FROM NUMERICAL EQUIVALENCE TO ALGEBRAIC EQUIVALENCE[1]

Rolene Liebenberg, Marlene Sasman and Alwyn Olivier

Mathematics Learning and Teaching Initiative (MALATI)

In this paper we describe Malati's approach to developing an understanding of the equivalence of numerical and algebraic expressions. We start with a procedural definition of numerical equivalence and ***at the same time*** *focus on a structural view of equivalent numerical expressions. This approach was monitored in our project schools. We report on the following: (1) grade 6 students' ability to judge equivalent numerical expressions* ***without doing a computation****; (2) two lessons observed in a grade 9 class in which pupils were asked to judge the equivalence of algebraic expressions.*

Introduction

Students' difficulties in learning the basic ideas of early algebra has been well documented. Kieran (1989) emphasises that an important aspect of this difficulty is students' difficulty to recognise and use *structure*. Structure includes the "surface" structure (e.g. that the expression $3(x + 2)$ means that the value of x is added to 2 and the result is then multiplied by 3) and the "systemic" structure (the equivalent forms of an expression according to the properties of operations, e.g. that $3(x + 2)$ can be expressed as $(x + 2) \times 3$ or as $3x + 6$).

Kieran sees algebra as the formulation and manipulation of general statements about numbers, and hence hypothesises that children's prior experience with the structure of numerical expressions in primary school should have an important effect on their ability to make sense of algebra. Booth expresses the same view:

> *...a major part of students' difficulties in algebra stems precisely from their lack of understanding of arithmetical relations. The ability to work meaningfully in algebra, and thereby handle the notational conventions with ease, requires that students first develop a semantic understanding of arithmetic.* (Booth,1989, p. 58)

From this perspective Booth formulates two tasks for research:

- To examine students' recognition and use of structure and how this recognition may develop.
- To devise new learning activities and environments to assist students in this development.

We share the same assumptions about the importance of understanding numerical structure as a prerequisite for understanding algebraic structure. This paper reports on some aspects of our attempts to address Booth's two research tasks.

[1] Liebenberg, R., Sasman, M. & Olivier, A. (1999). From numerical equivalence to algebraic equivalence. **Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa: Vol. 2**. (pp. 173-183). Port Elizabeth: Port Elizabeth Technikon.

Research on equivalence

Students' understanding of the concept of numerical equivalence has been researched extensively in the context of how they view the equal sign. Behr et al. (1976) has pointed out that students' view the equal sign as an operator or a "do-something signal". This perception is built in the earlier grades in which simple arithmetic equalities have the operations on the left and the result on the right, or they are required to complete open sentences like:

$$2 + 3 =$$
$$3 + 5 =$$
$$5 + 2 =$$

Herscovics and Kieran (1980) asked students to build numerical expressions with more than one operation on each side of the equal sign to in an effort to expand their understanding of the equal sign. In the latter research, while the students realised that the concept of equation indicated that the numerical expressions on each side have the same numerical value, the expressions which they constructed were often not equivalent and contradicted the order of operations. Booth (1982) conducted research that provided information on the kinds of expressions that students would perceive as being equivalent. Booth found that students regarded expressions such as $5 \times e + 2$ and $5 \times (e + 2)$ as being equivalent and that the students' interpretation of these expressions changed depending on the context. Hence in judging the equivalence of these algebraic expressions that are context based, students ignore the conventions about the order of operations as the context does not help them to recognise the ambiguity.

Other research studies investigated how students judge the equivalence of numerical expressions *without computing the answer* (e.g. Collis, 1974, 1975; Chaiklin and Lesgold, 1984; Cauzinille-Marmeche, Mathieu and Resnick, 1984) and found that students are not in a position to judge the equivalence of numerical expressions without computing. This research and that of Kieran (1989) suggests that students are not aware of the underlying structure of arithmetic operations and their properties and that this situation is most likely due a predominantly computational focus in the earlier grades.

Developing the concept of numerical equivalence

Our approach therefore to developing the concept of equivalent numerical expressions is set within the framework of developing a structural view of numerical expressions. Our starting point is to ask students to think of *different ways* in which three numbers, for example, 15, 8 and 4 and the operations + and $\times$ can be used, once only, so that the answer 47 is obtained. In the same task the students were asked to think of different ways in which the numbers 9, 6, 8, and 4 and the operations + and $\times$ could be used to obtain the answer 47. Incorporated in this task is the procedural definition of numerical equivalence:

Numerical expressions that have the same numerical value (the same answers) are called equivalent numerical expressions.

