RECONCEPTUALISING SCHOOL ALGEBRA

The invention of variables was, perhaps, the most important event in human evolution. The command of their use remains the most significant achievement in the history of he individual human being.

Percy Nunn, 1919

. . . the research brings Good News and Bad News. The Good News is that, basically, students are acting like creative young scientists, interpreting their lessons through their own generalizations. The Bad News is that their methods of generalizing are often faulty.

Steve Maurer, 1987

The symbolism of algebra is its glory. But it is also its curse.

William Betz, 1930

Introduction

Consider the following episode observed in a typical Grade 8 classroom. The pupils are working on "simplification" exercises, including:

$x + x$

$2x + 3x^2$

$3a + 4a$

Ame produces the following:

$x + x = x^2$

$2x + 3x^2 = 5x^3$

However, she is unsure about her answers. So she asks the teacher.

Ame: *Is it right, miss?*

The teacher wants the pupils to take responsibility for their own learning, to validate answers for themselves on logical ground. So she helps Ame by suggesting a strategy that would enable her to judge for herself whether her "simplifications" are correct.

Teacher: *Listen everyone. Check your answers! Choose any value for x and check whether the left-hand side is equal to the right-hand side.*

This seems like good advice. But Ame does not immediately follow her advice. She keeps brooding on the problem and does not seem to make any progress at all.

The researcher (R) now joins Ame to observe what is happening.

R: *So why do you not choose a value for x like the teacher suggested and check it?*

Ame: *Yes, I know she does it, but I cannot . . . I do not know what x is . . .*

(silence)

R: *But just choose <u>any</u> value . . . Lets choose x = 3. Then what is the value of the left-hand side?*

(silence . . . Ame continues staring at the exercise, but does not substitute a value for x.)

Ame: *But how do you know that x is 3? Is it?*

R: *It does not <u>have</u> to be 3. We can choose <u>any</u> value for x. Choose 5 if you like . . .*

Ame: *But how can we say x is 5 if we have not yet worked it out?*

(silence . . .)

Ame: *Ah! I worked it out! It is 2! See (referring to the first question), if x is 2, then the left-hand side is 2 + 2 = 4 and the right-hand side is $2^2 = 2 \times 2 = 4$. So its correct!*

R: *So? Does that mean that it is correct to say that $x + x = x^2$?*

Ame: *Yes!*

In the same class it is observed that Kenneth interprets the letter symbols in the exercises concretely as letters of the alphabet. He "simplifies" $3a + 4a$ by reasoning that "3 a's and

4 a's gives 7 a's". So these exercises are really very easy for Kenneth. He is getting all his simplifications correct and he consequently does not pay any attention to the teacher's advice of checking by substituting arbitrary values for the letters. However, one cannot but wonder how his conception of the letter *a* as a real letter of the alphabet will help him to give meaning to $a \times a$ or what sense he will make of solving an equation like $a + 3 = 5$.

Clearly, here is a disastrous mismatch between what the teacher intends to convey to the children, and the children's conception of the meaning of the letters and the nature of the task. Ame and Kenneth simply did not, and at this stage *cannot* understand the teacher's notion of "choose any value for x".

For the teacher, the task to "simplify" means to construct an *algebraic identity,* for example $x + x = 2x$. $x + x$ and $2x$ are two *equivalent algebraic expressions* that, although different, nevertheless yield the same value for the same value of x. This is true for any value of x. So for the teacher x represents an *arbitrary* but unspecified number, and therefore one can illustrate the validity of the identity with any *particular* number or set of numbers.

Ame on the other hand, is interpreting the letter x in the context of solving *equations*: In an equation like $2x + 3 = 15$, x is an *unknown* and the task is to find out what *specific* value of x will make the open sentence true. It is an unique solution, in this case $x = 6$. Therefore, the teacher's suggestion that she chooses *any* value for x is not understandable to her at all because her interpretation of the task is that she must *first* solve the *equation* $x + x = x^2$ in order to find out the unknown value of x. So how can she simply choose a value for x if she does not know what the value of x is, that is, she is concerned that she may choose an incorrect value of x, that is not a solution of the equation. Ame's struggle with the ambivalent x echoes that of the great philosopher Bertrand Russell:

> *When it comes to algebra we have to operate with x and y. There is a natural desire to know what x and y really are. That, at least, was my feeling. I always thought the teacher knew what they were but wouldn't tell me.*

It is important to realise that even if Ame had produced the correct identity $x + x = 2x$, she would interpret it as an equation, and therefore a question to find what value of x makes the equation true and she still would not be able to choose any value to check it, because she would not know what x is. However, her "simplification" $x + x = x^2$ is not generally true (is not an *identity*). She therefore has now produced an *equation*, which is true only for *some* values of x. She then proceeds to actually find a solution for this equation, but her confusion between the meaning of an identity and the meaning of an equation, leads her to now wrongly deduce that $x + x = x^2$ is an identity (is generally true)[1].

For Kenneth the letters do not represent *numbers*, but real *concrete letters of the alphabet*. So the teacher's suggestion to replace the letters with numbers does not make sense to him at all. His interpretation of the task is not to construct an equivalent algebraic expression, but he (and we suspect Ame too) is doing *calculations with letters* to get an *answer* in the same way he did calculations with numbers in the primary school (see also transforming to an equivalent expression later in this paper).

Situations like this are commonplace in classrooms throughout the country and, indeed, throughout the world. It is well-documented that children have severe difficulties in mastering the basic notions of algebra. Yet, Ame and Kenneth are average, hard-working pupils who pay attention in class and do their homework diligently. They have excelled at mathematics in the primary school. How come they are suddenly having trouble with algebra? How do we explain it?

