Students' Conceptions of Congruency Through the Use of Dynamic Geometry Software

Gloriana González · Patricio G. Herbst

Abstract This paper describes students' interactions with dynamic diagrams in the context of an American geometry class. Students used the dragging tool and the measuring tool in Cabri Geometry to make mathematical conjectures. The analysis, using the cK¢ model of conceptions, suggests that incorporating technology in mathematics classrooms enabled a measure-preserving conception of congruency with which students' could shift focus from shapes to properties. Students also interacted with dynamic diagrams in a novel way, which we call the *functional* mode of interaction with diagrams, relating outputs and inputs that result when dragging a figure. Students' participation in classroom interactions through discourse and through actions on diagrams provided evidence of learning using tools within dynamic geometry software.

Keywords Calculators · Dynamic geometry software · Congruency · Diagrams · Conceptions

Accounts in the history of mathematics have documented the crucial place of tools in developing new mathematical knowledge, particularly in the case of geometry (Eves 1990). The use of compass and straightedge, inherited from Euclidean geometry, has been a common practice in school classrooms through the history of geometry teaching. But tools can do more than aid people in expressing their knowledge; tools can also enable and sometimes constrain particular conceptions. The design of a contemporary compass, for example, allows us to take for granted a notion that Euclid did not take for granted: That it is possible to transport a distance from one location in the plane to another (cfr. Elements, Book I, Proposition 2).

Our argument is that dynamic geometry software (DGS), such as GeoGebra$^{©}$ (Hohenwarter 2001), Cabri^{TM} (Laborde and Bellemain 1993), or The Geometer's

G. González
University of Illinois at Urbana-Champaign, Champaign, USA

P. G. Herbst (✉)
University of Michigan, Ann Arbor, MI, USA
e-mail: pgherbst@umich.edu

Sketchpad[TM] (Jackiw 1991), includes features that similarly enable and constrain conceptions of geometric ideas; we investigate the case of congruency. While these DGS tools are much more complex than a compass, the design point made above about the compass also applies to them: They enable some geometric actions and by doing that they often conceal questions that one might ask were one not to have the chance to do those actions.

The designers of dynamic geometry software packages intend to provide new instructional tools for the study of geometry. While some of the features of DGS attempt to model the orthodox use of a compass and a straightedge in Euclidean geometry, other features such as measuring capabilities and the possibility to drag objects about the screen offer new affordances. According to Balacheff and Kaput (1996), computers in classroom mathematics create a new ontology of mathematical objects. "It is no longer geometry in the traditional sense, particularly as one is able to program sequences of constructions and variations in parameters into a given configuration and then 'run' the systems to create dynamic animations" (Balacheff and Kaput 1996, p. 492). Computers may also create constraints for the geometry student. Mathematics education researchers must continue to question how students learn mathematics through the use of new tools.

By examining students' conceptions of congruency when interacting with DGS, an underlying question in this paper is whether students' interactions with dynamic diagrams are different than interactions with static diagrams. By "dynamic diagram" we mean a diagram made with DGS and that has the potential to be changed in some way by dragging one or more of its parts. A diagram may appear static on a computer screen if students do nothing to it. However, inasmuch as a student could use the software to interact with the diagram, introducing changes to the diagram by dragging one of its component objects, we consider a diagram on a screen "dynamic".

1 Theoretical Framework

To describe how teachers and students interact with diagrams in geometry classes, Herbst (2004) has proposed four modes of interaction with diagrams: *empirical, representational, descriptive,* and *generative.* We use that framework to question whether affordances of DGS tools, particularly the possibility to drag and measure, expand students' "register of enabled action" (Vérillon 2000, p. 6), opening the range of possible actions upon diagrams in ways that justify a mode of interaction with diagrams unaccounted for in those four modes.

We focus on the way students used dynamic diagrams when solving problems that involved different conceptions of congruency. We use the expression *conception* following its usage in Balacheff and Gaudin (2003, 2009) to refer to stable ways in which a cognizing agent relates to their mathematical milieu when working on a problem. Building on Bourdieu's (1990, p. 83) notion of *sphere of practice,* Balacheff and Gaudin (2003) argue that *conception* can be used to refer to stable practices (in contrast with the canonical usage of the word "conception" to name mental structures). They propose that what the cognitive-based literature had alternatively called "misconceptions," "bugs," or "errors" are all comparable cases of adaptation between a cognizing agent (a student) and a milieu, under proscriptive conditions of practical viability. The cognitive literature has demonstrated the existence of contradictory conceptions in the same person. The relative independence of these different conceptions can be explained as sustained by different practical conditions and by different placements in time (including changes in time across settings such as in and out of school, or within a setting such as 'my class this year' versus

'my class last year'). To account for the different conceptions of a concept that may be present in a setting, this practice-based notion of conception uses the cK¢ model, where a conception is modeled as a quadruplet $C = (P_C, R_C, L_C, \Sigma_C)$,

P_C is a set of problems or tasks in which the conception is operational,
R_C is a set of operations that the agent could use to solve problems in that set,
L_C is a system of representation or semiotic register within which those problems are posed and their solution expressed, and
Σ_C is a control structure (for example, a set of statements accepted as true) used to validate the adequacy of the solution to the problem and warrant its solution

With this practice-based notion of conceptions we describe various conceptions of congruency that we observed in two high school geometry classes in lessons that were part of the same unit on quadrilaterals. We compare conceptions of congruency in the beginning 2 days of the unit, Lessons One and Two, when students had no access to DGS, with conceptions of congruency used in the following days, Lessons Three and Four. when they did work with DGS. We introduce some conceptions of congruency now that will be used in the analysis. Our main claim is that by incorporating DGS in Lessons Three and Four, students were enabled to use what we call a "measure-preserving" (MeaP) conception of congruency, which in turn enabled them to focus on the properties of figures.