This task allows for both a procedural approach as well as a structural approach to finding the different ways in which the numerical expressions can be created. A procedural approach would involve doing a calculation to find the four numerical expressions ($8 \times 4 + 15$; $4 \times 8 + 15$; $15 + 4 \times 8$; $15 + 8 \times 4$), whereas a structural approach would involve using the properties of the operations, in this case, the commutative property of addition and multiplication and the order of operations. We found that most students worked procedurally. This was to be expected, as they had not had many experiences with tasks involving a structural focus of numerical expressions at this stage. The tendency towards calculating is so strong that most students did not simply replace the 15 in the expressions with 9 and 6.

The students were also given tasks in which they were not allowed to calculate, for example:

Without calculating, insert an $=$ or $\neq$ between the number expressions:

1.	$28 + (2 \times 5)$	$28 + 2 \times 5$
2.	$18 + 54 - 4 + 25$	$18 + (54 - 4) + 25$
3.	$(1254 + 2973) \times 7$	$1254 + 2973 \times 7$
4.	$288 \div 32 \div 8$	$288 \div (32 \div 8)$
5.	$488 \div 8 - 6 \div 3$	$488 \div (8 - 6) \div 3$
6.	$(21 \times 13) \times (42 \div 6)$	$21 \times 13 \times 42 \div 6$

DON'T CALCULATE

Which of the following number expressions has the same answer as:
$367 + 68 \times 214 \times 1966 + 814 \times 45$

1. $367 + 68 \times 1966 \times 214 + 814 \times 45$
2. $214 \times 68 \times 1966 + 367 + 814 \times 45$
3. $45 \times 814 + 367 + 214 \times 68 \times 1966$
4. $68 + 367 \times 214 \times 1966 + 814 \times 45$
5. $1966 \times 214 \times 68 + 45 \times 814 + 367$
6. $367 + 68 \times 214 \times 814 + 1966 \times 45$

How do students judge the equivalence of numerical expressions?

We designed the following test based on one of the non-computational tasks to investigate the kinds of justifications the students would give in judging the equivalence of numerical expressions:

Without calculating, insert the symbol = or ≠ between the number expressions

In each case give the **reason/s** for the symbol you inserted between the expressions

Note: You may not give, as a reason that you *calculated the answers* as you must make your decision **without calculating.**

(a)	$(208 + 59) \times 61 \times 48$	$208 + 59 \times 61 \times 48$
(b)	$(415 \times 58) \times (232 \div 29)$	$415 \times 58 \times 232 \div 29$

This test was given to 40 grade 6 students in one of our project schools. An analysis of the students' justification for the equivalence of the numerical expressions is given:

In (a) there were 4 categories for the justification of the numerical equivalence:

	Classification of response	**Kind of responses**	**symbol chosen**	**number of pupils**
1	Applies the correct rules	*Because the first sum has brackets and the rule is start with the sum which have brackets around it. In the next sum we have to start with multiplication first because that is the rule and it does not have brackets like the first sum*	≠	29
2	Structural differences	*Because there is brackets and multiplication in the expression*	≠	2
3	Focuses on the numbers only	*The numbers are the same*	=	1
4	Incorrect rule applied	*Because in any case even when using brackets we do multiplication first.* *There are brackets around a sum with addition in, so it makes no difference if you calculate left to right or add first you will still end up with the same answer.* *Its equal because when there is × and + in a sum you can work from left to right.* *I say it is equal because you have to first times then add.*(ignores the role of brackets)	=	8

Table 1

In (b) there were 5 categories for the justification of the numerical equivalence:

	Classification of response	**Kinds of responses**	**Symbol chosen**	**Number of pupils**
1	Two different computational methods results in different answers	*Because if you have a number expression with multiplication and division you should multiply what you should and divide what you should or if want an easier way, you can work from left to right and they are not doing that in the first number expression.* *If you work from left to right you will have a certain answer, but, the first number expression you should first calculate the two number expressions in brackets and then multiply them together and it will leave you with a totally different answer than the other one.* *The sum is wrong because you have to times first and then divide by 29. The brackets are at the wrong places.* [puts brackets in (415 × 58 × 232) ÷ 29]	≠	22