Learning obstacles

Children's learning is influenced by a myriad of factors. *Cognitive* explanations for pupils' learning problems may be sought in three areas. One explanation may lie at the level of an *epistemological* obstacle – an obstacle intrinsically related to the *nature of the content* itself, that is, the content may be inherently difficult. We note in passing that in the historical development of algebra it took mankind, or more correctly, the best mathematicians

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[1] It is not at all simple! We may think that the transformation $x + x = x^2$ is "wrong". But it is *true* for $x = 0$ and $x = 2$! Yet, these are *special cases*, and we cannot conclude, like Ame does, that the statement is generally true.
To understand this, learners must understand that:

- $x + x = x^2$ is a quadratic *equation* which is true for *only* two values of x, while
- $x + x = 2x$ is an *identity* which is true for *all* values of x.

We are here making the same distinction mathematicians make when they use quantifiers:
$\exists\, x$ so that $x + x = x^2$ (there *exist some* x *such that*)
$\forall\, x,\ x + x = 2x$ (for *all* x . . .)

To *check* a transformation by checking with arbitrary values – the teacher's advice – one must realise that

- if you choose a value and the resulting *numerical* sentence is *false*, you know for sure that the transformation is wrong, e.g. for $x = 1$, $x + x = x^2$ is false and $x + x = x^2$ is an incorrect simplification (transformation).
- if you choose a value and the resulting *numerical* sentence is *true*, you cannot be sure the transformation is correct, e.g. for $x = 2$, $x + x = 2x$ is true, but remember, $x + x = x^2$ is also true for $x = 2$!

Here is an important and useful piece of knowledge: We know from the fundamental theorem of algebra that an nth degree polynomial equation has n solutions. The principle of pseudo induction says that if an nth degree equation has *more* than n solutions, it violates the fundamental theorem of algebra, unless it is an equation with all coefficients zero – in which case it is an *identity* which is *always true* for *all* values of the variable. For example, if the sentence $x(x + 1) + x + 1 = (x + 1)^2$ is a quadratic *equation*, it is true for *only two* values of x. But we can easily find three solutions (e.g. $x = -1; 0; 1$) and then conclude that it is true for *any* value of x. This is because the sentence can be reduced to the zero equation $0x^2 + 0x + 0 = 0$, which clearly is true for *any* value of x.

What this means practically, is that if you check an nth degree transformation by substituting arbitrary values, you can only be sure it is correct if it tests correctly for at least (n + 1) values of the variable. So, if you check $x + x = x^2$, you can find *two* values that make it true, but those are the *only* possible solutions for the quadratic equation! On the other hand, if you check $x^2 + x^2 = 2x^2$, you can easily find *three* values making it true, and that immediately *proves* that it is *generally* true! This is explained by the fact that the quadratic equation $x^2 + x^2 = 2x^2$ can be reduced to $2x^2 = 2x^2$ or $0x^2 = 0$ or $0 = 0$, which are all true *independently of the value of* x.

centuries to move from the idea of a letter symbol as an *unknown* in an equation (e.g. in $2x + 1 = 6$) to a letter symbol as an *arbitrary number* in an identity (e.g. in $2(x +1) = 2x +2$)) and a *parameter* in a formula (e.g. *a* and *b* in $y = ax + b$). One should therefore not underestimate the cognitive challenge in coming to grips with the basic concepts of algebra!

A second explanation may be that there is a *psycho-genetic* obstacle – an obstacle due to the intrinsic characteristics of the children's development. Some researchers may conclude that Ame may not be "ready", i.e. may not possess the necessary prerequisite cognitive structures . . .

Thirdly, Ame's dilemma may be the result or consequence of what the French call a *didactical* obstacle – the result of the teaching she has had in the past. In particular, we note that traditionally children's first encounter with letter symbols in the primary school is in the context of solving equations, i.e. the letter symbol as an *unknown*. This is the context familiar to her. It is possible that if we changed the teaching, Ame's problem would not arise.

Of course, these different kinds of obstacles may be related. For example, we note that the nature of letter symbols in algebra is that they are *abstract*. By the very nature of a *symbol*, a symbol does not refer to itself, but to something else; a symbol has a *referent*. In the case of school algebra letter symbols represent *numbers*. However, the very nature of learning is that we interpret the surface structure. Kenneth, for example, is interpreting the letters at the surface level, not at the deep level.

Lets now look closer at the nature of algebra.

There is general agreement that in the historical development of mathematics, and in the development of learners' understanding of mathematics, operational conception precedes structural conceptions. Sfard and Linchevski (1994) point out that eventually all mathematical conceptions are endowed with a "process-object" duality. For example, an algebraic expression, say $3(x + 5) + 1$ may be interpreted in several different ways, for example:

- it is a concise description of a *computational process*, a sequence of instructions: add 5 to the number at hand, multiply the result by 3 and add 1.
- it represent a certain *number*. It is a product of a computation rather than the computation itself. Even if this product cannot be specified at the moment because the number x is presently unknown, it is still a number and the whole expression should be expected to behave like one.
- it may be seen as a *function* – a mapping which translates every number x into another. The expression does not represent any fixed (even if it is unknown) value. Rather it reflects a change, it is a function of two variables.
- it may be taken at its face value, as a mere *string of symbols* which represents nothing. It is an algebraic object in itself. Although semantically empty, the expression may still be manipulated and combined with other expressions according to certain well-defined rules.

This *plurality* of perspectives may seem confusing, but it is actually necessary that the learner should develop *all* these different meanings, because they are used in different contexts.

It seems that it is this "process-object" conception of algebraic concepts that learners find difficult. The difficulties learners have with algebraic concepts reflect that they have difficulty in the ability to view a string of processes as a permanent entity or *object*. The history of the development of Algebra can help us understand this issue.