The measure-preserving conception of congruency describes the sphere of practice in which a student establishes that two objects (e.g. segments or angles) are congruent by way of checking that they have the same measure (as attested by a measurement instrument). For example, a student could determine that two segments are congruent by measuring them with a ruler and verifying that the readings on the ruler for those segments are equal numbers. A student could also make two segments drawn on DGS congruent by measuring both of them using the measure tool and dragging one extreme, watching how its length changes as it is being dragged, and stopping the drag when the two measures on the screen were equal. The control "two segments are congruent if their measures are (close to) equal" is a key control element of this conception.[1]

The measure-preserving conception of congruency contrasts with customary practices in geometry courses, which support another conception of congruency that could be called "correspondence" (CORR; see Herbst and Balacheff 2009). In this conception two objects (segments or angles) are congruent if they are corresponding parts in two triangles that are known to be congruent. The CORR conception of congruence can be considered a particular manifestation of the "transformation" conception of congruency (TRANS). The transformation conception of congruency establishes that two objects are congruent if there is a geometric transformation, mapping one to the other, which preserves metric invariants. Thus, two segments are congruent as long as there is a movement that maps one segment onto the other one. The usual work done to transport a segment using a compass illustrates the "transformation" (TRANS) conception—a student can create a segment congruent to a given one by drawing a line and marking one point on that line, then placing each of the two tips of the compass at the endpoints of the original segment and finally lifting the compass without changing its size to place the point-tip on the point marked on the line and marking a new point on the line with the pencil-tip. When using DGS a student could make two congruent segments by constructing a circle selecting a center and given radius

---

[1] Note that we are here describing a conception of congruency that might help explain some behaviors of students when they are determining congruence. We are not asserting that such conception is correct or even adapted to describe all practices related to congruency.

(taking as radius the original segment) or by translating a point using the original segment as a vector. In general, the TRANS conception of congruency does not rely on equal measurements to control the correctness of the solution of the problem.

A fourth conception of congruency is what we call the "perceptual" conception of congruency (PERC). This conception relies on visual perception to control the correctness of a solution to the problem of determining if two objects (or more) are congruent. That is, within this conception, two objects are congruent if they appear to be so by looking at them. In contrast with the other conceptions of congruency, the PERC conception of congruency depends upon spatial and graphical features of the diagram. The orientation, size, or position of the diagram may affect students' ability to perceive whether the objects are congruent or not.

In this paper we study students' conceptions of congruency when combining two tools—a conceptual artifact and a technological tool. The question is: What conceptions of congruency were apparent in the discourse of a geometry class as students interacted with dynamic diagrams? We present evidence for the emergence of the measuring-conception (MeaP) of congruency, different from the perceptual (PERC) conception of congruency pervasive in class during the first two days of the unit. We also argue that students interacted with diagrams in a novel way, which we call the *functional* mode of interaction. In this study, students used the measuring and dragging capabilities of DGS to collect evidence about relationships between geometrical objects. The episodes show that the tools helped students make new inferences about congruence by interacting with dynamic diagrams, inferences that they could not make when interacting with static diagrams earlier the same week.

The significance of observing a MeaP conception of congruency in the context of students' interaction with dynamic diagrams is that we could foresee the instructional possibilities that could be built on that basis. Students' inferences supported by the measuring conception of congruency could become the seeds of "reasoned conjectures" (Herbst 2004, 2006)—bridging between students' empirical statements and theoretical arguments that connected them. The teacher could foster students' opportunities to learn by using students' inferences to envision a *hypothetical learning trajectory* (Simon 1995; Simon and Tzur 2004) and by designing future instructional tasks (Gravemeijer 2004) that might move students from a measuring conception of congruency to a transformation conception of congruency. The road between perception and theorization may not need to pass through measurement. But since much of students' prior knowledge of geometry is grounded on measurement, one should care to know whether measurement-based knowledge can evolve toward theoretical knowledge.

2 Research on Students' Use of Dynamic Diagrams

In this section we first review past research on students' interactions with static and dynamic diagrams. Secondly, we review how DGS tools enable students to discover characteristics of geometric objects. Thirdly, we present arguments found in the literature about the making of constructions with DGS tools. Finally, we explore the place of measuring in geometry, particularly in reference to measuring tools of DGS.

2.1 Differences Between Static and Dynamic Diagrams

The works of Fischbein (1993) and Herbst (2004) could help in understanding students' interactions with diagrams in the context of geometry. According to Fischbein, the

interplay between concepts and images is characteristic of the work of geometry students. Fischbein's notion of *figural concepts* denotes a relationship where mental images shape interactions with physical objects. Figural concepts "reflect spatial properties (shape, position, magnitude), and at the same time, possess conceptual qualities—like ideality, abstractness, generality, perfection" (Fischbein 1993, p. 143). According to Fischbein, geometrical reasoning is the study of figural concepts and general statements about figures must be supported by geometrical concepts and definitions rather than by visual perception.

The interplay between visual perception and geometrical notions is present also in Herbst's (2004) description of four modes of interaction with diagrams. In proposing these modes of interaction, Herbst (2004) attempted to describe epistemic relationships between geometric objects of discourse, the action of an agent, and material artifacts, in particular diagrams. Within the *empirical* mode of interaction, the agent (e.g., a student) uses proximal, physical experiences with diagrams as resources for making statements about geometric objects of discourse. These statements are the symbols that point to the properties of diagrams as referents. Conversely, within the *representational* mode, the agent uses distal physical experiences (oral declarations and questions) to describe how diagrams as symbols represent abstract geometric objects of discourse. These two modes of interaction, the empirical and the representational, portray two opposite views about how students may work with diagrams when solving problems in geometry. While the empirical mode takes physically experienced attributes of a diagram as crucial in making statements about geometric objects, the representational mode tends to arrange those attributes to fit definitions and properties of geometric concepts. For example, measurements could play an important role within the empirical mode in determining that two segments are congruent, while measurements would be irrelevant to show that two segments are congruent within the representational mode.

Herbst (2004) has proposed two other modes of interaction with diagrams that help describe current and possible practices in geometry classes: the *descriptive* mode and the *generative* mode. Within the *descriptive* mode, students maintain a distal relationship with the diagram but still use it as the source of statements that characterize the geometric object. They do so by using visual perception to anticipate what might be true and by correcting that perception through interpreting additional symbols (hash marks or arcs). In contrast with the descriptive mode, within the *generative* mode, students interact proximally with diagrams by introducing changes or by studying the feedback that diagrams provide on the possibility that a geometric object had a target property. Rather than reading off from that experience what the properties of the geometric objects are, students who interact generatively with a diagram use the definition and properties of the geometric objects to guide how they read the information given by a diagram, to decide what changes are sensible to make, and what possible interpretations to give to those physical experiences. Herbst has argued that by interacting generatively with diagrams students can start making reasoned conjectures, that is conjectures that spring from an analysis of the connections among the objects at play in a figure.

