Malati

Mathematics learning and teaching initiative

Introductory Calculus

Module 1

Functions and graphs

Grade 10 and 11

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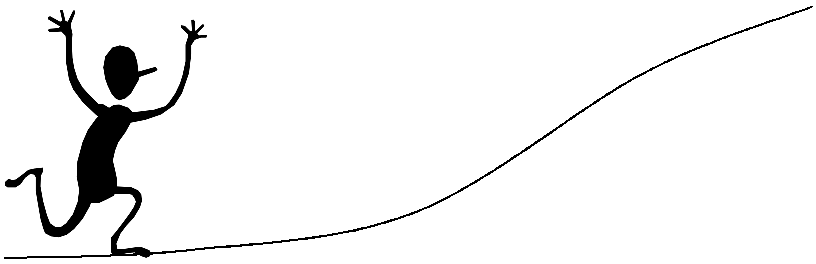
December 1999

Activities for Learners

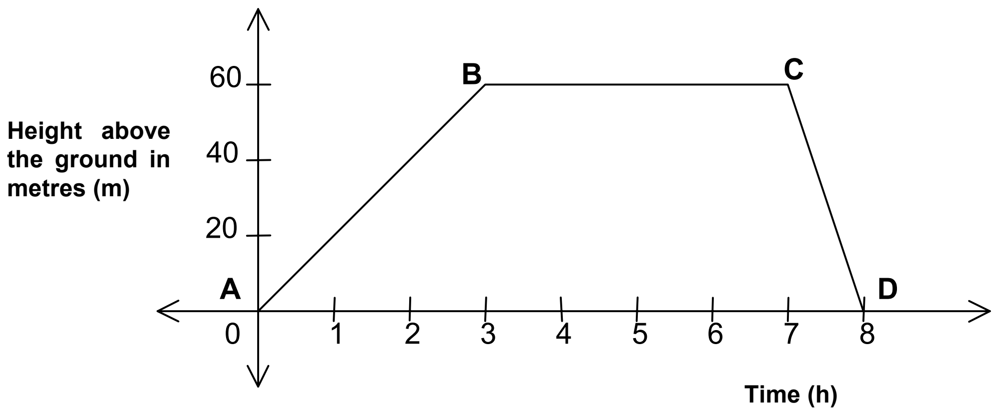
Activity 1

What do graphs tell us?

A) Tom is walking up a hill



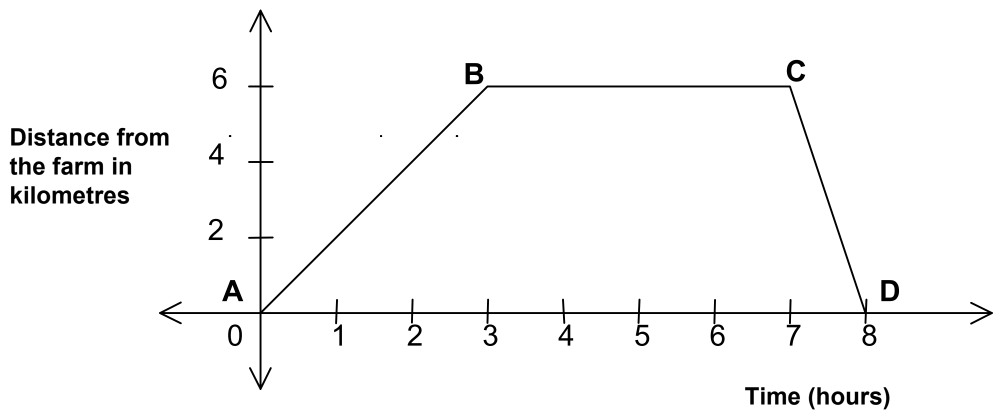
The graph below shows the height above the ground of Tom who has spent some time walking in the hills near his home.



1. What measurement does the numbers on the horisontal axis show?
2. What measurement does the numbers on the vertical axis show?
3. Account for at least two things that happened between points A and C in the graph above.
4. What do you think Tom was doing between points B and C in the graph?
5. What do you think Tom was doing between points C and D in the graph?
6. Do you think Tom returned to his starting point? Why?
7. Comment on the speed with which Tom was walking up and down the hill.

B) Going to the farm

The graph below represents the distance walked during a day by Shaik who lives on a farm.



1. What do the units on the horisontal axis represent?
2. What do the units on the vertical axis represent?
3. How long do you think Shaik spent at the place to which he walked?
4. What must have happened to Shaik between points A and C in the graph above? Account for at least two things that happened.
5. What do you think Shaik was doing in the first 3 hours after he had left the farm?
6. Looking at the graph, do you think Shaik returned to his starting point?
7. Compare Shaik's journey from A to B and from C to D.
8. Did Shaik walk up and down a hill?

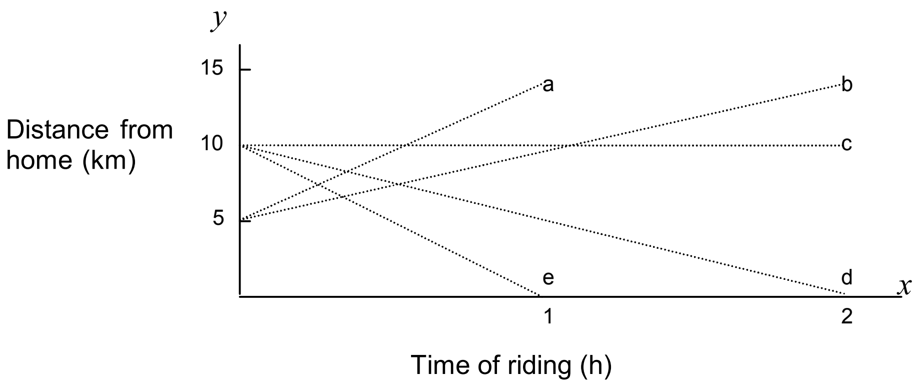
C) Comparing the two graphs

What is similar and what is different about the two graphs in A and B above?

