A vision for the learning and teaching of school *geometry*

Geometry as traditionally taught in South African schools can be summarised as follows:

(a) The description and classification of plane figures (for example parallelograms and cyclic quadrilaterals)
(b) The study of the properties of these plane figures
(c) The direct comparison of these figures and their properties
(d) Deduction using congruence of figures as a basic tool (some of the properties are deduced from others). This is done within the specific axiomatic deductive system originally used by Euclid. Proof is used as a form of verification.

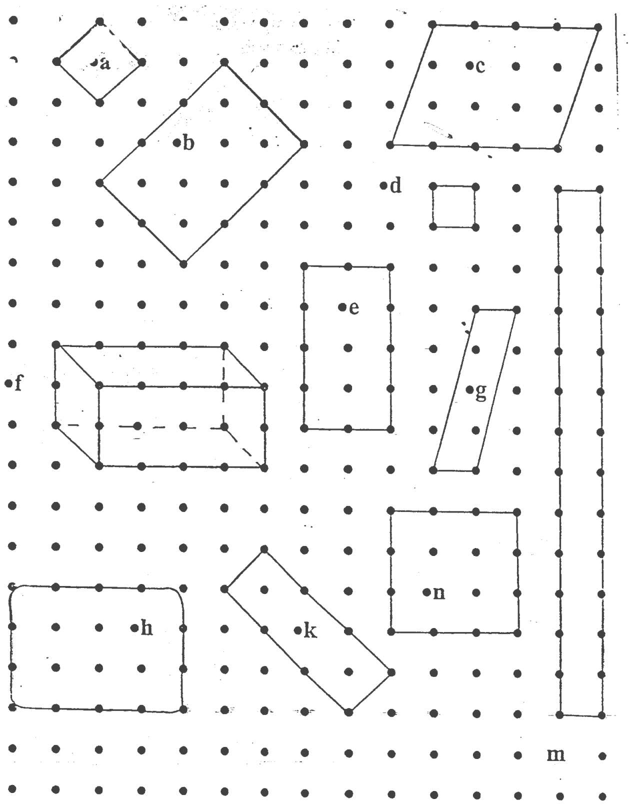
Aspects (a) and (b) are undertaken at the primary school in preparation for the more formal geometry in the senior grades. Our observations of work with primary teachers have suggested, too, that many of these teachers teach little or no geometry to their classes. Where this is done in the primary school, this is usually restricted to the identification and naming of simple geometric figures such as a square, triangle, circle and rectangle.

Many high school teachers can attest to the difficulties they have teaching geometry of this nature. Despite the best efforts of teachers, learners continue to have difficulty with deduction and proof. It appears, firstly, that in trying to cope with this form of geometry, learners and teachers turn it into something algorithmic: Many learners simply memorise proofs or rules.

Furthermore, research has shown that learners are inappropriately prepared for the formal geometry demanded by the curriculum. According to the van Hiele theory, a learner needs to be on the ordering van Hiele level to cope meaningfully with the an axiomatic system. Research in South Africa (De Villiers & Njisane, 1987; Smith, 1987) and elsewhere (Senk, 1989; Usiskin, 1982; Shaughnessy & Burger, 1985) has shown that many school learners are only on the van Hiele visual or analysis levels. As a result learners cannot find a meaningful interpretation of the activities required at high school and resort to memorisation.

We have identified low van Hiele levels in the responses of learners at MALATI project schools. We illustrate some examples of Grade 9 learners' work. When reading these examples it should be noted that this data was collected at the end of the Grade 9 year and that these learners would be expected to perform on the van Hiele ordering level in Grade 10 geometry.

This learner was asked to identify, giving reasons, all the rectangles in this picture. The responses are typical visual level responses as the learner is looking at the figures as "wholes".