<u>A History of Algebra</u>

Researchers describe three stages in the development of algebraic notation. In the first stage, <u>rhetorical algebra</u>, no symbols were used at all, and all equations and problems were posed and solved completely in prose form. This type of algebra was used in Europe until approximately the sixteenth century. The following is an example of rhetorical algebra, provided by the Hindu mathematician Aryabhata, AD 499 (Sfard, 1995):

> *To find the number of elements in the arithmetic progression the sum of which is given: Multiply the sum of the progression by eight times the common difference, add the square of the difference between twice the first term, and the common difference, take the square root of this, subtract twice the first term, divide by the common difference, add one, divide by two.* (p. 19)

The second stage, <u>syncopated algebra</u>, was characterized by the use of some abbreviations for the frequently recurring quantities and operations. Although Diophantus is generally credited with the introduction of syncopated algebra in 240 AD, syncopated algebra did not become the standard until approximately the sixteenth century. Girolamo Cardano (1501-1576), for example, wrote:

4 cubus aequantur 6 quadratus & 2 res & 3

which in modern notation is

$$4x^3 = 6x^2 + 2x + 3$$

The third stage in the development of algebra was <u>symbolic algebra</u>, which is the algebra we use today. Symbolic algebra was developed through contributions by Francois Viete (1540-1603) and Rene Descartes (1596-1650), among others, and gained widespread use by the middle of the seventeenth century. It was with the development of symbolic algebra that something very remarkable happened. With symbolic algebra, the symbols became objects of manipulation in their own right, rather than simply a shorthand for describing computational procedures. Sfard (1995) describes the impact of symbolic algebra as follows:

> *Employing letters as givens, together with the subsequent symbolism for operations and relations, condensed and reified the whole of existing algebraic knowledge in a way that made it possible to handle it almost effortlessly, and thus to use it as a convenient basis for entirely new layers of mathematics. In algebra itself, symbolically represented equations soon turned into objects of investigation in their own right and the purely operational method of solving problems by reverse calculations was replaced by formal manipulations on propositional formulas*. (p. 24)

Sfard's theory is that the historical development of algebra from rhetorical to symbolic must be reproduced in the individual to achieve understanding of algebra. More specifically, Sfard (1991, 1995) describes three stages that characterise the development of mathematical understanding in any area of mathematics, not just algebra. In the first stage, interiorization, some process is performed on a familiar mathematical object. In rhetorical algebra, for example, numbers are effectively manipulated and those manipulations are described in prose. In the second stage, condensation, the process is refined and made more manageable, as in syncopated algebra. In the third stage, reification, a giant ontological leap is taken. "Reification is an act of turning computational operations into permanent object-like entities" (Sfard, 1994). In the third stage, the individual must move from an operational or computational orientation – for example, seeing $x + 8$ as the process whereby 8 is added to the number x – to a structural orientation -- for example, seeing $x + 8$ as an *object*, a "whole", a "thing", an "answer".

In examining the difficulties students encounter in moving from arithmetic to algebra, Sfard (1991, 1994), Kieran (1989, 1992), and Herscovics (1989) describe a number of obstacles that can be connected directly to the difficulty of reification as described by Sfard. For example, children usually have difficulty accepting an algebraic expression as an *answer*; they see an answer as a specific number, a numerical product of a computational operation. Furthermore, the equal sign is usually interpreted as requiring some *action* rather than signifying equivalence between two expressions, leading to the howler that $x + 8 = 8x$.

Kieran (1992) proposed that the problem with modern algebra is that we *impose* symbolic algebra on students without taking them through the stages of rhetorical and syncopated algebra. Thus, as many educators and students have observed, students often emerge from algebra with a feeling that they've been taught an abstract system of operations on letters and numbers with no meaning. Herscovics (1989) describes the situation by stating that the students have been taught the *syntax* of a language without the *semantics*; in other words, they know all the rules of grammar, but do not understand the meaning of the words. Sfard and Kieran would very logically argue that this situation has resulted from jumping to symbolic algebra without exploring rhetorical and syncopated algebra.

Sfard's three-stage process seems to repeat itself historically and perhaps cognitively in the development of understanding of other mathematical concepts. For example, negative numbers were originally considered the *absurd* answer of the process of subtracting a larger number from a smaller one. It took hundreds of years for mathematicians to see negative numbers as objects representing direction rather than the waste products from a process on counting numbers. Complex numbers, again originally defined in terms of a process, for 300 years appeared to be as useless to the greatest mathematicians of the time as they seem useless today to algebra students. By interpreting the numbers as a way of referencing the plane -- visualizing these numbers as objects -- they eventually became indispensable in engineering.

Reification was thus an historically difficult process; it is no wonder that it is a difficult process in the classroom. Sfard's research indicates that reification does not build slowly over time, but is a sudden flash of insight, a "big bang," a "discontinuity" (Sfard, 1994, p. 54).

The real voyage of discovery consists not in seeking new landscapes but in having new eyes. Marcel Proust

The difficulty of finding a precise difference between arithmetic and algebra (as these terms are commonly understood) is well known. It is due to the fact that the distinction between them consists not so much in a difference of subject-matter as in a difference of attitude towards the same subject matter.

Nunn, 1919, p. 1

The move into algebra from arithmetic is not an easy continuous, gradual shift – it is a traumatic change in the *frame of reference*, a *rapture*. **The move into algebra from arithmetic is not simply a continuation of arithmetic, but it requires a break with arithmetical conceptions**. In Piagetian terms: what is required is not *assimilation* of new knowledge into the existing arithmetic conceptual structures, but a major *accommodation* (change) of these conceptual structures.

An approach to Algebra

Goals for mathematics instruction depend on one's conceptualization of what mathematics is, and what it means to understand mathematics. Alan Schoenfeld, 1999

Different approaches to the introduction of school algebra is currently being proposed in the research community, each associated with different perspectives on the nature of algebra (e.g. generalisation; problem-solving, modelling, functions). Traditionally the approach to the introduction of algebra in South African schools was through the letter as an *unknown* in the context of solving equations. This approach reduced much of the introductory teaching of algebra to rules for transforming and solving (linear) equations. The interim syllabus of 1995 and the current Curriculum 2005 reflect a shift in approach to algebra. In the interim syllabus algebra is viewed as “the study of relationship between variables” which means the letter is introduced as a *variable* in the context of *formulae*. Curriculum 2005, while not explicitly defining algebra as a mathematical domain, emphasises the importance of “observing, representing and investigating patterns in social and physical phenomena and within mathematical relationships". An analysis of the performance indicators in Specific Outcome 2 indicates that the letter is introduced as a *variable*. The didactical motivation for this shift from the introduction of the letter as unknown to the letter as variable was mainly to facilitate or support an *operational* understanding of the notion of *function*, from the perspective that the notion of function is central in mathematics. We support the importance and centrality of the function concept in mathematics and especially in algebra. However, which is the best *approach* to develop learners' understanding of function is not clear.