Past research suggests that students' interactions with dynamic diagrams are different than those with static diagrams. A main difference is that students can drag dynamic diagrams and see how measures change when dragging. It has been argued that this allows students to find characteristics of geometric objects that might remain hidden in static diagrams. Mariotti (2000) posits that static diagrams possess a dual, ambiguous nature. On the one hand, static diagrams show a particular figure, and on the other hand they present generic features of a class of objects: "Ambiguity between drawings and figures may represent an obstacle: although the question is about the drawing, its sense concerns

the geometrical figure it represents" (Mariotti 2000, p. 32). Thus, students' interactions with diagrams involve the interplay between drawings (which we call diagrams) and the geometric objects of discourse they represent, which we call figures (Parzysz 1988).

Laborde (2005) coined two terms to identify different properties of figures: spatio-graphical (SG) and theoretical (T). Spatio-graphical properties refer to those characteristics that are contingent properties of a figure but evoked by a particular diagram through perception or measurement (e.g., orientation in the plane, particular angle measures, etc.) whereas the theoretical properties refer to properties necessitated by the geometric relations that define the figure. A salient feature of dynamic diagrams is that by dragging their vertices students could create bridges across diagrams that might help apprehend figures. For example students could rapidly examine a collection of related diagrams and hypothesize abstract invariants that they could attribute to the figures that those diagrams represent; this could be especially powerful in the case where dynamic diagrams have been constructed using the software construction tools, which enables them to maintain some characteristics of the figure upon dragging. These kinds of diagrams may afford a more controlled variation, which may help students see changes on the diagram that correspond to different possibilities in the figure.

2.2 Possibilities of DGS Tools for Interacting with Diagrams

DGS adds novel tools to possible interactions that students have with diagrams. Operations such as drawing auxiliary lines or measuring are possible ways of interacting with paper and pencil diagrams that could also be used when working with dynamic diagrams. However, the dragging tool allows students to move objects and to see the effects of moving them. By dragging objects students can trace the locus of points, collect data about measurements, and perceive relationships between elements of a configuration. According to Goldenberg and Cuoco (1998), dragging allows a "continuous real-time transformation" (p. 351). De Villiers adds that dragging models continuous transformations by making discrete changes, since computers make discrete calculations (De Villiers 1998, p. 372). Those discrete changes might not be evident to the user, especially in new versions of DGS, which make changes in the screen appear to be smooth.

Some researchers have found special characteristics in the way students work with dynamic diagrams by using dragging. Hölzl (1996) found that students used the heuristic of *drag & link*. By dragging, they controlled the position of geometric objects to solve a problem. Then by linking, they created a dependency between geometric objects. *Drag & link* allowed students to manipulate the dynamic diagram so that it would appear to have properties that the diagram did not possess when it was initially created. Arzarello et al. (2002, 2007) identified different dragging modalities that students perform when working on geometric problems. These modalities exemplify how dragging enables students to manipulate dynamic diagrams in order to solve a problem differently than when working with static diagrams. Moreover, the dragging modalities show that students interpret actions that they can perform with the same tool—the dragging tool—differently, as they try to solve a problem (Zbiek et al. 2007). Hollebrands (2007) used the dragging modalities to characterize students' actions with dynamic diagrams and then focused on students' interpretations of results from dragging. She found two ways of reasoning when dragging: students relied on theoretical properties of geometric objects to explain how they should have constructed the diagram, and students explained the outcomes of dragging for their diagram without using theoretical properties. According to Hollebrands, students used *reactive* or *proactive* strategies when dragging, either in response to or in anticipation of

the effects on dragging. Most students in her study used reactive strategies. She concluded that teachers can help students use proactive strategies, and in doing so, they would have to rely on theoretical properties of geometric objects. These studies show that dragging enables students to apply novel strategies to interact with diagrams.

The interplay between figures and drawings is of interest to researchers who study how students interact with geometric objects with novel tools such as DGS. Laborde (2005) argues that dragging becomes an essential tool as students strive to connect spatio-graphical and theoretical properties of figures represented by dynamic diagrams. According to Laborde, students' interpretation of feedback from dragging dynamic diagrams provides new affordances when compared to their work with static diagrams. Moreover, she states that students combine visual perception with theoretical understandings of geometrical relationships when looking for invariants that result from dragging.

Students' reliance on spatio-graphical properties of dynamic diagrams, particularly when interpreting feedback from dragging, is one of the main differences with static diagrams. Some researchers have pointed to a distinction among diagrams, between *drawings* and *constructions*, as essential for students to go beyond visual perception and have a deeper understanding of relationships between geometrical objects (Chazan and Yerushalmy 1998; Hoyles and Noss 1994; Jones 2000; Mariotti 2001). This interest in geometric constructions seems to be one of the strengths of instructional uses of dynamic diagrams.

2.3 Constructions Versus Drawings

Dynamic geometry software includes tools that intend to model how compass and straightedge can be used to do geometric constructions. At the same time, students are also enabled to do free drawings with other tools, without having to use a dynamic version of a compass and a straightedge. Free drawing tools could enable students to make representations quickly without going through an elaborate process. A teacher who wanted to instill in students the need to construct geometric figures might see these free drawing tools as problematic.

In a research project where students used Cabri Geometry, Hoyles and Noss (1994) required students to make dynamic diagrams that could not be "messed up" (p. 717) by dragging. They argue that distinctions between drawings that are unrecognizable if you "mess them up" and drawings that could not be messed up allowed students to rely on dependence relationships between geometric objects. Moreover, by introducing the idea of drawings that could be messed up, researchers accomplished three goals— "It has given us a language in common with our students, provided a rationale for *constructing* rather than *drawing,* and afforded us all a mutually acceptable mode of validation for construction" (Hoyles and Noss 1994, p. 717). The notion that there are drawings that cannot be messed up underscores how dragging can help validate constructions. In that sense, students get direct feedback from DGS by dragging and by looking at the results from dragging. As long as students can compare different versions of a dynamic diagram that is subject to dragging to verify that some of the features of the diagram do not change, then students would validate their constructions without depending on the teacher.