Reasons:

1. M= It has a very long shape sc it has to be a rectangle.

2. K = Rectangle because it has a sh and size of a door

3. G = It has a shape of a buses re

4. E = Could be square but the siz is longer. It looks like a smaller drawer cupboard

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In the question below, learners were required to give the shortest possible definition of a rhombus:

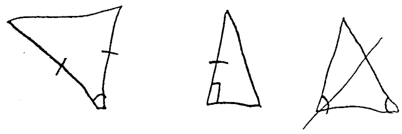
> *Mark and Jacob are playing a guessing game. Mark thinks of a quadrilateral and Jacob must ask questions about the <u>properties of the quadrilateral.</u>*
>
> *Mark thinks of a rhombus. What is the* ***minimum*** *number of questions that Jacob needs to ask to find out that the quadrilateral is a rhombus? Write down the questions.*

This response is a typical analysis level response as the learner lists a number of properties, some of which are not necessary in an economical definition:

Are all the sides equal ?
Are all the angles equal ?
Are the adjacent sides the same ?
Are the opposite angles equal ?
Does the diagonals bisect eachother equally ?
Are both diagonals equal to each other ?

The next example illustrates how a learner trying to work on the ordering level can resort to memorisation. The question required that learners draw three **different** triangles. The learner tired to relate this to congruency, but could not remember the details:

> ***"Side angle side. There are four different kinds of congruency. Hypotenuse...and something else...angle angle side"***



It appears therefore, that with respect to the study of plane figures, learners require better preparation for the deductive approach. Shaughnessy and Burger (1985) suggest for example, that more geometry should be introduced in the lower grades and that more informal geometry should be done in secondary schools before the more formal approach is introduced.

Furthermore, in reconceptualising the geometry curriculum it should be noted that, while the traditional approach to geometry as described above is undoubtedly important, there is much more to the study of geometry than this and this can realistically be explored at school level:

Geometry is relevant as a body of knowledge that supports interaction in physical space

Recent curricular innovations reflect the recognition that spatial sense is important not only in everyday interaction in physical space, but in the study of geometry and mathematics in general. For example, the NCTM Draft "Standards 2000" Document suggests that mathematics instruction programmes should pay attention to geometry and spatial sense so that all students, among other things, "use visualisation and spatial reasoning to solve problems both within and outside of mathematics". In South Africa, one of the ten Specific Outcomes for Mathematical Literacy, Mathematics and Mathematical Sciences in Curriculum 2005 requires that learners:

> *Describe and represent experiences with shape, space, time and motion, using all available senses: Mathematics enhances and helps to formalise the ability to be able to grasp, visualise and represent the space in which we live. In the real world, space and shape do not exist in isolation from motion and time.*

Different ways in which people interact in physical space may be distinguished. These include:

1. Observing spatial objects in a discriminating way, that is, two- and three-dimensional figures and the properties of these figures
2. Generating information that cannot be directly observed, for example, determining distances, elevations, area and volumes
3. Designing spatial objects and configurations, for example, gardens, furniture arrangements, furniture, buildings and artistic designs
4. Representing spatial configurations with plane drawings
5. Interpreting plane representations of spatial configurations.

Traditional school geometry in South Africa has attempted to address the first three aspects, but is singularly lean on the rich domain of geometrical ideas pertaining to aspects (4) and (5). Considering that so much of our interaction in physical space involves dealing with two-dimensional representations of this space, MALATI believes that these aspects should be explored in school geometry. Some work on simple projections will not only strongly enrich the utilitarian value of school geometry, but will extend the content beyond the domain of describing the properties of plane figures.

For further details see the MALATI Spatial Sense Rationale.

Geometry is relevant as a domain of purely mathematical activity

"Formal" geometry as done in traditional South African school geometry derives its educational rationale from this perspective. But it seems that the mathematical activity can be richer, and undertaken at lower levels, than currently experienced by learners. Martin (1993) notes that this view is consistent with a constructivist philosophy of learning – if we support the notion that mathematical knowledge cannot be transferred ready-made from one person to another, but must be built up by the learner, then it is appropriate that learners be given opportunities to engage in meaningful mathematical activity.

De Villiers (1997) distinguishes between the products (axioms, theorems, definitions, classifications) and the processes of geometry. These he identifies as axiomatising, proving defining, experimenting, refuting, pattern finding, generalising, specialising, classifying and theorem-finding. We would add systematising to this list. This perspective points to the rich mathematical thinking required by geometric activity at different levels.