Activity 2

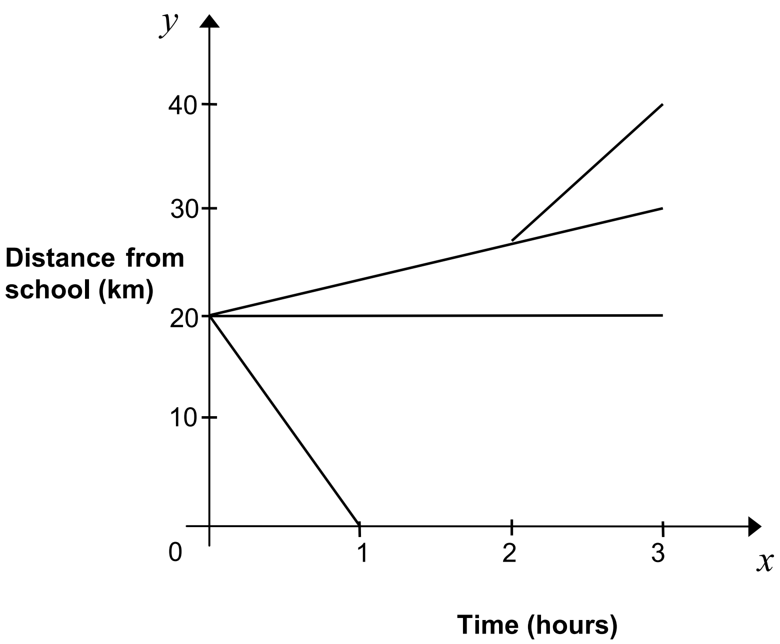
Stories

1. Consider the following stories about five different people, Kate, Karen, Kenneth, Godfrey and Marlene who have each gone for a bike ride. Match each of the stories to one of the graphs given in the figure below.



(a) Kate starts out five kilometres from home and rides five kilometres per hour away from home
(b) Karen starts out 10 km from home and rides 10 km per hour away from home.
(c) Kenneth started out 10 km from home and rides towards home, arriving after one hour.
(d) Godfrey starts out 10 km from home and is halfway home after one hour.
(e) Marlene starts out 5 km from home and is 10 km from home after one hour.

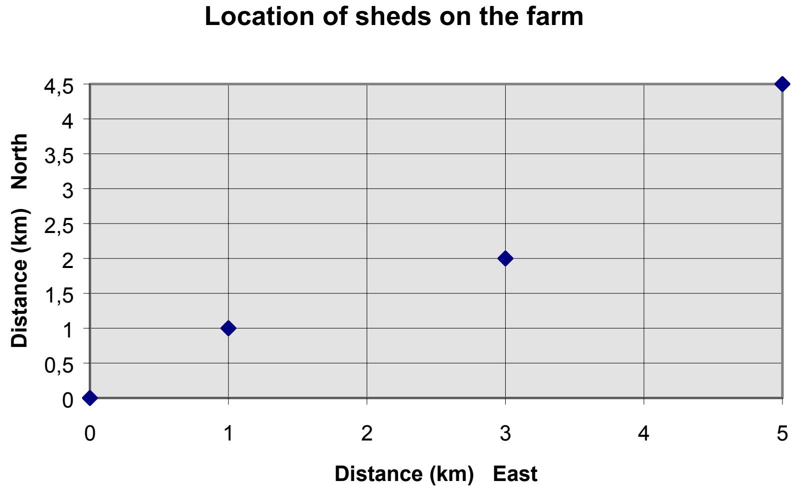
2. Four friends decided to cycle to a picnic site 20 km from school. The graph below represents their movements *after* the picnic.
   Explain what happened after the picnic.



Activity 3

Sheds on a farm

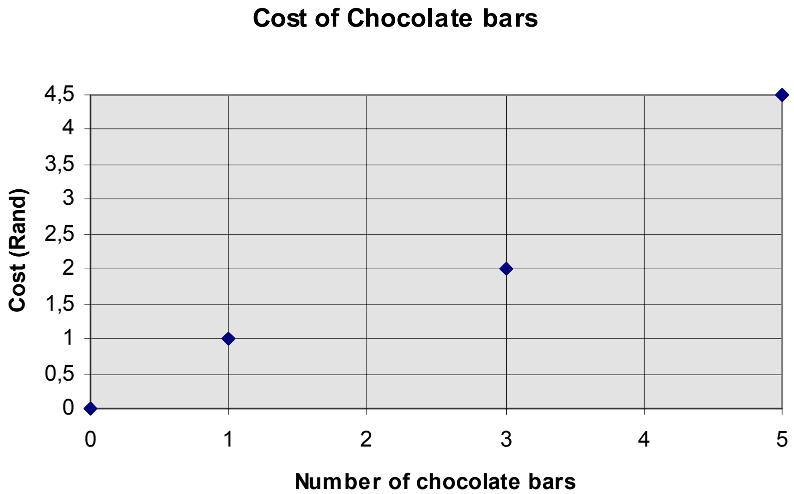
A) Below is a graph showing the location of different sheds on a farm relative to the main farmhouse that is located at the origin (0; 0).



1. Describe where the three sheds are located?

2. Calculate the distance between the farmhouse and each of the sheds?

B) Below is a graph showing the cost of various packs of chocolate bars. Each pack contains a different number of chocolate bars.



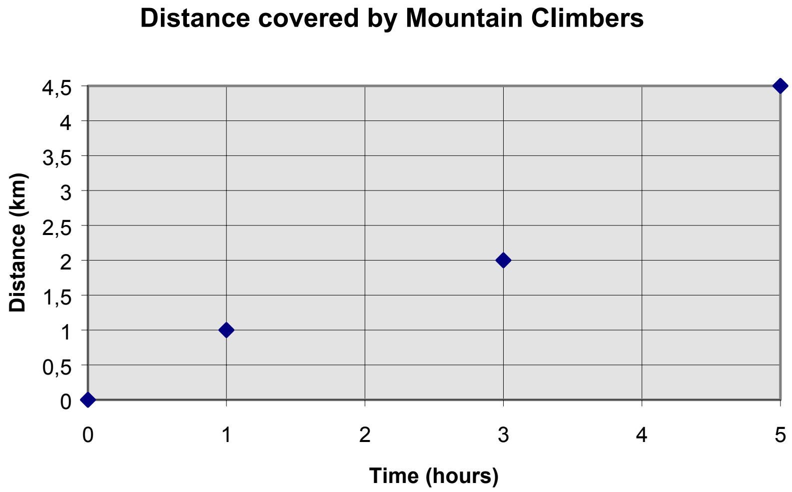
1. What is the cost if you buy a pack containing 3 chocolate bars?