We should be clear that there are many different kinds of algebras in mathematics, for example the geometric algebras in which the symbols denote points, lines, transformations or other geometrical elements and their compositions. In Boolean algebra the letters denote propositions, and the operations are *and* and *or.* There is vector algebra and there is the algebra of sets. *School algebra*, however, is concerned mainly with only one kind of algebra, namely the *algebra of numbers*. We view school algebra as the “science” of numbers, in the sense that we are "experimenting" with numbers, trying to formulate generalisation and rules and methods of conjecturing, conviction and proof.

> *Historically, algebra grew out of arithmetic, and it ought so to grow afresh for each individual.* Mathematical Association, 1934

In MALATI, we approach our experimentation with numbers through the following three vehicles:
1. generalisation
2. structure
3. statements about numbers and operations

These are vehicles through which we build an understanding of the *letter as a number* from the perspective of known and unknown and variable. A fundamental pedagogical principle, for us, is that the development of an understanding of the letter as a number is not hierarchical. We therefore provide for all the different perspectives in the introduction to algebra. This development must reflect a flexibility to shift from one perspective to another as well as the ability to cope with all the perspectives in a specific context.

Experimenting within the context of number, we construct meaning for the:
1. Generalisations of number patterns expressed in different ways, for example, in tables or pictures.
2. Algebraic language
3. Structure of algebraic expressions
4. Transformation (manipulation) of algebraic expressions

and all these aspects are in the service of developing the notion of *function*.

Another fundamental pedagogical principle for MALATI is that the *need* for each of these aspects should be motivated by the context.

We now briefly comment on each of these aspects.

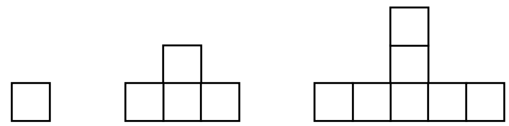
Generalisation

Generalisation as a thinking process is not unique to algebra. Its centrality in mathematics is highlighted in Curriculum 2005 as is evident from the following outcomes:

SO 2:	Manipulate numbers and number patterns in different ways
SO 9:	Use mathematical language to communicate mathematical ideas, concepts, generalisations and thought processes.
SO 10:	Use various logical processes to formulate, test and justify conjectures.

Mason (1996) concurs with this view and describes an "awareness of generality", namely, "seeing a generality in the particular and seeing the particular in the general" as the essence of mathematics and of algebra. From this perspective therefore, the expressions of generality in number patterns only serve to highlight this process. We however consider the need for an epistemological analysis of the nature of generalisation in geometric-number patters to be fundamental from a didactical perspective. Radford (1996) considers an analysis of the logical base inherent in the generalisations of number patterns. This analysis begins by considering the goal of such generalisations which is to "see a pattern" in the set of data ("observed facts") and to obtain a "new result" (conclusion or rule). Firstly, the recognition of a pattern can lead to different kinds of representations due to the way in which the pattern is perceived or interpreted, for example:

"observed facts":



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This will lead to "seeing the facts" in different ways and the emergence of new representational systems of these facts, for example:

1; $1 + 3\times1$; $1 + 3\times2$; ...

or

$3\times1 - 2$; $3\times2 - 2$; $3\times3 - 2$; ...

In finding the number of squares in the 100^{th} picture, the generalisation involves extracting what is variant and invariant from the syntactic structure of these *new* representations. If the number of squares is, for example, obtained by $1 + 3\times99$, the question of the *validity* of this invariance is open to debate. We believe that the underlying logic of generalisation, namely the process of *justification* of a new rule for the generalisation, therefore should be incorporated in the didactical design.

Note, that while generalisation activities like these can lead to tasks, for example, finding the rank of the picture in which there are 346 squares, this task does not involve generalisation. The logical base of the reasoning process is different in that we now start with a hypothesis about an unknown number, but handle this number as if it were known, in the process of finding it. This is the analytical dimension of algebraic thinking.

The kind of generalisation activity that focuses on number pattern recognition is therefore a vehicle in which to confront the notion of a *general number* leading to the development of the notion of a *variable*.
Not all problems that involve generalisation require the same kind of "seeing a pattern" as in the square's problem, in which we look for the next. For example:

What happens if we add two consecutive numbers?

This kind of activity stimulates an experience of experimenting with numerical facts that leads to the identification of a property that can be expressed as a generalisation. Firstly, the generalisation here involves knowledge about numbers, for example:

1. *The order of numbers: consecutive numbers involve one even and one odd number.*
2. *The property of even numbers: even numbers are divisible by 2.*

The production of the statement: the sum of two consecutive numbers is always an odd number, involves generality in that the nature of the sum does not vary with the particular pair of consecutive numbers selected. The learner needs to be challenged to explain this generality. Representing any pair of consecutive numbers, for example, 3 + 4, as 3 + 3 + 1, immediately exposes the algebraic structure ($2\times3 + 1$) of the problem. In fact, learners will find it difficult to appreciate the need to validate or prove the generality of such obvious statements. We, however, believe that the learners must distinguish between conjecture and proof, justification or validation to develop an understanding of the logical nature of generalisation.

The logical aspects of generalisation provides a rich context in which learners can engage in the logical processes highlighted in Curriculum 2005, Specific Outcome 10 which states "*Use various logical processes to formulate, test and justify conjectures".*

While an understanding of the symbolic form of the algebraic language is not the first goal of generalisation, its inevitable role in equivalent transformations, should be the motivation for its introduction.