Researchers have proposed that students' work with dynamic diagrams enables them to do a different kind of mathematical work than when working with static diagrams. Chazan and Yerushalmy (1998) have argued that students uncover general characteristics of geometric objects through dragging constructions. By allowing students to identify

invariants in a multitude of cases, dynamic diagrams have the potential of engaging students in the process of making conjectures. Jones (2000) takes students' abilities to make constructions using tools from dynamic geometry as a sign of students' "progressive mathematisation," because, with the aid of the tools, they have a deeper understanding of geometric properties. Similarly, Mariotti (2001) found that students who made constructions with DGS tools drew upon their experiences making those constructions to develop better arguments to craft proofs.

In sum, when students do constructions using DGS tools they can: (1) align their understanding of relationships between geometric objects with theoretical notions in geometry, (2) move towards abstractions and generalizations, and (3) get experience with geometric objects and their connections that can enable them to justify relationships between geometric objects and eventually produce proofs of theorems. The achievement of those objectives relies on dragging to test the validity of constructions. Researchers' emphasis on construction tools as essential for students' learning in geometry contrasts with worries about students' use of the measuring capabilities of DGS.

2.4 DGS and the Appropriateness of Measurements in Geometry

According to conventional wisdom, geometry students ought not to rely on measurements in order to make statements about geometrical objects. Within this view, measurements inhibit students from establishing conditions that would hold for general cases and students could mistakenly use empirical results as the basis of generalizations. Noss and Hoyles (1996) caution about a trend in the use of dynamic geometry in the U.K. where students are driven to make empirical statements about mathematical objects as part of instructional design. In particular, they worry that the use of dynamic geometry leads students to base their results upon measurements instead of theoretical considerations. Similarly, in a National Council of Teachers of Mathematics publication introducing the use of computers to teach geometry, Chazan and Houde (1989) advise teachers not to use measurements as the basis of statements, as part of the understandings and beliefs in a geometry class. They make strict distinctions between measuring and proving, arguing that one activity does not necessarily constitute evidence for the other one. According to Chazan and Houde, measure-preservation could suggest congruency. However, the value of measure-preservation is questionable because it could be the case that students, by means of using measurements, would not be relying on geometric properties. In contrast, if students were to use geometric properties to establish congruency, they would appeal to reason by relying upon theoretical relationships.

Research on students' use of measuring tools in DGS shows that some students confuse theoretical expectations and feedback from measuring. Olivero and Robutti (2001, 2007) propose using measurements in classrooms in two activities: for students to develop their conjectures and then for students to verify their findings. They state that design elements of DGS impose some limitations to students. For example, the measurements that appear in the screen are the result of calculations that users do not see. Olivero and Robutti suggest differences in students' ways of interpreting measurements depending upon the tools at hand, whether paper and pencil or DGS tools. They argue that, "Cabri in a way resembles the paper and pencil environment, because students can draw and measure (with a ruler) figures on paper too. The difference is that students can control the ruler better than the 'measuring tools' in Cabri" (p. 5). That is, students assign a margin of error to their measurements with rulers. However, students usually do not read the measurements in Cabri as intervals of values but as single values. Olivero and Robutti (2007) suggest that,

"Cabri acts as a mediator between the quantity to measure and the result of the measurements, and users' capacity to manage this depends on their awareness of the epistemological status of the objects on the one side, and the potentialities of tool on the other side" (p. 139). The invisibility of the measurement operations within dynamic geometry software requires students to have an understanding of the values that result from measuring, since they do not have to make decisions about approximations as they do when using physical tools such as ruler and protractor.

Reports of research in mathematics education on students' use of DGS measurements tools vary. Despite the possible danger of using dynamic geometry in schools solely as a tool for collecting numerical data, Noss and Hoyles (1996, p. 236) see the potential in using measurement tools within DGS in novel ways. Laborde (2001) reports that measuring did not help students to move forward in solving problems—showing a disconnected view of measuring from other activities that involved mathematical reasoning. Hollebrands (2007, p. 188) found that students applied reactive strategies, using measurements to formulate conjectures. But also, students used measurements as a proactive strategy to confirm that their construction had the desired properties. In addition, Olivero and Robutti (2007) identified different measuring modalities as students move from focusing on spatio-graphical properties to focusing on theoretical properties and vice versa. These measuring modalities are useful for understanding how students establish relationships between two processes: Making measurements and proving. So, on the one hand, researchers see measurements as a new affordance for students, while on the other hand, researchers also worry that measurements prevent students from engaging in exploring mathematical ideas deductively.

Dynamic geometry software packages include other tools besides measuring which students could combine when making explorations. Particularly, the combination of dragging and measuring can be an alternative for students who want to establish the initial conditions of a given diagram without necessarily making a construction. Healy and Hoyles (2001) present the case of two students, Tim and Richard, who were investigating quadrilaterals with two consecutive angle bisectors intersecting at a right angle. Tim and Richard made a general quadrilateral, *ABCD*, and two angle bisectors for angles *B* and *C*. They did not make a construction, but used measuring and dragging to set the angle bisectors at a right angle, and then made conjectures about the original quadrilateral.

> Thus the constraints of the required quadrilateral were not constructed but simply created 'by eye'. At the moment at which the angle between the two bisectors measured 90°, the boys noticed that BA was parallel to CD and they almost immediately conjectured, on the basis of this one example, that whenever the two angle bisectors were at right angles, BA must be parallel to CD. (Healy and Hoyles 2001, p. 244)

This account is relevant because Tim and Richard did not need to rely on constructions in order to fulfill the requirements of the investigation—but on a drawing that could be "messed up." Just dragging or measuring alone would not have sufficed. By combining dragging and measuring Tim and Richard set two perpendicular angle bisectors of a quadrilateral. Once the initial conditions were set—they had a quadrilateral with two perpendicular angle bisectors—Tim and Richard could focus on other features of the diagram. Using Laborde's (2005) terms, spatio-graphical properties, and not theoretical properties of the diagram led Tim and Richard to make a conjecture.