2. You need to buy 15 chocolate bars for a party is it better to buy 5 packs of 3 bars or 3 packs of 5 bars?

Activity 4

Mountain climbers

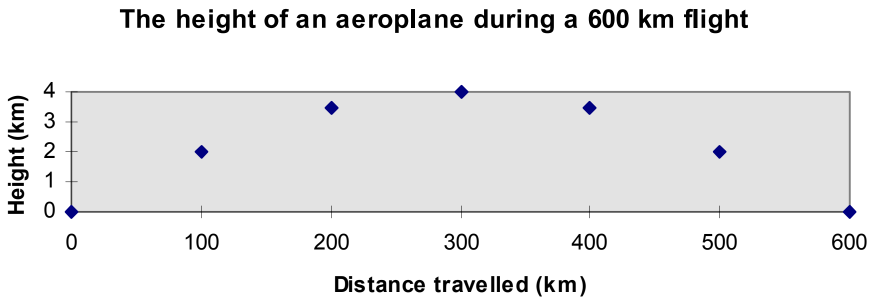
Below is a graph of showing the distance covered by a group of mountain climbers during a difficult climb. The climbers estimated the distance they had covered after the first, third and fifth hour.

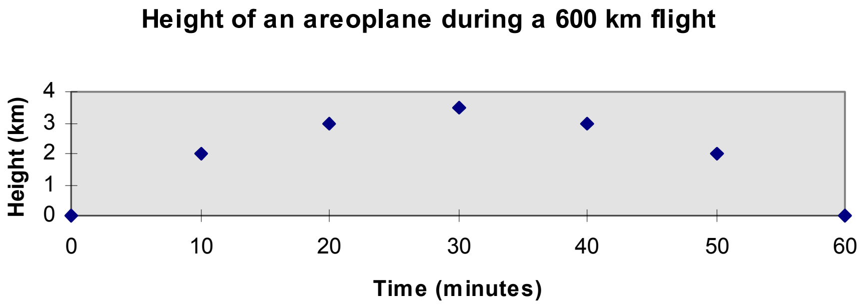


1. Which part of the climb do you think was the most difficult? Discuss.
2. What was their average speed during the last two hours of the climb?
3. What was their average speed during the whole climb?

Recap

Look carefully at all three graphs above. Explain their similarities and their differences. Below are two graphs or maps (or pictures). One is showing where the aeroplane is, and the other one is showing how two variables are related.





Activity 5

Different but the same !

In each of the following exercises you are given one representation (words, formula, table or graph) of how two variables are related and you will need to give this same relationship in another representation. In order to make the formulas easier to write call the input "x" and the output "y".

1. *Words:* The output is the sum of twice the input and five

   Table:

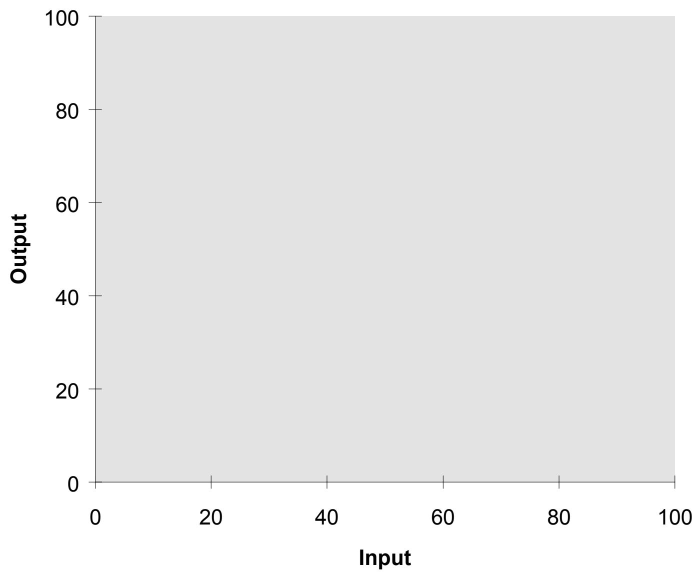
Input	0	1	3	8	10	25	100
Output							

2. *Words*: The output is the cube of the input

   Formula:

3. *Words:* The output is the result when 100 is divided by the input.

   Graph:



4. *Formula:* (Remember input is "x" and output is "y")

$$y = 2(x + 1)$$

Words:

5. *Formula:*

$$y = x^2$$

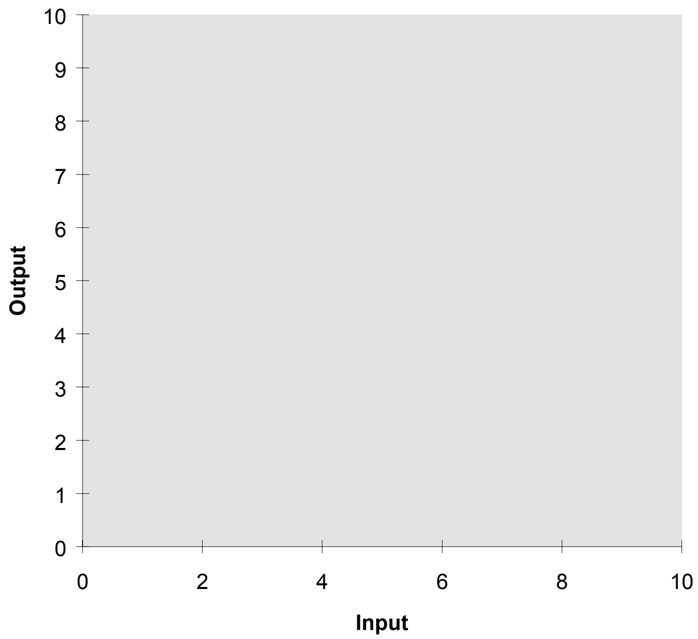
Table:

Input	0	1	3	8	10	25	100
Output							

6. *Formula:*

$$y = 10 - x$$

Graph:



7. *Table:*

Input	0	1	2	4	10	20	100
Output	3	4	5	7	13	23	103

Words:

8. *Table:*

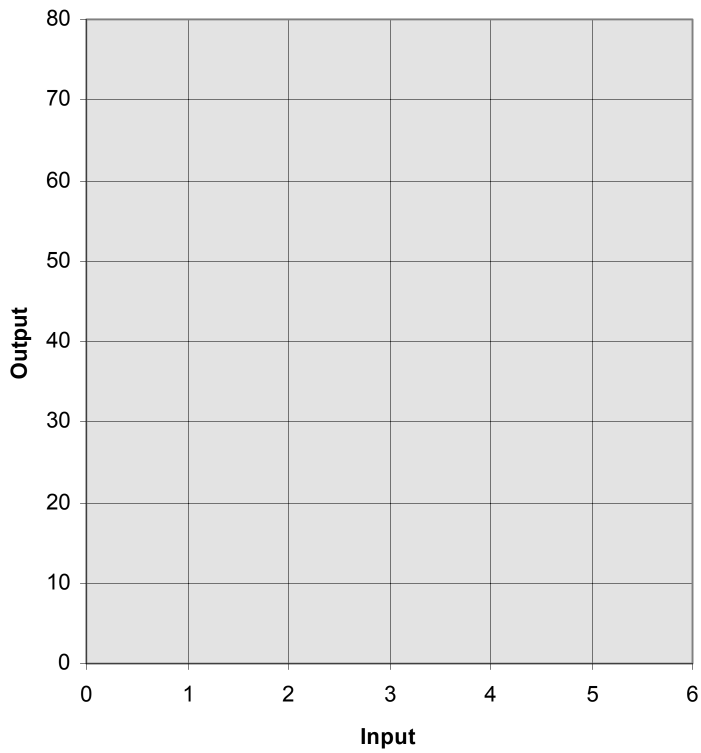
Input	0	1	2	3	5	10	100
Output	0	3	6	9	15	30	300

Formula:

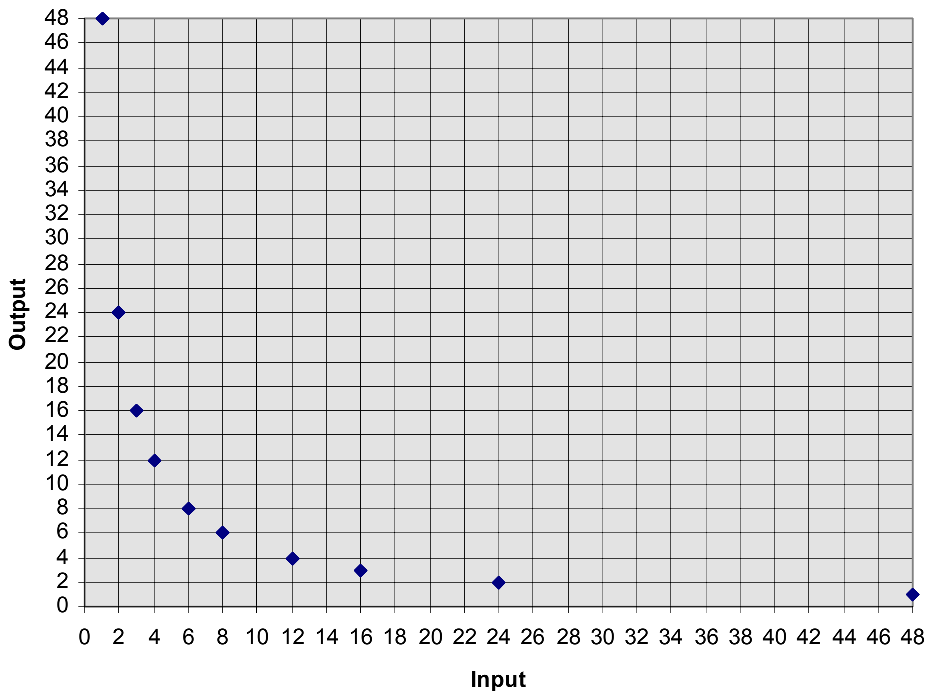
9. *Table:*

Input	0	1	2	3	4	5	6
Output	1	2	4	8	16	32	64

Graph:

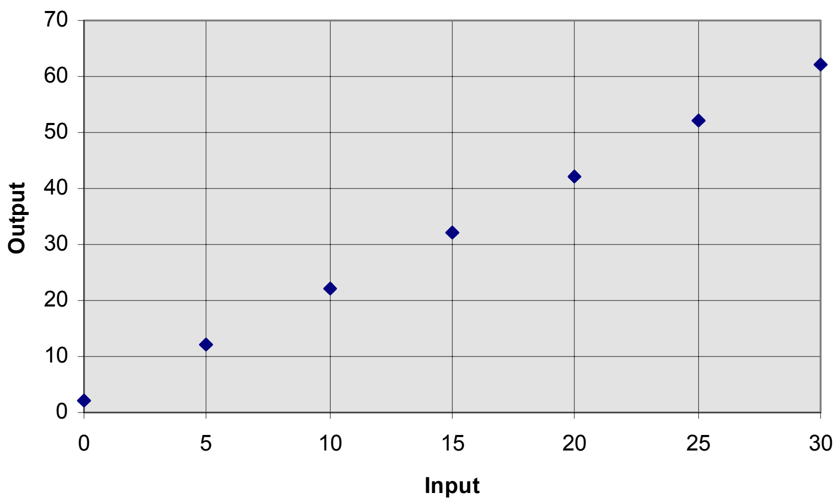


10. *Graph:*



Words:

11. *Graph:*



Formula:

12. *Graph:*

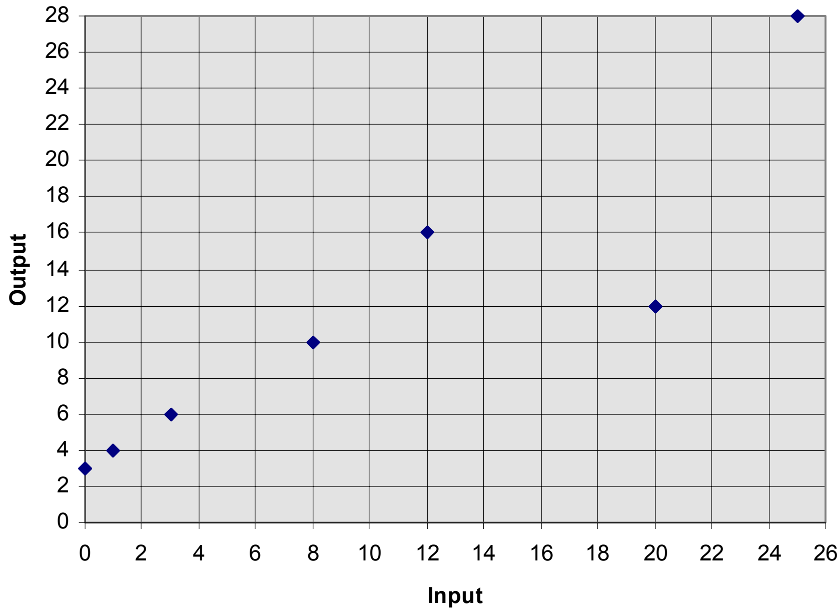


Table:

Input							
Output							

Activity 6

Salary increase

Imagine having been offered a job with the starting salary of R30 000 per annum. Your employer says to you that he will guarantee a cost of living increase of 10% per annum for the next 8 years. Your annual increase could be more than 10% if your work is good.