Table 2

	Classification of response	Kinds of responses	Symbol chosen	Number of pupils
2	Recognises that the second expression can be computed in the same way as the first.	*It is the same because the first one has brackets and that means you can start with any one first and the other one does not have brackets so that means you can also start with anyone.* *I say it is equal because the second method you first do 415 × 58 and you get your answer then you say 232 ÷ 29 and you get another answer then you say your first answer × your second answer and it is the same as working form left to right.*	=	8
3	Over-generalise the rule that × is done before addition	*Because here we are starting with multiplication first and that is the rule.* *The first sum is exactly the same as the second one and you have to multiply first and then divide.*	=	8
4	Focuses on the numbers only	*The numbers are the same*	=	1
5	Focuses on the operations only	*It has × and ÷*	=	1

Table 2 (continued)

The students found it easier to provide a justification for the non-equivalence of the first numerical expressions. The students at this stage were not familiar with the distributive property in a formal sense and had only focussed on activities dealing explicitly with the commutative property of addition and multiplication. Hence their only means of a syntactical or structural justification (i.e. based on the rules or the positions of the numbers) was based on the order of operations. See in Table 1, response category 1: Applies the correct rules.

The students had greater difficulty in judging the equivalence of the second pair of numerical expressions. A common justification was based on an over-generalistion of the rule that multiplication is done before addition. In a follow up test, to further investigate this over-generalisation, the students were asked to judge whether the expressions (415 × 58) ÷ (232 ÷ 29) and 415 × 58 ÷ 232 ÷ 29 were equivalent. The students' knowledge of the role of brackets at this stage was limited to that of "do the calculation in the bracket first". The students were therefore not in a position to provide a syntactical justification in terms of the other roles of the brackets, for example, if there is a subtraction sign in front of a bracket the signs inside the bracket changes if the bracket is removed. From the responses in table 2 it is evident that students relied on a syntactical justification based on the rule for calculating the expressions.

Developing the concept of algebraic equivalence.

Our aim is to develop two dimensions of understanding the equivalence of algebraic expressions. The first dimension of understanding is that two algebraic structures are equivalent if the numerical expressions are equivalent for ***all*** values of the variable. The students are given tasks in which they explore certain algebraic expressions numerically and based on the procedural definition of equivalence, establish whether the algebraic structures are equivalent. In order to develop the notion of algebraic equivalence as expressions that must be numerically equivalent for all the values of the variable, the students are also challenged to find numerical situations for which the algebraic expressions are numerically equivalent for ***some*** but ***not all*** values of the variable, for example:

$x + x = x^2$ is numerically equivalent for $x = 0 : 0 + 0 = 0^2 = 0$ and for $x = 2 : 2 + 2 = 2^2 = 4$ but for no other values.

In the tasks it is however made explicit that those expressions that are not equivalent for all the values of the variable are ***not algebraically equivalent***[2]**.** The initial tasks that students are given to judge algebraic equivalence *do not* involve any manipulation of the letters. For example, students are asked to explain why $a - (50 + 25)$ and $a - 50 - 25$ are equivalent. The students are encouraged to provide both syntactical and semantic[3] justification for the equivalence of the expressions.

The second dimension of understanding involves the function or *usefulness* of algebraic equivalence so that the transformation of one algebraic expression into another becomes meaningful for the students.

[1] This definition should later be adapted to include $\frac{1}{x} + \frac{1}{x} = \frac{2}{x}$ for all ***sensible*** values of x.

[3] Semantic applies to the meaning of the numbers, for example, the students may say that in the expressions $a - (50 + 25)$ and $a - 50 - 25$ the same value is being subtracted from a.

The tasks designed to develop this also reinforces the first dimension of understanding, for example:

Complete the following table:

x	1	2	5	12	19	37	45
$2x + 5x$							
$3x + 4x$							
$12x - 5$							
$7x$							
$6x + x$							
$9x - 2x$							

(a) What do you notice in the table?
(b) Determine the value of $2x + 5x$ if $x = 19$. Discuss your method.
(c) Determine the value of x if $9x - 2x = 35$. Discuss your method.