Algebraic Language

The algebraic language refers to the representations or tools used to describe the generalisations (rules) about numbers or the solution of problems. Our theoretical framework of the developmental stages of algebra from operational to structural conceptions relates to the various algebraic representations in this development. The earliest form of algebraic expression was verbal (*rhetoric algebra)*, as is evident in the following problem:

Al-Khwarizmi, A.D. 825

Problem: What is the square which combined with ten of its roots will give a sum of 39?

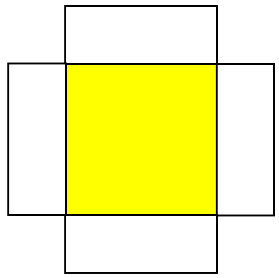
Solution: ... take one half of roots just mentioned. Now the roots in the problem before us is 10. Therefore take 5, which multiplied by itself gives 25, an amount which you add to 39, giving 64. Having taken then the square root of this which is 8, subtract from it the half of the roots, 5, leaving 3. The number three therefore represents one root of this square, which itself, of course is 9.

The algebraic character of this solution is reflected firstly in the kind of problem that is posed, in that we now need to find the side/root of the unknown square on which an operation was performed.

The rule given in words by Al-Khwarizimi, corresponds to the solution:

$$\text{the root} = \sqrt{\left(\frac{10}{2}\right)^2 + 39} - \frac{10}{2}$$

The generality of this solution was established by considering an unknown square to which four rectangles of base $10 \div 4$ and length the side of the unknown square is added to the square as represented below:



This shape still has an area of 39. Al-Kharizimi, proceeds by completing this to find the area of the larger square which is 39 + 4 $\left(\frac{10}{4}\right)^2$ and therefore the side of the new square is $\sqrt{39+\left(\frac{10}{2}\right)^2}$ and this is equivalent to the unknown side plus 2$\left(\frac{10}{4}\right)$ or unknown side plus $\frac{10}{2}$.

This solution is clearly algebraic because of its analytical nature but it focuses only on numerical processes. Based on our theoretical analysis of algebra we will regard this solution as being operational in that its focus is only on processes. A structural approach to this solution would require a new form of representation, the letter as number, resulting in the symbolic form of the solution:

$$\begin{aligned} x^2+10x &= 39 \\ x^2+10x+(5)^2 &= 39+25 \\ (x+5)^2 &= 64 \\ x+5 &= 8 \\ x &= 3 \end{aligned}$$

The symbolic nature of the solution does not, however, indicate a more sophisticated form of thinking. In fact, the symbolism makes the manipulation of the algorithms represented verbally much easier to handle. The transition from an operational approach to a structural approach is not so much in the use of letters instead of numbers. Rather, it lies in the ability to perceive a process, for example, "add five to a number", as a new entity or *object* and the ability to operate on this object – referred to as “reification”. Because it is impossible to operate or manipulate these objects when expressed verbally, symbolism was necessary for the historical transition to structural thinking. However, we want to emphasise that symbolism is not *sufficient* to attain structural thinking, as operational conceptions can also be conveyed through symbols. For example, learners who are not able to think *structurally* will view an expression like x + 5 simply as a computational process (often leading to the "answer" 5x). The ability of learners, for example, to successfully handle the kind of problem shown below is not an indication of whether they have achieved reification, as there is no need to operate on or manipulate “objects”:

> Evaluate the following if $x = 4$:
>
> $$x + 2x^2$$
>
> Note: In this problem the letter **is not** an unknown number and it **is not operated on.**

Once $x = 4$ has been substituted, it is merely a set of computational procedures that is performed: $4 + 2\times4\times4$.

The problem below, however, requires a structural approach in that the learner needs to handle the algebraic expression $x + 2x^2$ as an "object":

> For what of x is $x + 2x^2 = 36$?
>
> Note: In this problem the letter is an unknown number and by operating on the letter we treat the unknown number as a "known".

Research has shown that learners who may succeed in solving the above equation often lack an understanding of the underlying algebraic notions, such as the meaning of solution and the equivalence of equations. For many of these learners, solving an equation is just the performance of a certain algorithm. Furthermore, very few learners are able to perceive $x + 2x^2$ and 36 as functions (mathematical objects), in the given context.
In the traditional approach far too much time and effort was used in the mastery of symbolic manipulations without sufficient reflection on the difficulties that confront learners in developing a structural understanding of algebraic notions. While it may be true that, for the mathematician, the original meaning of the symbols is lost and they are able to work with the symbols as objects (things) according to known rules, we believe that learners must struggle for meaning of these symbols in an effort to *make sense* of the mathematical activities in which they engage.

Our view of the role of algebraic symbolism is:

> *Algebraic symbolism should be introduced from the very beginning in situations in which learners can appreciate how empowering symbols can be in expressing generalities and justifications of arithmetical phenomena......in tasks of this nature, manipulations are at the service of structure and meanings.* (Arcavi, 1994, p. 23)

Structure of algebraic expressions

In arithmetic we can many times bypass the properties and conventions related to the algebraic structure and replace them with an operational approach. However, in algebra those properties and conventions prove to be *essential*. For example, if it had been agreed that every possible pair of brackets should be inserted in each arithmetic string, we could have avoided the need for a convention about the order of operations in most cases (e.g.

$3\times5 + 3\times7 + 5$ would be written as $(3\times5) + (3\times7) + 5$ and the only rule needed to calculate the answer would have been “calculate between brackets first”). In algebra, however, even a simple equation like $6 + 9\times x = 60$ or $6\times x + 9\times x = 60$ cannot be handled without a convention about the order of operations. As a result, at the beginning of algebra the focus shifts to a great extent to the *structural properties* of the expressions. Learners must therefore be exposed to the structure of algebraic expressions so that it enables them to develop *structure sense*. This means that they will be able to use *alternative* structures of an expression flexibly and creatively. The teaching program should promote the search for de-composition and re-composition of expressions and guarantee that the need for manipulating algebraic expressions will make sense.

