have not been exposed to a learning environment in which their misconceptions are resolved and challenged. For example, many learners responded as follows, having been exposed to division *only* as sharing and believing that division 'makes smaller':

$$4 \div 8 = 2$$

We therefore recommend a *classroom culture* in which learners are given the opportunity to *make sense* of the problems and to *reflect* on the problems individually and by discussing their strategies with each other (Hiebert, Carpenter, Fennema, Fuson, Wearne, Murray, Olivier & Human, 1997). Through such discussion, errors are identified in a non-threatening way, and learners are encouraged to develop more sophisticated strategies.

Introducing decimal fractions

Finally, we suggest that the introduction of decimal fractions should be delayed until the learners have a relatively good concept of equivalence (at least halfway through Phase 2). Decimal fractions are an alternative notation for fractions, using only certain denominators to express all fractions. An understanding of finding equivalent fractions in tenths, hundredths and thousandths is thus a pre-requisite. Please see our "Reconceptualising the Teaching and Learning of Decimal Fractions" document for further details.

Other papers

Readers should also consult the following MALATI research papers, which form a backdrop for the design of the materials and the teaching approach:

Jooste, Z. (1999). **How grade 3 & 4 learners deal with fraction problems in context**. **Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa: Vol. 1.** (pp. 64-75). Port Elizabeth: Port Elizabeth Technikon.

Lukhele, R.B., Murray, H. & Olivier, A. (1999). **Learners' understanding of the addition of fractions**. **Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa: Vol. 1.** (pp. 87-97). Port Elizabeth: Port Elizabeth Technikon.

Murray, H., Olivier, A. & De Beer, T. (1999). **Reteaching fractions for understanding**. In O. Zaslavsky (Ed.), **Proceedings of the Twenty-third International Conference for the Psychology of Mathematics Education: Vol. 3**. (pp. 305-312). Haifa, Israel.

Newstead, K. and Murray, H. (1998). **Young students' constructions of fractions**. In A. Olivier & K. Newstead (Eds.), **Proceedings of the Twenty-second International Conference for the Psychology of Mathematics Education: Vol. 3**. (pp. 295-302). Stellenbosch, South Africa.

Newstead, K. and Olivier, A. (1999). **Addressing students' conceptions of common fractions**. In O. Zaslavsky (Ed.), **Proceedings of the Twenty-third International Conference for the Psychology of Mathematics Education: Vol. 3**. (pp.329-337). Haifa, Israel.

Van Niekerk, T., Newstead, K., Murray, H. & Olivier, A. (1999). **Successes and obstacles in the development of grade 6 learners' conceptions of fractions**. **Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa: Vol. 1**. (pp. 221-232). Port Elizabeth: Port Elizabeth Technikon.

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De Beer, T. & Newstead, K. (1998). Grade 1 learners' strategies for solving sharing problems with remainders. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, *4*, 327. Stellenbosch, South Africa.

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Kamii, C. & Clark, F.B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematical Behavior, 14*, 365-378.

Mack, N.K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*, 16-32.

Mack, N.K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422-441.

Murray, H., Olivier, A. & Human, P. (1996). Young learners' informal knowledge of fractions. In L. Puig & A. Gutiérrez (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, 4*, 43-50. Valencia, Spain.

Newstead, K. & Murray, H. (1998). Young students' constructions of fractions. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, 295-302. Stellenbosch, South Africa.

Newstead, K. & Olivier, A. (1999). Addressing students' conceptions of common fractions. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education*, *3*, 329 –336. Haifa, Israel.

Piel, J.A. & Green, M. (1994). De-mystifying division of fractions: The convergence of quantitative and referential meaning. *Focus on learning problems in mathematics*, *16*, 44-50.

Sáenz-Ludlow, A. (1995). Ann's fraction schemes. *Educational Studies in Mathematics*, *28*, 101-132.

Streefland, L. (1991). *Fractions in Realistic Mathematics Education: A Paradigm of Developmental Research*. Dordrecht: Kluwer Academic Publishers.

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