Olivero and Robutti (2007) report a similar case where students used measuring to confirm the results of Varignon's problem for a square. Students had proven that the

midpoint quadrilateral of a square is a rhombus. Then, they used Cabri tools to verify their result. Similar to Tim and Richard, they used dragging and measuring to make a "square," without *constructing* a square. Even though they checked that all sides of the avowed square were congruent, they did not check that all the angles were congruent. As a result, the measurements of the sides of the midpoint quadrilateral, which they had proven was a rhombus, were not the same. Their inspection of the differences in the measurements led them to conclude (incorrectly) that their proof was incorrect. The disagreement between the results from the measurements and the proof made them question the proof, instead of provoking them to question their work with the dynamic diagram. With this case, Olivero and Robutti show a case where students attributed more credibility to spatio-graphical properties—the results of measuring using DGS tools—than to theoretical properties—the result of their proof (see Chazan 1993).

In this paper we present episodes where students used dragging and measuring in an approach similar to that of Tim and Richard. We argue that the combination of dragging and measuring enabled students to investigate relationships between a quadrilateral and the quadrilateral resulting by connecting the midpoints of the original one. With paper and pencil, students would have had to draw or construct many individual cases by hand. In contrast, a dynamic diagram has the potential to show many examples at once by dragging. Also, by measuring, students could attend to properties other than those apparent by visual perception and use the numerical values as feedback to manipulate the dynamic diagram until they could guarantee that the initial conditions are set. In that way, students would be starting the investigation by making diagrams that are not constructions, supported by a measure-preserving conception of congruency. The use of dragging and measuring in the geometry class contrasts with teachers' usual reluctance to measuring and their support of a correspondence or a transformation conception of congruency. At the same time, by dragging and measuring students were not just relying on visual perception. Students used numerical values in various ways to support their investigation. We argue that students' "register of enabled action" (Vérillon and Rabardel 1995, p. 6) in our data included mathematical and non-mathematical elements that imposed constraints and possibilities on students' way of relating to the problem at hand. In analyzing students' conceptions of congruency, we focus on ways in which students acted upon dynamic diagrams and talked about noticeable changes in the diagram as they participated in classroom discourse. Classroom discourse changed as students tried to incorporate feedback from their interaction with dynamic diagrams in their public presentations of results of their findings.

3 Data Collection and Analysis

3.1 Data Sources

The main data sources in this work are video records of an instructional intervention in two sections of honors geometry (2nd period and 7th period) taught by the same teacher, Ms. K.[2] Other data sources such as student interviews, homework assignments, and audio recordings of students' work in groups give contextual clues for the analysis of the videos. The intervention included the replacement of the usual unit on quadrilaterals by a set of problems and activities designed in collaboration between the teacher and the authors. During these lessons we set two cameras in the room. One camera was in the back of the

---

[2] All names are pseudonyms.

room and followed whoever was at the front making presentations, usually the teacher. The other camera was in the front of the room, taking a wide angle shot at the class.

During the first two days of the 12-day unit, students created a table of properties relating characteristics of a quadrilateral to the characteristics of its "M-Quad" (the dual quadrilateral, formed by connecting successive midpoints of the original quadrilateral). Students tried to answer the question: "What quadrilateral would you need to start from in order to get an interesting M-Quad?" This paper focuses on the 3rd and 4th days of the unit where students used two artifacts:

1. A conceptual artifact: A table relating properties of an outer quadrilateral to properties of the quadrilateral resulting from connecting the midpoints of the original one created in common by all students in each class during the first two days of the unit. Each student had a copy of this table at the beginning of the 3rd day (see Appendix 1).
2. A technological artifact: A calculator TI-Voyage^{TM} 200 with Cabri^{TM} for each student that they could keep in class and take home during the unit.

The table and the calculator challenged students' normal ways of doing mathematics in Ms. K's class. The table required students to talk about quadrilaterals in terms of their properties instead of referring to special quadrilaterals by their names (e.g. square, rhombus, trapezoid, which they knew previously as shown in a test administered before starting the unit). The combination of properties allowed students to explore quadrilaterals that may not even have a special name, but it constrained students from referring to known quadrilaterals as they usually do. Students did however use the names of special quadrilaterals in identifying the resulting figure.

The use of dynamic geometry was another novel artifact in Ms. K's class. Usually students sketched their diagrams by hand, drew diagrams with measuring tools such as a protractor and a ruler, or did elementary constructions using a compass and a straightedge. Dynamic diagrams were new to students and different from diagrams that one could do with other software tools, in the sense that consecutive sides of a polygon would remain connected when dragged instead of turning into disjoint pieces. The table of properties was supposed to set the initial conditions of the outer quadrilateral guiding students' exploration with the calculator. Students had to manage how to relate the table of properties and the calculator.

3.2 Criteria for Selecting the Episodes

We chose episodes where students used the dragging and measuring features of DGS to manipulate dynamic diagrams in their public presentation of results. Measuring and dragging appeared to provide new affordances for students to discover relationships between a quadrilateral and the quadrilateral formed by connecting successive midpoints of the sides of the former. Even though the teacher usually did not encourage measuring in this class, measuring and dragging became essential in the lessons chosen for students to pursue their findings. In the selected episodes, two students presented results to the class combining the dragging and measuring features of DGS.

3.3 Methodology

Methodological approaches in this study include the analysis of videos of the lessons using discourse analysis tools (Erickson 2004; Halliday and Hasan 1976; Halliday and Matthiessen 2004; Pimm 1987). In particular we looked at the way that speech acts and gestures

aligned with changes introduced to dynamic diagrams during students' public presentations building on Rowland's approach to examining the use of pronouns and hedges in mathematics classrooms. Rowland's (1999) study of pronouns shows that these have different functions in mathematics talk. Most times, pronouns show ambiguity in communication. Other times, pronouns are clear about their referent. In mathematics, pronouns are tools for generalizations. Rowland's (1995) study of hedges shows how these are used depending upon the context to signal either predictions or generalizations. The uncertainty of hedges could also be used in categorizing numerical values. Pimm's (1987) discussions of mathematical register in classrooms and a lexical cohesion analysis (Halliday and Hasan 1976; Halliday and Matthiessen 2004, pp. 535–538) were helpful in understanding how mathematical and non-mathematical terms provided affordances and constraints for students.