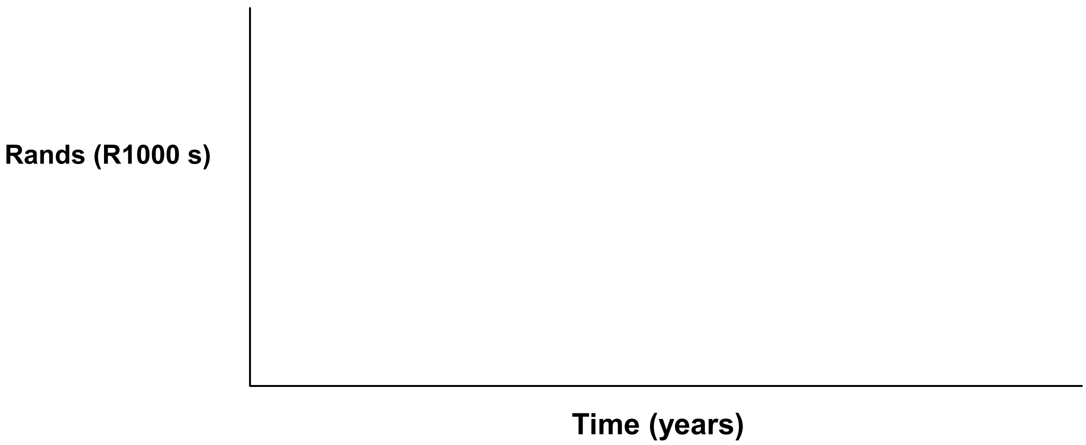
Case 1: Assume that you are working and only receive cost of living increase each.

1. What would your salary be in 5 years?
2. When would your annual increase be more than R5 000 for the first time?
3. Complete the following table. It will assist you in answering question 1 and 2 above.

Time (years) t_n	Annual salary (rands) $S(t_n)$	Annual increase (rands) $I(t_1) = S(t_{n+1}) - S(t_n)$
t_0	30 000	3 000
t_1	33 000	3 300
t_2	36 300	3 630
t_3	39 930	
t_4		
t_5		
t_6		
t_7		
t_8		

3. Plot on the same set of axes your annual salary as well as your annual increase in the graph provided below.

4. What type of functions do you expect your salary, $S(t)$, and your annual increase, $I(t)$ to be?
5. Is there any mathematical way of verifying your answer to question 3?



Case 2: Assume that you are a consistent hard worker and consequently your annual salary increase is 15%. Complete the following table to determine what effect this will have on your salary.

Time (years) t_n	**Annual salary (rands)** $S(t_n)$	**Annual increase (rands)** $I(t_1) = S(t_{n+1}) - S(t_n)$
t_0	30 000	4 500
t_1	34 500	5 175
t_2	39 675	
t_3		
t_4		
t_5		
t_6		
t_7		
t_8		

5. Is your answer for question 3 still valid? Explain.

6. Describe one way of determining whether a sequence is a geometric or not.

7. Show that in Case 1 and in Case 2, your salary, $S(t)$, and your annual increase, $I(t)$ are both geometric sequences.

8. What do you notice about the common ratio for the salary, $S(t)$, and your annual increase, $I(t)$ in:
   (a) Case 1?
   (b) Case 2?

9. Explain in words how the annual increase and the salary are related.

10. Give formulas for determining both $S(t)$ and $I(t)$, and then write an expression to show the relationship between the two functions.

11. Check to see whether your relationship holds for any geometric sequence.

Activity 7

Storm-water

Johannesburg municipality has been doing a survey on the amount of water that flows into a reservoir following a common summer afternoon thunderstorm. They measured the flow through numerous storm-water drains after several thunderstorms. We will just study the data for one of the storm-water drains.

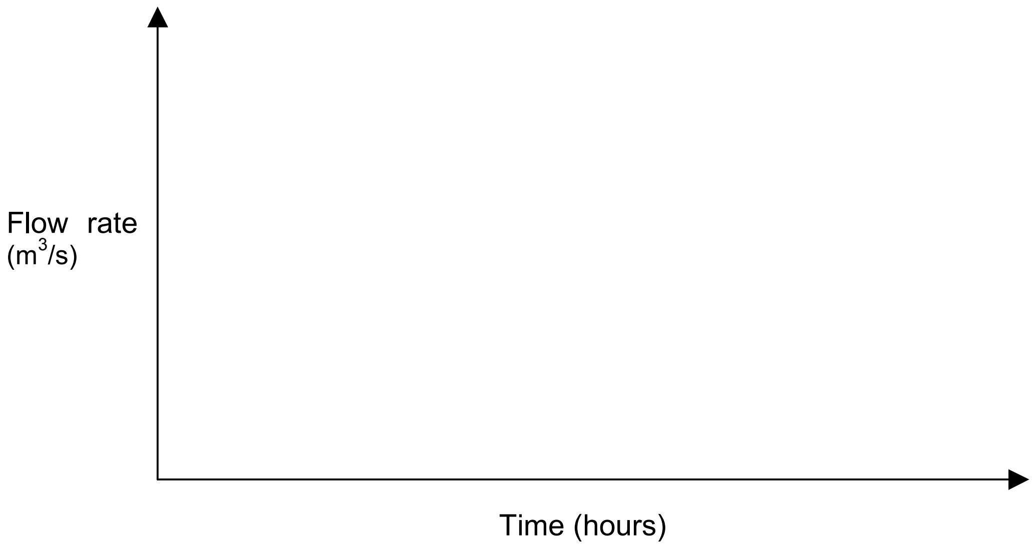
The following data was collected immediately after a normal storm.