These tasks are followed by a definition of algebraic equivalence:

We say that the expressions in the tables are *equivalent* if they produce the same output for the same input numbers.
Therefore $4x + 2x$ and $6x$ are equivalent expressions, because they produce the same output values for the same input values.
We can explain it in this way:
$4x + 2x$ means $4 \times x + 2 \times x = (x + x + x + x) + (x + x) = 6 \times x = 6x$

While the earlier tasks do not involve any manipulation of the letters, the students are now encouraged to focus on the properties of the operations to build equivalent algebraic expressions. The notion of algebraic expressions as replacements to make the task at hand easier is reinforced in these tasks.

The discussion on algebraic equivalence is now extended to introduce the different meanings of algebraic statements, for example:

1. $2x + 3x = 5x$, a statement which is true for all values of x (algebraic identity)
2. $4x + 12 = 7x + 50$, a statement which is true for only one value of x (equation)
3. $10x + 40 = 10x + 50$, a statement for which no values of x exist.

The tasks in which the meaning of these algebraic statements are explored do not involve any manipulation of the letters either. The numerical approach is followed to reinforce the first dimension of understanding of algebraic equivalence. For example:

Complete the following table.

x	1	3	5	8	11	17	38
$4x + 12$							
$7x + 3$							

(a) Is $4x + 12 = 7x + 3$ an algebraic identity? Discuss by using the table.

(b) Are there any values of x where the two expressions have the same numerical value.

The following note is given to the students after this task:

Sometimes the values of the two expressions are not equal for all values of the variable.
$4x + 12$ and $7x + 3$ are not equivalent.

How the students judged the equivalence of algebraic expressions

In the grade 9 class that we observed the students had nearly completed our unit called "Making Life Easier" (Towards Manipulation) that focussed on the development of the concept of algebraic equivalence. The students were working on tasks that involved finding the value of an algebraic expression for a specific value of the variable, for example:

What is the value of the expression if $x = 23{,}45$:
$25x – (7x + 5)$

In this task the students first transformed $25x – (7x + 5)$ into $18x – 5$ and then substituted $x = 23{,}45$: It became evident however, from the response below that they were uncertain whether the two algebraic expressions were equivalent:

Teacher: "*Are these expressions equivalent?*"
Students: "*Yes, **if** the answers are the same.*"

This may seem contradictory in the sense that while executing the "simplification", the students did in fact not realise that the transformation of one algebraic expression into another, meant that these expressions can **replace** one another and are therefore equivalent (our second dimension of understanding). The students did not accept the transformation process (i.e. structural justification) as sufficient proof for the equivalence of two algebraic expressions. The need to check that "*the answers are the same*" might be due to our emphasis on the procedural justification in which the students analysed input - output tables to deduce the equivalence of algebraic expressions.

We also observed that the students' perception of judging algebraic equivalence was influenced by tasks in which they were asked to determine the value of the variable that would make two algebraic expressions ***numerically equivalent***. The process of solving an equation was seen as a way of judging the equivalence of two algebraic expressions. This is evident in the following response of a student when asked whether the expressions $3x + 2$ and $5x + 3$ are equivalent:

Student: "*No, it **depends** on the value of x*"

We also observed that some students were solving the equations, for example, $3x + 2 = 5x + 3$ and when they found one value of the variable for which the answers were the same, concluded that the expressions were equivalent. Unless it is made explicit that the two algebraic expressions are ***numerically equivalent*** for a specific value of the variable and that the two expressions are ***not algebraically equivalent***, a serious misconception can result.

An analysis of our unit, "Making Life Easier"(Towards Manipulation), suggests that we need more tasks that focus explicitly on the notion of the transformation of algebraic expression as a ***sufficient condition*** to judge the equivalence of two algebraic expressions. In most of our tasks we asked the students to use a table to check the equivalence of two algebraic expressions, for example:

Are $12x - (7x + 5)$ and $12x - 7x - 5$ equivalent? Use a table to check.

There is also clearly an imbalance between tasks of a procedural nature and those that have a structural focus. By the latter we mean that students not simply engage in simplifying algebraic expressions but focus explicitly on the properties of the operations that make it possible to carry out transformations.

Conclusion

We should note that the analyses presented here for grade 6 and grade 9 are for different students. It is to be hoped that when the students from grade 6 start with the algebraic work in grade 9 they will have a better foundation and grasp the difficult concept of algebraic equivalence easier.

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