Our pedagogical approach therefore focuses on *structures*. This, however, is done within the context of **numerical expressions**, following Sfard and Linchevski's (1994) recommendation that the development of algebraic concepts should be from an “operational (process-oriented) conception to a structural conception”. We focus on numerical expressions whose structures have a possible input on the necessary and relevant algebraic expressions (with which learners will be dealing at the beginning of algebra).

MALATI's decision to start Algebra in the context of numbers is based on the following reasons:

1. The structure of the algebraic system is based on the properties of the number system;
2. It is a context with which the learners are familiar;
3. It is a meaningful context through situations for the construction of schema;
4. It allows meaningful reflection and verification procedures through calculations.

Research studies (Chaiklin & Lesgold,1984) has shown that many learners have a limited understanding of arithmetical structure. In this research learners had difficulty in judging the equivalence of the following numerical expressions without calculating:

$$685 - 492 + 947$$
$$947 + 492 - 685$$
$$947 - 685 + 492$$
$$947 - 492 + 685$$

Kieran (1991) proposed that this could possibly explain why, in algebraic contexts, learners have difficulty in understanding that $a + b - c$ is equivalent to $a - b + c$ if $b = c$.

The research of Linchevski & Livneh (1996) has also shown that difficulties (cognitive obstacles) that learners have with *algebraic* structure were present in *numerical* contexts. For example:

<u>Detachment</u> (Splitting the expression in incorrect places)
Examples:
$186 - 20 + 26$ is understood as $186 - (20 + 26)$
$165 \div 8 \div 4$ is understood as $165 \div (8 \div 4)$

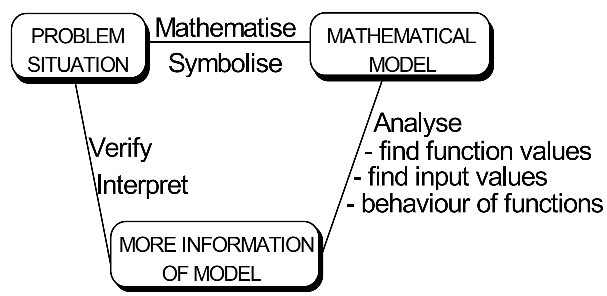
A high rate of detachment was noted for 50 – 10 + 10 + 10, due to the primitive model of multiplication as repeated addition. Learners therefore tend to reorganise the expression as 50 – 3×10.

<u>Competition between number sense and structure sense</u>
Some numerical expressions will induce an incorrect response, whereas other will support the correct computational procedure. For example, although the following two expressions have the same *structure* $a - b + c$,

- in 127 – 20 + 20 learners are inclined to first add;
- in 127 – 27 +16 learners are inclined to first subtract, and this is not motivated by the "left-to-right rule".

Some remarks on functions
There are many situations involving two variables where the one variable is *dependent* on the other variable, i.e. where a *change* in the value of one (the independent) variable causes a deterministic change in the value of the other (the dependent) variable.

Algebra is a language and a tool to study the nature of the relationship between specific variables in a situation. The power of Algebra is that it provides us with *models* to describe and analyse such situations and that it provides us with the analytical tools to obtain additional, unknown information about the situation. We often need such information as a basis for *reasoning* about problem situations and as a basis for *decision-making*. This view of algebra can be described as *applied problem solving*.



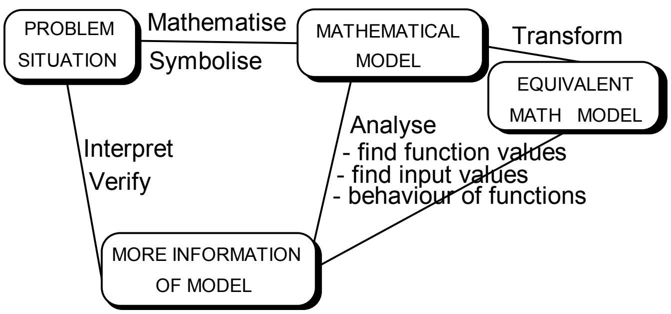
Learners should:

- experience the *modelling process*,
- experience and *value* the power of algebraic models to generate new information, and
- *understand* the concepts, procedures and techniques involved in modelling.

The mathematical model we construct of the situation may be represented in different ways: in words, as a table of values, as a graph or as a computational procedure (formula or expression). The additional information we need to generate is mostly of the following three *types*:

1. *finding values of the dependent variable* (finding function values)
2. *finding values of the independent variable* (solving equations)
3. *describing and using the behaviour of function values* (increasing and decreasing functions, rate of change, gradient, derivative, maxima and minima, periodicity, . . .)

The additional information is obtained by different techniques in different representations of the model (e.g. finding function values by reading from a table, reading from a graph, or substituting into a formula). Some techniques are easier than others and/or yield more accurate results. Therefore, an important aspect of algebraic know-how is transforming from one representation of the model (e.g. a table) to another, equivalent, representation (e.g. a formula) which is more convenient to solve problems of the above three types. These transformations are summarised in the following diagram and table:



From...to	**Words**	**Table**	**Graph**	**Formula**
Words				
Table				
Graph				
Formula				

In particular, the above three problem types are handled more easily when a formula or function rule is available. This can be viewed as a fourth important problem type:

4. *finding a function rule* (formula)

Learners should be able to find function rules in different representations of the model, i.e. to find the function rule from words, from a table or from a graph. The processes involved in finding function rules include induction (recognising a pattern in a table of values) and analytical processes (deduction, e.g. solving simultaneous equations). The relationships between tables, graphs and formulae are here of particular importance (e.g. the relationship between a recursive common difference of 2 in a table, a gradient of 2 in the graph and the coefficient 2 in the formula $y = 2x + 3$).