Goldenberg (1996) suggests how a geometry curriculum can be organised around what he calls "habits of mind" that is, mathematical ways of thinking. Along with the ability to interpret diagrams, he identifies these inclinations as important "habits of mind" in geometry: the inclination to visualise, to describe formally and informally, to translate between visually and verbally presented information, to 'tinker", to look for invariants, to mix experiments with deductions, to build systematic explanation and proof, to construct and reason about algorithms, and to reason by continuity. He suggests that these "habits of mind" could be the basis of curricula with varying content.

The importance of thinking skills in geometry is reflected in Curriculum 2005. Specific Outcome 10 for Mathematical Literacy, Mathematics and Mathematical Sciences requires that learners "use various logical processes to formulate, test and justify conjectures". It is stated that:

> *Reasoning is fundamental to mathematical activity. Active learners question, examine, conjecture and experiment. Mathematics programmes should provide opportunities for learners to develop and employ their reasoning skills. Learners need varied experiences to construct arguments in problem settings and to evaluate the arguments of others.*

The importance of pattern finding and mathematical communication is also stressed in this curriculum.

MALATI believes that geometry offers an excellent context for learners to experience mathematical activity and that this can be done at the primary **and** secondary levels.

Geometry is relevant as a particularly illuminating example of mathematical activity
Mathematicians have learnt much about the nature of mathematics from geometry. Smith (1997) points out that historically geometry was the first branch of mathematics to be axiomatised and remains one of the more accessible examples of such a system for school learners. He stresses that this does not mean that high school geometry should be developed as a complete axiomatic system. Rather, learners in the further education phase could explore the nature and purpose of a deductive system through an approach of local axiomatisation, that is, by studying short sequences of Euclidean geometry theorems in which they explore the role of assumptions and the consequences of replacing these.

Geometry is relevant in the sense that it provides models and analogies of other areas of mathematics
Through visual representations such as graphs, geometric intuitions and knowledge is made accessible as sources for understanding ideas in algebra and analysis (calculus). But the potential for using geometry in other areas of mathematics is often not recognised, for example, transformations can be used to enhance the study of graphs in algebra.

Geometry is relevant as a domain for exploring certain classifications
This can be illustrated using an example from the study of plane geometry in which the objects of study vary: On one level one studies the plane figures as wholes, but on another level one studies the properties of these figures.

But the objects of study are not restricted to geometrical figures:
- Transformations are often regarded as a vehicle for exploring plane figures and explaining the properties of these figures, but these movements themselves can be the objects of study. We can study the properties of these transformations.
- The importance of being able to represent and interpret representations of spatial information has been noted. But these different representations, for example, parallel projection or perspective can themselves also be objects of study.

Geometry as studied by modern geometers

Malkevitch (1998) notes that nowadays most professional geometers are not professionally interested in the axiomatic development of geometry. He notes, rather, that for most geometers, "geometry has become the study one is led to by mathematical training when one studies visual phenomena". Geometry has led to a number of rich applications currently used in modern technology, for example, in computer technology, medical imaging, communications technology (codes in fax technology etc) and image processing. Malkevitch has suggested the following topic for study at school: Graph theory; compression codes and error-correcting codes; frieze patterns, wallpaper patterns, fabric patterns; knots; and polyhedra and tilings.

While acknowledging the implications of the inclusion of such topics in the curriculum for teacher development we feel that it is imperative that learners be afforded the opportunity to study these topics in preparation for participation in a technological society.

Proof

Proof at school level has traditionally been used as a form of *verification* or *justification*, that is, to convince one of the truth of a proposition. Research has shown that learners have difficulty with the notion of proof (Senk, 1985; Usiskin, 1982; Bell, 1976). This could, once again, be explained in terms of the van Hiele theory, for in order to understand this form of proof, a learner needs to be on the van Hiele ordering level. As Schoenfeld (1986) points out, proof is not meaningful until the entities manipulated in the proof (for example, the plane figures and the properties of these figures) are meaningful.

Furthermore, many learners do not see the need for proof as verification, for example, learners will question why it is necessary to proof a proposition when the result appears "obvious". Proof does not appear to bring conviction for learners. De Villiers (1997) proposes that there be less focus on this form of proof in geometry. After all, this can be done with algebra.