Time elapsed since storm (hours)	1	2	3	4	5	6
Flow rate (m^3/s)	40	20	10	5	2,5	1,25

1. What type of function is the flow rate? Justify your answer.

2. Give a formula that could be used to determine the flow rate, f(t) at any given time, t, in hours.

3. Plot the above information on the following set of axes.



4. Does it make sense to join the points in the above graph? Explain.

5. Devise and explain a method of estimating the total flow through the pipe during the first hour following the thunderstorm.

6. Complete the following table using your method for question 5.

Time elapsed since storm (hours)	1	2	3	4	5	6
Flow during the given hour (m^3)						

7. What type of function is the flow, F(t)? Justify your answer mathematically.

8. Determine the common ratio for both f(t) and F(t) to two decimal places. Are they similar, and is this what you would expect?

9. If you used 10 minute intervals instead of hourly intervals, would your function F(t) behave in the same manner?

10. Will your estimate of the flow become more or less accurate with a decrease in the time interval. Explain your answer.

11. Describe in words the relationship between f(t) and F(t). Make sure you understand clearly what f(t) and F(t) represent.

Recap
In the salary increase problem we had exponential function, and we discovered that the rate of change of this function is also an exponential function with the same common ratio as the original function. In the storm-water problem the rate of change of the function is exponential, and from this we discovered that the function itself was exponential again with the same common ratio.

Although we have not proved mathematically that the rate of change of an exponential function are directly proportional, or in other words

Rate of change of function = constant x function

We have shown several cases where it holds.

A geometric function can be written in the form

$$f(t) = a.b^x, \text{ where a and } b \in R,\ b > 0 \text{ and } x \in N_0.$$

If a function is geometric, then the rate of change of the function is directly proportional to the function. In other words,

Rate of change of function = constant x function

The constant depends on the value of b.

Geometric functions are a subset of exponential functions. Exponential functions can be written in exactly the same way as geometric functions, except that the exponent x is now an element of the real numbers. The question is, does the relationship between the rate of change of the function and the function remain true if $x \in R$?

Complete the following table for the function (the change in the function values),

$$f(t) = 2^x, \text{ where } \Delta f(x) = \Delta f(x_n) - \Delta f(x_{n-1}).$$

x	-3	-2,5	-2	-1,5	-1	-0,5	0	0,5	1	1,5	2	2,5	3
f(x)													
Δf(x)													
$\Delta f(x_n)/\Delta f(x_{n-1})$													

12. What do you notice about all the entries in the last row? Explain slight differences in value.

13. What type of function do you think the rate of change will be?

14. If you took more points between -3 and 3, for example -3; -2,75; -2,25; …; 2,5; 2,75; 3, would your findings be the same?

15. If you took a larger domain, for example –10; -9,5; … 9,5; 10, would your findings be the same?

16. Do you think that

    the rate of change of function = constant x function

    holds for all exponential functions?

Activity 8

Inflation

The price if a loaf if bread in South Africa in 1999 is R2,05 and the inflation rate is quoted at 9,8% per annum. This means that a person living in South Africa can expect to pay 9,8% more the following year for the same bread. The price of bread, P, can thus be calculated as follows:

$$\mathbf{P = 2{,}05 + 0{,}098 \times 2{,}05} \qquad \text{(after 1 year)}$$

This expression can be factorised to

$$\mathbf{P = 2{,}05(1 + 0{,}098) = 2{,}05 \times 1{,}098 = 2{,}25} \qquad \text{(after 1 year)}$$

So you can expect to pay R2,25 for a loaf of bread. *If the inflation rate remains constant*, the price will rise by 9,8% again the following year. Thus the price after 2 years will be

$$\begin{aligned} \mathbf{P} &\mathbf{= 2{,}25 + 0{,}098 \times 2{,}25} \\ &\mathbf{= 2{,}25(1 + 0{,}98)} \\ &\mathbf{= 2{,}25 \times 1{,}098} && ** \\ &\mathbf{= 2{,}47} && \text{(after 2 years)} \end{aligned}$$

If you look carefully at the line **, you can see that this could be written as

$$\begin{aligned} \mathbf{P} &\mathbf{= 2{,}05 + 1{,}098 \times 1{,}098} \\ &\mathbf{= 2{,}05 \times 1{,}098^2} \end{aligned}$$

During the 3rd year, the price will rise again by 9,8% so that after 3 years, you can expect to pay

$$\begin{aligned} \mathbf{P} &\mathbf{= 2{,}47 + 2{,}47 \times 0{,}098} \\ &\mathbf{= 2{,}47(1 + 0{,}098)} \\ &\mathbf{= 2{,}47 \times 1{,}098} && *** \\ &\mathbf{= 2{,}71} \end{aligned}$$

Again look at the line ***. It could be written as

$$\begin{aligned} &\mathbf{= 2{,}05 \times 1{,}098^2 \times 1{,}098} \\ &\mathbf{= 2{,}05 \times 1{,}098^3} \end{aligned}$$

Summary:

Currently, the price is	$\mathbf{P = 2{,}05}$
1 year later	$\mathbf{P = 2{,}05 \times 1{,}098 = 2{,}25}$
2 years later	$\mathbf{P = 2{,}05 \times 1{,}098^2 = 2{,}47}$
3 years later	$\mathbf{P = 2{,}05 \times 1{,}098^3 = 2{,}71}$

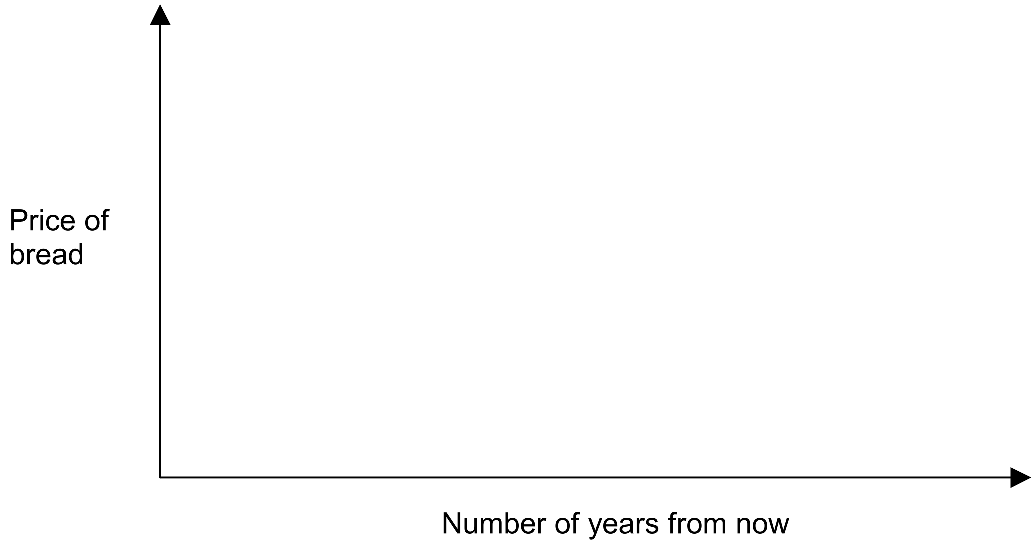
What would be the price of bread in **k** years?