In problem solving, the choice of model (formula) depends on the properties of the model and the characteristics of the situation. In studying the relationship between variables, it is important to analyse the different behaviours (i.e. *properties*) of different *models* (functions), including:

simple direct proportion $y = mx$

linear function $y = mx + c$

inverse proportion: $xy = k$

quadratic: $y = ax^2 + bx + c$

exponential: $y = ab^x$

logarithmic: $y = \log_a x$

Sometimes the formula describing the relationship between variables may be "complex", making the first three problem types above very "complex". In such cases it is convenient to first transform the formula to an *equivalent*, but more convenient form for a specific task (from formula to formula in the previous table). To use the simplest of examples: To find the value of $2x + 8x$ for $x = 7{,}53$ requires three "complex" calculations. But if we *first* transform the expression to $10x$ the calculation is easy! This defines our fifth important problem type:

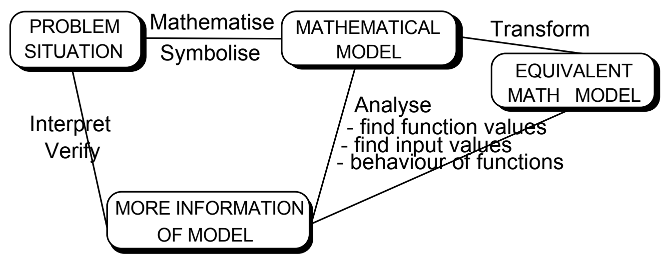
5. *transforming to an equivalent expression* ("manipulation" of algebraic expressions)

We should beware of superficial interpretations of algebraic manipulation, e.g. that it is "calculation with letters", in the same way as arithmetic involves "calculation with numbers". However, in $2(x + 3) = 2x + 2\times3$, $2x + 2\times3$ is not the *answer* or *result* of *multiplying* 2 with $x + 3$. x represents a *number*, so if for example $x = 7$, all that the statement $2(x + 3) = 2x + 2\times3$ says, is that $2(7 + 3) = 2\times7 + 2\times3$. This is not "multiplication" or an "*answer"* as we know it in arithmetic! The difference between the processes in arithmetic and algebra should be clear from the following example:

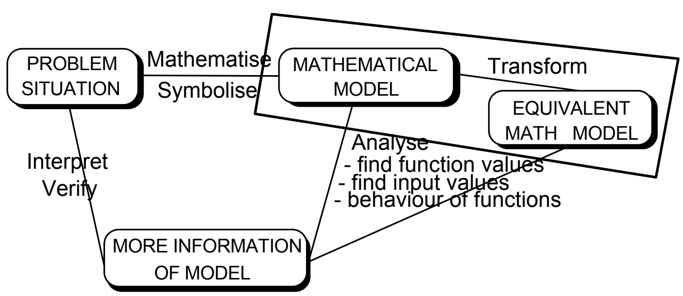
	Arithmetic	Process		Algebra
	12×17			
=	$12(10 + 7)$	*renaming the numbers*		$2(x + 3)$
=	$12 \times 10 + 12\times7$	*equivalent transformation*	=	$2x + 2\times3$
=	$120 + 84$	*sub-calculations*		
=	204	*final calculation*		

Algebraic "manipulation" involves *equivalent transformations:* $2(x + 3)$ and $2x + 2\times3$ represent two *different "methods"*, and we may in a particular context choose any of the two because they are the *same* in the sense that they yield the same *values* for the same values of x*:*

x	1	2	3	4	5
$2(x + 3)$	8	10	12	14	16
$2x + 6$	8	10	12	14	16



Traditional teaching of algebra put a lot of emphasis on "manipulation", i.e. the transformation of algebraic expressions to more convenient equivalent expressions. However, such manipulation was done mainly in isolation of the other important processes, as illustrated in the following representation.



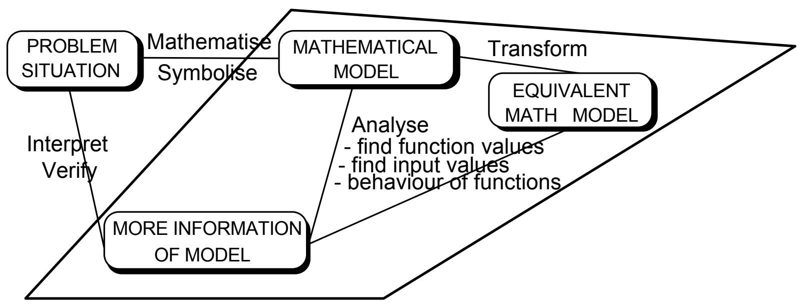
Algebraic manipulation is a useful *tool in problem solving*, but should not be elevated to a *skill for its own sake*. Our teaching should develop the meaning of equivalent expressions and learners should *experience the usefulness* (significance) of such transformations to make the first three problem types easier. For example, these two "complex" problems:

If $x = 7,3$ evaluate $\frac{4x^2 + 2x}{2x}$

Solve for x if $\frac{4x^2 + 2x}{2x} = 11$

become fairly easy once we transform $\frac{4x^2 + 2x}{2x}$ to $2x + 1$.

Therefore algebraic manipulation should be developed and practised in the context of *equivalent transformations*. While it certainly is not necessary to follow the full modelling process every time, i.e. to start with a real-life situation *every* time, once learners realise that the models we work with come from real situations, it is nevertheless preferable to at least include the analysis part as indicated in the following diagram, so that learners can experience the significance (purpose) of manipulation:



In traditionally handling manipulation in isolation, we have *denied* learners the *opportunity* to even begin to understand the *meaning* and the *significance* of the construction of equivalent expressions (manipulation).

It is important that the teaching programme provides for appropriate experiences of all five problem-types and that it develops the underlying concepts and techniques to enable learners to experience the power of algebra as a tool to solve problems. However, it should be emphasised that our objective should be to solve *problems*, not to master isolated skills for its own sake (say *factorisation*).