Furthemore, Bell (1976) notes that conviction in mathematics is usually reached by other means proof. Research (Mudaly, 1999) has shown that, by engaging in appropriate exploratory activities (in their case using computer software), learners can gain conviction. This can be followed by a need for explanation. Proof can thus be regarded as a means of *explanation* or *illumination*, providing insight into why a proposition is true.

De Villiers (1997) has identified other meanings of proof, namely *systematisation* (the organisation of results into a deductive system of axioms and theorems) of *discovery* of new results, of *communication* (in transmitting and making mathematical knowledge public), and of *self-realisation*.

We believe that proof should be retained in the geometry curriculum – it is an important mathematical thinking skill and part of a mathematical culture, that is, in mathematics we rely on proof rather than experience or simple hear-say. The challenge is to select a meaning of proof that makes sense to learners for example, using proof as *explanation*, and to provide learners with appropriate learning experiences so that proof can become meaningful.

The MALATI Geometry Curriculum

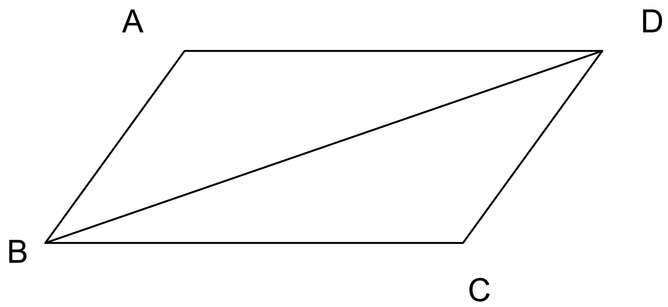
Bearing these different aspects of geometry in mind, MALATI has formulated the following objectives for the study of geometry at school. Learners who study geometry should:

1. Develop spatial sense (see MALATI Spatial Sense Rationale)
2. Engage in the process of mathematisation (in both the primary and secondary school)
3. Learn to use a number of tools to solve problems
4. Develop a sense of the structure of mathematics (in the Further Education Phase).

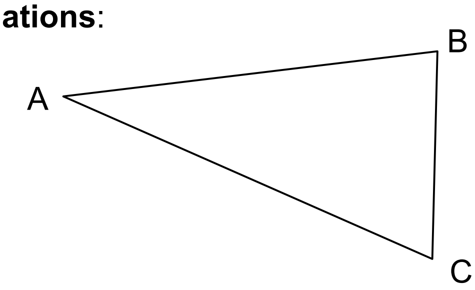
With regard to objective (3) above, we are regarding synthetic geometry (as traditionally used at school), co-ordinate geometry, transformation geometry, vectors and trigonometry as different notation systems / methods by which plane figures can be explored. Consider, for example, the following:

Show that the opposite sides of a parallelogram are equal

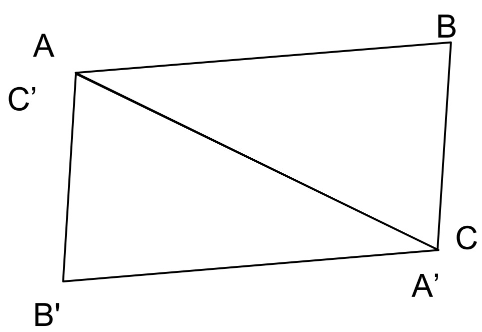
1. Using **synthetic geometry:**
   We can use congruency to prove $\Delta ABD \equiv \Delta BDC$ (SAA).



2. Using **transformations**:



We can rotate the triangle about the midpoint of AC to get:



In this way we can show: AB' = BC and AB = B'A'.

3. Using **Co-ordinate Geometry** we could place the parallelogram on a system of axes and use the distance formula.

Although co-ordinate geometry and transformation geometry have been studied at school, this has traditionally been done as isolated topics, with no relationship being forged between these and traditional synthetic geometry. We would like learners to be able use these aspects as **tools for exploring plane figures**.