We coded the episodes where students made public presentations of results using dragging and measuring by identifying the mathematical terms and other terms introduced by the teacher in the class. This analysis helped us track the use of mathematical language as a tool in students' explanations. We also coded the episodes by looking at the function of hedges in characterizing numerical values and in talking about dynamic diagrams. In addition, the study of pronouns was helpful for understanding students' generalizations and predictions.

In explaining students' conceptions of congruency the cK¢ model (Balacheff and Gaudin 2003, 2009; see also Herbst 2005; Mesa 2004), guided our analysis of students' claims to congruency about figures using dynamic diagrams. We characterized students' conceptions of congruency using the quadruplet (P, R, L, $\sum$) proposed in the cK¢ model. Students made active use of dynamic diagrams and of DGS tools such as measuring and dragging. In that sense, discourse was multi-modal because students' gestures compensated for the lack of technical language. Also, other actions such as dragging and measuring became important when talking about noticeable features of the diagram. The teacher's feedback allowed students to make public interpretation of results in the emergent moment-by-moment interaction.

4 Analysis and Results

4.1 Students' Initial Conception of Congruency

During the first two days of the unit, students had drawn paper and pencil diagrams to complete the table of properties. Students had made their assertions either by measuring one diagram with a ruler and a protractor or by using visual perception. During the 2nd day of the unit, each class managed to produce a proof for some of the relationships in the table. Most of their conjectures can be described as responding to a perceptual (PERC) conception of congruency.[3]

The attribution of a perceptual conception of congruency (PERC) to students' actions is based on the evidence of students' work during the first two days of the unit. Students

---

[3] In the second period, the teacher recorded in the table students' results after a brief discussion, and made diagrams to illustrate one of the ten entries of the table. In the seventh period, the teacher made diagrams for two out of the nine entries of the table. In addition, she made diagrams for one case that was not recorded on the table. In those cases, the teacher asked students to look at the diagram and identify the midpoint quadrilateral.

started with a diagram of a quadrilateral that had some specified properties. Then, students created the midpoint quadrilateral and named that quadrilateral according to the special quadrilateral it appeared to be (e.g., square, rhombus, parallelogram, rectangle). After some time for the investigation in small groups, the teacher led class discussions where students talked about properties of the midpoint quadrilateral. Each class gathered the results of their investigation in the table (see Appendix 1). The perceptual conception of congruency was apparent when students controlled how they assigned a name to the resulting figure, considering which parts (e.g. sides and angles) appeared to be congruent. We propose the following as a characterization of the perceptual (PERC) conception of congruency:

P_{PERC}	Problems	To recognize a given quadrilateral as belonging to a class of special quadrilaterals
R_{PERC}	Operations	To observe the figure resulting from connecting midpoints—rotating the paper, ignoring the original quadrilateral. Compare the aspect of the reoriented quadrilateral with the prototypes of special quadrilaterals. Assign a name to the midpoint quadrilateral
L_{PERC}	Semiotic register	The diagram is the medium for the presentation of the problem. The main elements of the semiotic register here are strokes on paper
		Discourse is the medium for the solution
		Mathematical terms such as congruent, angles, midpoint, parallel, and quadrilateral
		Mathematical terms for special quadrilaterals such as parallelograms, rhombus, squares, and rectangles
		Indexicals and gestures to relate discourse to parts of the diagram
$\sum_{PERC}$	Control structure	If it looks like a diamond, with equally long slanted sides, it is a rhombus
		If it has equally long horizontal and vertical sides, it is a square
		If it has unequally long horizontal and vertical sides, it is a rectangle

Most students used visual perception to identify congruent parts and to name the resulting quadrilateral. Some students, however, had used measuring tools (rulers and protractors) in their investigation to create the quadrilateral they started with and possibly used these to support their findings. At the beginning of the 3rd day, the teacher explained how to combine the table of properties and the calculator. Her explanation opened the possibility for students to measure using DGS tools. This new affordance changed the nature of the *task*. We use *task* to mean the system of dialectical interactions between a cognizing agent and a mathematical problem (Herbst 2006). Even when the problem did not change from the first 2 days of the replacement unit, students could apply novel operations, use other resources, and possibly get new products (Doyle 1988), all of which effectively changed the task.

Ms. K. gave specific suggestions on how to combine the table and the calculator. She endorsed measuring as an operation that students could perform in order to translate the initial conditions set in the table into properties of a dynamic diagram. Measuring became one of the affordances of DGS in this class, opening students' register of enabled action. At the same time, Ms. K. did not emphasize making constructions to meet the conditions of the table. Thus, students' diagrams were such that they could be "messed up."

In the 2nd period, the teacher referred to different ways of verifying their conjectures—by drawing on paper and pencil, by proving, or by measuring. The focus of the investigation was on the resulting figure. Measuring and dragging enabled students to go a step
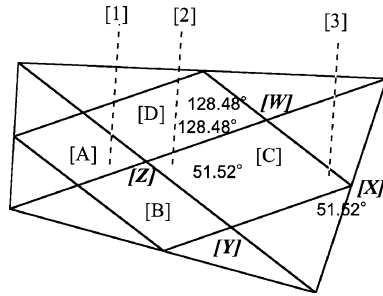
Fig. 1 Brett's diagram on the board. Added features in *brackets* show parts of the diagram that Brett pointed to during his presentation

further than looking at just one case on paper and pencil. At the same time, measuring allowed students to study other cases where they had not written a proof. In the 7th period, the teacher endorsed measuring as a way to verify that the quadrilateral had two congruent sides. The focus of the class was to match the givens in a dynamic diagram with the characteristics of the outer quadrilateral in the table of properties. This action is analogous to the episode described by Healy and Hoyles (2001) where Tim and Richard used dragging and measuring to set the initial conditions of the object at stake—a quadrilateral with a set of consecutive perpendicular angle bisectors. However, in the 7th period nobody made explicit references to dragging as the other action that should be combined with measuring in order to set the initial conditions of the quadrilateral. In the following episodes, two students combined dragging and measuring in their public presentation of results to class.