Answer: $\mathbf{P = 2{,}05 + 0{,}098^k}$

1. Complete the following table

Number of years from now	Price of bread (to the nearest cent)
1	2,25
2	2,47
3	2,71
4	
5	
6	
7	
8	
9	
10	

2. Now plot this information on the graph below.



3. Does it make sense to join the points on the graph? Explain.

4. How long will it take for the price of bread to double?

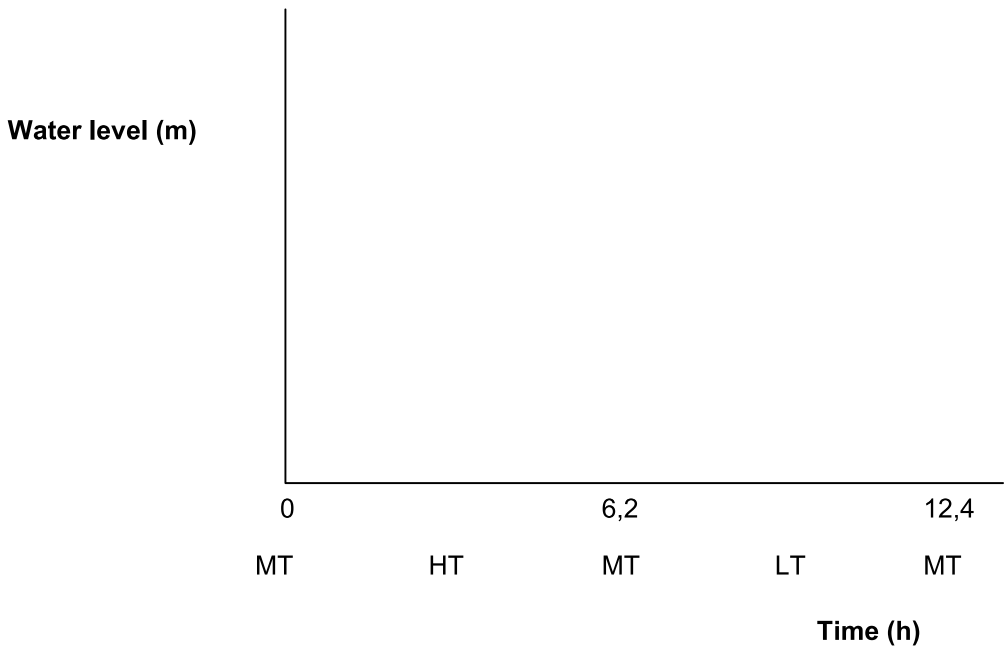
Activity 9 (Trigonometric graphs)

St Lucia Estuary 1

This section of activities is designed in the context of St Lucia Estuary. When developing a hydrodynamic model of an estuary, the water flow and water level at the interface within the estuary are effected by the tides. Tidal motion is a periodic function and is thus best represented by either the sine or cosine function.

On the side of a fixed jetty in the estuary there is a measuring stick. We can read off the depth of the estuary at any time by looking at where the water surface touches the measuring stick. At low tide (LT), the reading is 1 m and 2,2 m at high tide (HT). A complete tidal cycle is 12,4 hours long. We will start and finish measuring the water level half way between low tide and high tide.

(a) Plot on this graph what you think the water level reading (depth of the water) would be during a complete tidal cycle.



(b) Write a short paragraph explaining your predictions of the water level during the tidal cycle.

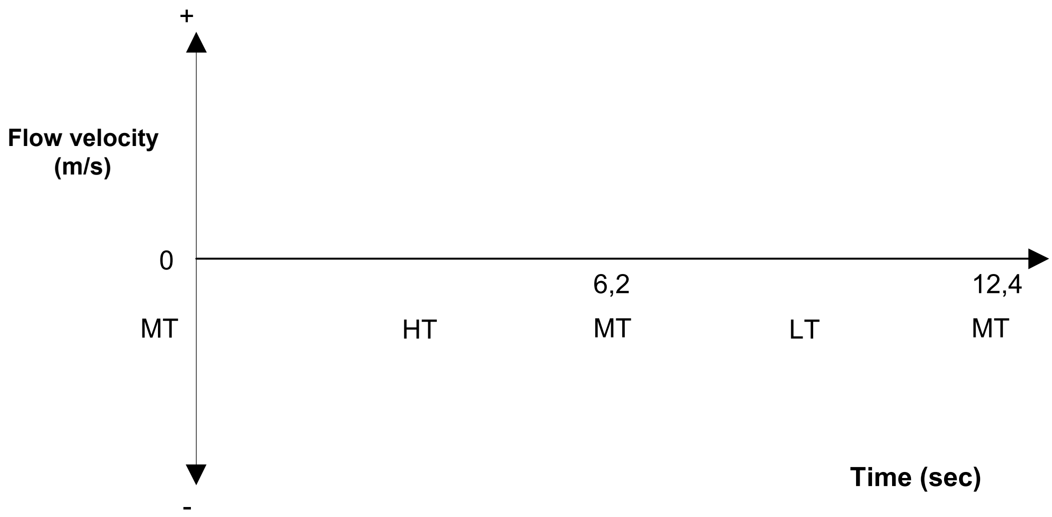
(c) The direction and the speed of the flow of water within the estuary are also effected by the tides. In the next table explain in which direction the water will flow under the given conditions.

Condition	**Flow direction**
Water level in the estuary higher than the water level in the sea	
Water level in the sea higher than the water level in the estuary	
The water level in the sea and the estuary are the same	

To cope with these possible changes in the flow direction, we decided to describe water flowing into the sea as flowing in a positive direction and into the estuary in a negative direction. For example water flowing at 0.3m/s into the estuary from the sea has a flow velocity of –0.3m/s.