Furthermore, the study of the properties of plane figures in traditional geometry in South African schools has been restricted to congruence arguments, that is, two figures are regarded as equivalent if they are congruent. Another form of equivalence argument, similarity (parallel projection), is studied at school, but this explored using congruence! Another form of equivalence which is not addressed at school level is non parallel projection.

MALATI proposes that these different techniques can be used **informally** to explore plane figures. For example, congruence can be explored through pattern-making, similarity through scale drawings / photocopies, and non-parallel projection through the study of photographs. The actual differences between the different techniques should only be made explicit in the Further Education Phase.

The structure of the MALATI geometry materials

The MALATI geometry materials have been designed with the above framework in mind with the aim of facilitating learners' movement through the van Hiele levels. It should be noted that MALATI is using the van Hiele theory, together with subsequent research, as a guide (and not as a rigid framework). We have found the theory useful in designing learner materials and in working with teachers to assess learner responses and to select appropriate activities (see MALATI Van Hiele Theory Document). Our interaction with teachers and our classroom observations have suggested that the provision of appropriate geometry activities and support for different learners is a complex process.

Within the time-constraints of the project it has only been possible to design modules to address some of the above-mentioned aspects of geometry, but we feel that the broader MALATI vision as described here could be used in curricular design in both the General Education Phase as well as in the Further Education Phase

Rationale documents related to specific aspects of geometry are included at the beginning of each module where appropriate.

PTO for a summary of MALATI geometry modules.

The content of the MALATI geometry materials

Primary materials (Grades 4 to 7)

Module	**Content**
Prim 01	Interaction in Physical Space
Prim 02	Representations of 3D objects in 2 dimensions – drawings
Prim 03	Representations of 3D objects in 2 dimensions – nets and cross-sections
Prim 04	Position and Location
Prim 05	2-dimensional figures – includes transformations
Prim 06	Similarity / Enlargement
Prim 07	Area

Secondary materials (Grades 8 and 9)

Module	**Content**
Sec 01	Similarity – Grade 8
Sec 02	Exploring lines and angles using transformations – Grades 8 and 9
Sec 03	Polygons – Grades 8 and 9
Sec 04	Area – Grade 8

We also provide guidelines on how these modules can be developed for higher grades.

A review of the literature on the van Hiele theory suggests that much of the work done within this framework relates specifically to the classification of plane figures. We have extended this and also structured the work on the study of three-dimensional figures, similarity, area and transformations within the van Hiele framework.

The reader should also consult the following MALATI research papers, which form a background to the design of the material and the approach:

Bennie, K. (1998). An analysis of the geometric understanding of Grade 9 pupils using Fuys et al's interpretation of the Van Hiele theory. In N.A. Ogude & C. Bohlmann (Eds.), **Proceedings of the Sixth Annual Meeting of the Southern African Association for Research in Mathematics Education** (pp. 64-69). Pretoria: Universtiy of South Africa.

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MALATI Geometry and Curriculum 2005

As noted briefly in the discussion, Curriculum 2005 paints a broader picture of school geometry than the traditional curriculum. The MALATI materials are in line with the following aspects of this curriculum:

Specific Outcome	Assessment Criteria
SO7 **Describe and represent experiences with shape, space , time and motion using all available senses**	1. Descriptions of the position of an object in space 2. Descriptions of changes in shape of an object 3. Descriptions of the orientation of an object 4. Demonstrate an understanding of the inter-connectedness between shape, space and time
SO8 **Analyse natural forms, cultural products and processes as representations of shape, space and time**	1. Recognition of natural forms, cultural products and processes and their value 2. Representation of natural forms, cultural products and processes in a mathematical form 3. Generation of ideas through natural forms, cultural products and processes
SO2 **Manipulate number patterns in different ways**	2. Evidence that number patterns and geometric patterns are recognised and identified using a variety of media 3. Completion and generation of patterns 4. Exploration of patterns in abstract and natural contexts using mathematical processes
SO4 **Critically analyse how mathematical relationships are used in social, political and economic relations**	4. Demonstrate knowledge of the use of mathematics in determining location

In addition it should be noted that mathematical thinking skills (SO10) as discussed above are developed and required in the MALATI activities and geometrical terminology (SO9) is introduced and reinforced in the materials.

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