5 Students' Presentations of Results

5.1 The Parallelogram Episode

Approximately 45 minutes into the 2nd period class, during the 4th day of the replacement unit, Brett presented to class a quadrilateral with its diagonals and its midpoint quadrilateral. He said that one of the figures determined by the midpoint quadrilateral and the diagonals of the original quadrilateral was a parallelogram (see Fig. 1). Brett had applied the Cabri measuring tool to measure specific angles in the diagram. His diagram showed that the measurements of opposite angles of the avowed parallelogram were congruent. Prior to the replacement unit on quadrilaterals this class had studied properties of parallelograms and what conditions would allow one to prove that a quadrilateral is a parallelogram. Brett's conjecture was supported by a theorem they had proved: If opposite angles of a quadrilateral are congruent, then the quadrilateral is a parallelogram.[4]

Brett: You know, uh, when you draw the diagonals [traces the diagonals; see Fig. 1] I realized (that)

Ms. K.: (The diagonals) of what?

Brett: Oh, of the o-quad and you make the m-quad of the o-quad, uh, the diag or the [points to 2], the inner angle of like these two angles [points to 1 and 2] or this angle

---

[4] Transcription conventions include descriptive comments in brackets. These describe actions or visual elements on the board.

[points to 2] is congruent to this angle [points to 3]. So, it makes a parallelogram [traces quadrilateral *WXYZ*] on the inside.

Ms. K.: They are all parallelograms. (Right?)

Brett: (They are all, yeah.) Each one of these [points to regions *A*, *B*, *C*, and *D*] is a parallelogram.

Ms. K.: It is sort of a cool thing 'cause they are not all equal. You can see—here drag it around and see if you can make them even more different sizes. You know how that inside ones are.

Brett: Uhm. [Drags upper left vertex of outer quadrilateral.]

Brett started dragging and while the figure momentarily disappeared from the screen, he managed to pull it back. The teacher commented on this new version of the dynamic diagram as more evidence for stating that there were four parallelograms.

Ms. K.: Okay. Here, stop. That's pretty good. You can see those four things he has inside there are still parallelograms. But they are not anywhere close to the same size or anything. It's sort a cool thing.

In analyzing the parallelogram episode we point to four main themes. First, we analyze Brett's active interaction with the diagram through actions such as drawing the diagonals of the outer quadrilateral, pointing, tracing, and talking about the diagram. Secondly, we focus on the teacher's feedback to Brett's presentation and her request to drag the diagram. Thirdly, we study the effects of dragging and the kinds of assertions that the teacher makes about the dynamic diagram. Finally, we look at what conception of congruency can be ascribed to Brett's interaction with the diagram.

Brett's use of the diagram reminds us of the *generative* mode of interaction with diagrams (Herbst 2004). Brett added the diagonals of the original quadrilateral, thereby generating new figures on which he made claims that helped him address the question posed. In addition, Brett measured the angles of the conjectured parallelogram. By pointing and tracing, Brett emphasized which parts of his diagram were relevant in posing his conjecture that one of the inner figures was a parallelogram. The angle measurements supported his conjecture as evidenced in Brett's use of "so" when he said, "So, it makes a parallelogram." The conjunction "so" has the purpose of denoting causal-conditional relationships (Halliday and Matthiessen 2004, p. 543). "So," which in mathematics means, "therefore," signaled the conclusion of Brett's claim: *The inner quadrilateral is a parallelogram because opposite angles are congruent*. Brett used the diagram with the angle measurements to make his claim.

Brett's mathematical register included vocabulary terms that had been introduced in this class such as parallelograms, diagonals, and angles. Brett also used the terms *o-quads* and *m-quads* introduced in this class by the teacher during the 1st day of the unit. These terms, which would not appear in a mathematics textbook, allowed Brett to refer to the given and the resulting figure as he spoke. However, Brett noticed something else that was not part of the original problem assigned by the teacher. He said that there was a parallelogram on the inside. This conjectured parallelogram was not the focus of the investigation—the new figure was the result of Brett's initiative to add the diagonals of the outer figure. At the end of the previous day, the class had discussed ideas about diagonals of special quadrilaterals. Yet, the teacher had not assigned a name to this figure as she had done with *m-quads,* which were anticipated figures. Brett's use of indexicals when he said, "the inner angle of like these two angles or this angle is congruent to this angle" shows how Brett used geometrical objects which do not have an official name, but which he could point to.

"This" and "these" are part of the language that he spontaneously used as a "bricoleur" (Erickson 2004, p. 166) taking advantage of the moment-to-moment interactions. Deictics such as "this" and "that" helped Brett in keeping up with the flow of his explanation, by also integrating gestures, which compensated for the lack of technical language to refer to the new objects generated in this kind of interaction with the diagram (see Herbst 2004, p. 134).

Brett's diagram did not show its dynamic features before the teacher requested him to drag the outer quadrilateral. In spite of being static, Brett's diagram illustrated how he used some of the features of DGS to pursue his investigation. By adding the diagonals of the outer quadrilateral and by measuring the angles of the inner figure, Brett appeared to have made a shift in his use of the calculator from an *artifact* to an *instrument*. The notion of a shift from artifact to instrument is used to denote changes in how users accommodate to new situations because of the use of tools (Rabardel 1995; Vérillon and Rabardel 1995). The actions of adding auxiliary lines and measuring are not exclusive to DGS. Brett could have done these two actions with paper and pencil. However, it may be the case that the availability of DGS tools (in the various menus) prompted Brett to draw the diagonals and to measure the angles of the inner figure.

This shift in using the tools of DGS to modify the diagram and to use it as a partner in developing new conjectures is an important feature of what Herbst (2004, p. 135) calls the *generative* mode of interaction with diagrams. The calculator was an affordance for Brett's generative interaction with the diagram because he used the menus and functions of the calculator to introduce changes to the diagram. At the same time, Brett incorporated these modifications and interpreted the feedback from measuring in proposing his conjecture.