The definition of a function

We should be aware that the definition of a function as currently used in mathematics is a modern definition that has evolved over centuries. This evolution is described by Sfard(1991) as "from an operational notion as a process to a structural notion as an object".

Two essential features of the modern definition of function are

- **arbitrariness**
  This refers to both the relationship between two sets and to the sets themselves. As regards the arbitrariness of the relationship, it means that functions do not have to show regularity, be described by any expression/formula or graph, in other words, the two variables need not be *dependent* on each other. The arbitrariness between two sets means that the sets do not have to be numbers. Both these aspects makes the function concept more general.

- **univalence**
  This refers to the restriction that for each element of the domain there is *only one* element in the range. Mathematicians included this restriction to make operating on functions manageable. For example:

  If $f = \{ (2 ; 4), (3 ;5), (7 ; 9) \}$ and $g = \{ (2 ; 1), (3; 4), (7 ; 1)\}$
  then $f + g = \{ (2 ; 5), (3 ;9), (7 ; 10) \}$

  But if $f = \{ (2 ; 4), (3 ;5), (3 ; 8), (7 ; 9) \}$ and $g = \{ (2 ; 1), (3; 4), (7 ; 1)\}$
  how would we find $f + g$?

From this perspective, the following two sets of data (variables) from the index of a book are both functions, although the variables are not dependent on each other we cannot *predict* the page number or the author of Chapter 6 (if it exists). The distinguishing features in both cases are that there is a correspondence between the two sets and the value of the range is unique for each value of the domain (in the example on the left there is a one-to-one correspondence and in the example to the right there is a many-to-one correspondence).

Chapter	**Page**
Chapter 1	1
Chapter 2	18
Chapter 3	30
Chapter 4	44
Chapter 5	64

Chapter	**Author**
Chapter 1	Baxter
Chapter 2	Baxter
Chapter 3	Liddle
Chapter 4	Meyer
Chapter 5	James

The teaching program should ensure that learners also experience such examples in order to develop such a broader conception of function.

Pedagogical Strategy
With the above analysis as background, we now briefly outline our pedagogic approach to introducing "algebra" and the principles behind the design of our algebra materials.

1. Developing structure sense

From a procedural view to an explicit structural view of certain numerical strings
Our designed activities focus on the order of operations and on other structures within the context of numerical expressions that have a possible input to the understanding of the necessary and relevant algebraic expressions. The development of the structural view will be based on the procedural knowledge the learners have and their ability to make sense of the activities. The activities will present the learners with conflicts between their number sense and the structure sense.

2. Developing algebraic-language

Introducing algebra as the "science" of numbers and the letter as a number
The activities outlined above will naturally lead to discussions about "rules" of operating on numbers and to the introduction of a letter as standing for "a number". The learners might

experiment expressing some of the rules they "discovered" in the above prepared activities in the algebraic language (while the exploration of numerical strings continues).

Learners might suggest rules about operating on numbers that they already know and try to formulate them in the algebraic language. For example, multiplying a number by zero gives zero. The rules should also include inequalities (sentences with the greater than or smaller than signs). The learners might be challenged to come with sentences about numbers that only subgroup of the numbers (known to them) "obeys" and to express them in the algebraic language e.g. "a number multiplied by itself equals to the number", "a number multiplies by itself is greater than the number". If the learners at this stage are already familiar with integers (negative numbers) they will probably have more interesting examples. We would like to open the discussion to the idea that in all cases we were saying "something" about numbers; however, there is a difference between cases in which what we have said refers to *all* numbers and cases in which it refers only to *some* numbers. Some sentences say something about numbers but there are *no* numbers that obey this rule.

3. Towards the notion of function

We build on the learners' procedural view of function, namely generalising certain input-output relations given by situations, tables, graphs and mathematical rules. The activities address the aspects of function we find essential while keeping in mind the aspect of rate of change and vise versa, i.e. the rate of change as leading to a family of functions. The choice of examples should allow the construction of a broader notion of function. We will integrate reflection sessions and discussions to help learners to make the concepts part of their explicit cognitive schema. We will try to highlight each aspect of functions; we will address, in each one of the four presentations of functions and point to the connections between them in the different presentations. The concept of variable, function, domain, range, algebraic expression and more are dripped as we go along. Teachers will have to decide which, how and when to make them explicit.

We introduce the Cartesian plane to study graphs as a representation of functions. No doubt this is a classic place for differential teaching. Probably not all learners can pick it up while dealing informally with functions. However, many may do so (or simply come with the knowledge from other contexts).

4. Unpacking the notion of the function on the way to linear functions

At a certain moment we narrow the activities and the discussion to the linear function and the quadratic function and consolidate most of the concepts developed previously in the broader context in the context of these two families of functions. We will, later on, narrow down the discussion to the linear function and add more properties specific to this family. Of course all this will be done in the four representations of functions with a bit more emphasis on "abstract" cases, i.e. mathematical rules and graphs.

5. Equivalent transformations

The notion of equivalent algebraic expressions is developed to give learners opportunities to appreciate the significance and usefulness of constructing equivalent expressions.

6. Equations

The notion of equation will be developed out of the concept of function.

An equation like $3x$+ 5 = 7 will be viewed as the value of x for which the value of y (or $3x$ + 5) is equal to 7.

Solving Linear Equations in an analytical way

The process for solving linear equations is introduced via the notion of equivalent equations. The concept of equivalent equations should be a part of the cognitive schema. The concept of solution will be developed. The equation will be viewed as computational processes to be balanced by "the solution", e.g. in $3x + 5 = x + 12$.

7. Unpacking the notion of function on the way to quadratic functions

Going back to verbal situations, tables, etc.

8. The quadratic equation

The notion of an equation is revisited in the context of quadratic of the quadratic equation. The notion of a solution is also revisited and the number of solutions will be introduced. Out of this we will extract the notion of a quadratic so as to distinguish it from linear and other equation. The solution process is via the notion of equivalent equations.

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