(d) Using this convention, describe how you would expect the flow velocity to change during a complete tidal cycle.

(e) Plot your prediction of the flow velocity on the graph below.



Activity 10

St Lucia Estuary 2

A) Complete tide

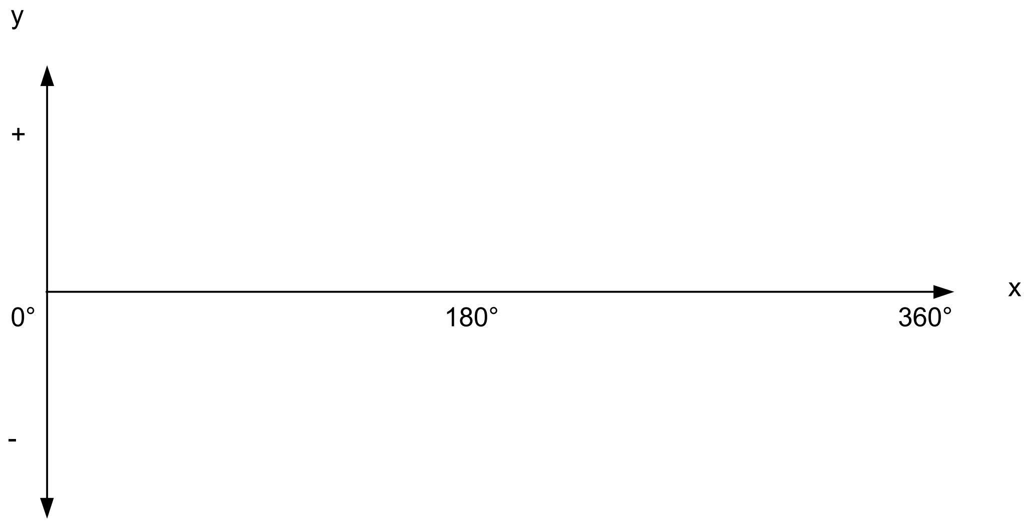
The data below is set for the St Lucia Estuary over a complete tidal cycle. Plot this data on the corresponding axes above but use a different colour plot your predicted graph.

Time (hours and minutes)	Water level (m)	Flow velocity (m/s)
0	1,6	0,1
1h 2min	1,9	-0,1
2h 4min	2,12	-0,25
3h 6min	2,2	-0,3
4h 8min	2,12	-0,25
5h 10min	1,9	-0,1
6h 12min	1,6	0,1
7h 14min	1,3	0,3
8h 16min	1,08	0,45
9h 18min	1,0	0,5
10h 20min	1,08	0,45
11h 22min	1,3	0,3
12h 24min	1,6	0,1

(a) Is your predicted graph similar to what actually happens? Can you explain any differences?

(b) Which type of function has a similar shaped graph to both the water level and flow velocity graphs?

(c) Write down the formula for this function and plot its graph on the axes below.



(d) Write down all the differences between the sine graph and the graph of the water level.

(e) How can we change $y = \sin x$ so that it can be used to describe the water level during a tidal cycle? Looking at the water level graph and the sine graph you should have noticed that the period, amplitude and range of the two graphs are different. We will investigate each of these in order to get a formula for the water level function.

Note: To change the shape of graphs we introduce "parameters". For example:

- to change the shape of the straight line graph $y = x$, we introduce m and c to get $y = mx + c$. What are the effects on the straight line if $m = 2$ and $c = -3$?

- to change the shape of the parabola $y = x^2$, we introduce a, b and c to get $y = a(x-b)^2 + c$. What are the effects on the parabola if $a = -2$, $b = 1$ and $c = 3$?

B) Period

Complete the following table where degrees are represented by x and hours by h.

Condition	**Sine graph** (x = …)	**Water level** (h = …)
When does the graph have a maximum value?		
When does the graph have a minimum value?		
When does the graph have values that are exactly mid-way between the minimum and maximum values?		
When does one complete cycle elapse?		

1. What is the relationship between degrees and hours?

2. If 4 hours had elapsed, what would be the corresponding value of x on the sine graph?

3. If x = 220^0, what would be the corresponding time on the water level graph?

4. Give an expression that could be used to convert hours to degrees.

C) Amplitude and range

Complete the following table

Condition	**Sine graph**	**Water level**
Maximum value		
Minimum value		
Amplitude		
Range		

1. What do you notice about the amplitude of the two graphs?

2. How could we adapt the sine function $y = \sin x$ in order for it to have the same amplitude as the water level graph?

3. Would this new function have the same range as the water level graph? If not, explain the differences.

4. To move the graph of a function up or down relative to the x-axis, what must we do to the function?

5. How could we move the graph of the function $y = 3x$ in order for the new graph to have the same gradient as the given function but having a y-intercept of 4?

6. How could we move the sine graph we created above to have the same range as the water level graph?

D) Finding a formula for the water level

For the function $y = a\sin(bx) + c$, match the correct parameter with the statements given

Parameter	**Statement**
a	Changes the range by moving the graph up or down
b	Changes the amplitude
c	Changes the period

Use all your calculations and the above information to determine the formula for the water level within the estuary in the form $y = a\sin(bx) + c$.

E) Flow velocity

Analyse the flow velocity graph in the same manner that we analysed the water level and determine a formula for the flow velocity.

Activity 11

Extension Activity

Further applications (extending to cosine graphs)

1. Profits from the sale of swimming costumes in a small shop during one-year period are given by $P(t) = 6 - 6\cos(6{,}926t)$, where *t* is in weeks and *P* is the profit in R1000s.

   (a) What is the maximum profit in any given week during the year?

   (b) During which weeks is there no profit? Explain.

   (c) What is the period of this graph?

   (d) Why do you think the profit function would have this period?

2. An adult at rest breathes in and exhales approximately 0,8 ℓ of air. The time between consecutive inhalations is 8 seconds. The volume of air, V(*t*) in the lungs, *t* seconds after exhaling is given by

$$V(t) = 0{,}45 - 0{,}35\cos(45t)$$

   (a) What is the minimum amount of air in an adult lung under rest conditions? Does this make sense?

   (b) What is the period between inhaling and exhaling?

   (c) During the breathing cycle, when would an adult have 0,6 ℓ of air in their lungs?

[Adapted from Barnet and Ziegler, 1990]