With her feedback, Ms. K. approved and extended Brett's conjecture. Ms. K. repeated geometrical terms introduced by Brett. Ms. K. also noticed new things about the inner figures on the spot. Then, Ms. K. requested Brett to drag the diagram, apparently to stress that not all the avowed parallelograms were congruent to each other. As Brett dragged his quadrilateral, the measurements of opposite angles of Brett's parallelogram continued to be congruent to each other in each of the different versions of the dynamic diagram. This showed more evidence in support of his conjecture: While the angle measures changed, the class could see in the display that the equality of those angle measures persisted in spite of dragging.

Brett's way of supporting his conjecture illustrates Laborde's idea of the interplay between theoretical and spatio-graphical properties of dynamic diagrams (Laborde 2005, p. 172). On the one hand, spatio-graphical properties of the inner figure may have prompted Brett to measure the inner angles. Brett could have used measurements to check a conjecture that he might have formulated by using visual perception (Olivero and Robutti 2001, p. 7). On the other hand, Brett's use of measuring to assert that the inner figure is a parallelogram, had a theoretical component, based upon the property that parallelograms have opposite congruent angles.

5.2 Mathematical Ideas Underlying in Brett's Conjecture

Brett's conjecture, that the dual quadrilateral is a parallelogram, is true for any quadrilateral.[5] The justification of this conjecture lies on what Ms. K. and her class would later call the "medial-line theorem." This theorem[6] states that the segment connecting the

---

[5] Also known as Varignon's theorem.

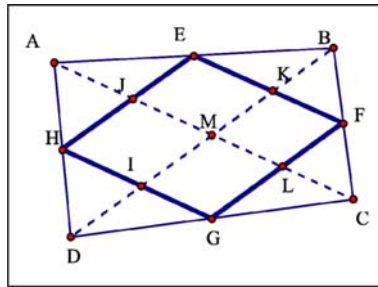[6] Also known as the midpoint connector theorem.

Fig. 2 Analysis of Brett's conjecture

midpoints of two sides of a triangle is parallel to the third side and half its length. We refer the reader to Fig. 2 for an analysis of the underlying mathematical ideas in Brett's conjecture. Applying the "medial-line theorem" to triangles *ABD* and *CBD*, and noticing that *H*, *E*, *F*, and *G* are midpoints, $\overline{BD}$ is parallel to and twice as long as $\overline{HE}$ and $\overline{GF}$. Similarly, it follows that $\overline{AC}$ is parallel to and twice as long as $\overline{HG}$ and $\overline{EF}$. Thus $\overline{HG}$ and $\overline{EF}$ are parallel and congruent and, by the same argument, $\overline{HE}$ and $\overline{GF}$ are parallel and congruent. Since $\overline{MK}$ is contained in $\overline{BD}$ and $\overline{LF}$ is contained in $\overline{GF}$, then $\overline{MK}$ and $\overline{LF}$ are parallel. By a similar argument, $\overline{ML}$ and $\overline{KF}$ are parallel. Since opposite sides of the quadrilateral *MKFL* are parallel then the quadrilateral is a parallelogram. The same would hold for the other four quadrilaterals that make the dual quadrilateral, *HIMJ*, *MIGL*, and *JMKE*.

5.3 The Measure-Preserving Conception of Congruency as Seen in Brett's Work

The Measure-Preserving (MeaP) conception of congruency is useful to explain Brett's interaction with the diagram. We model this conception using the (P, R, L, $\sum$) quadruplet proposed in the cK¢ model.

P_{MeaP}	Problems	To determine whether a quadrilateral has the congruence properties needed to deserve being identified as a special quadrilateral
R_{MeaP}	Operations	To measure the pairs of opposite angles of the quadrilateral using the measuring tool in Cabri. This involves selecting the measuring tool and then selecting the angles to measure
		To drag the quadrilateral while keeping its angle measures on the screen
L_{MeaP}	Semiotic register	The calculator screen is the medium for the problem and its solution
		Icons and buttons in Cabri for constructing segments and for measuring the angles
		Numerical values for the measurements of angles (in degrees)
		Indexicals and gestures to refer to parts of the diagram such as opposite angles, diagonals, and inner parallelograms
$\sum_{MeaP}$	Control structure	If angles have the same measure they are congruent
		If two angles continue to be equal to each other, although different in value, upon dragging, they are always congruent
		If pairs of opposite angles in a quadrilateral are congruent then the quadrilateral is a parallelogram

The solution using a dynamic diagram allows seeing many examples with just one diagram. The original quadrilateral must be connected in the sense that points and segments preserve the conditions of a quadrilateral even when dragging. The original quadrilateral does not need to have special characteristics because the conjecture holds for any quadrilateral. Ms. K.'s request for dragging allowed students to see many examples where Brett's conjecture held. However, nobody in class made explicit references to this. The control structure was part of the practices but was not remarked upon. We say that there is a measure-preserving conception of congruency because the invariance of the relationship between measures of opposite angles was used to decide on the properties of those angles.

The notion that two angles were congruent because they had equal measure supported Brett's conjecture about the inner quadrilateral. Having congruent opposite angles implies that the figure is a parallelogram, a theorem that the class had already studied. Behaviors interpretable with the measure-preserving conception of congruency (MeaP) surfaced as a result of using measuring tools in DGS. This conception is different from the visual perception conception of congruency (PERC), which helps understand what students did in the first 2 days of the replacement unit. Brett's use of the measuring tool turned the calculator screen from an artifact into an instrument, because he went beyond visual perception to using the dynamic affordances of the calculator to make a reasoned conjecture about the inner figure.

5.4 The Square Episode

Anil's presentation in the 7th period followed Ms. K.'s explanation on how to combine the table of properties and the calculator. When Ms. K. asked him to describe what he had found, Anil said that he had started with a diagram that was "like a square." He combined measuring and dragging to make a square—setting the initial conditions of the quadrilateral to pursue his investigation. Then, he used measuring and dragging to generate other squares. Anil found an interesting numerical pattern for the ratio of sides of a square and its dual, which he verified in several different squares.

Teacher: Okay, who found something that maybe wasn't something that we had seen before? Anil, you said you found something interesting, what was that?
Anil: Well, like, um, my thing I made like a square.
Teacher: Yeah, okay, here, plug it in.

Anil went to the front of the room, connected his calculator to the overhead projector, and showed a diagram of a square (see Fig. 3).
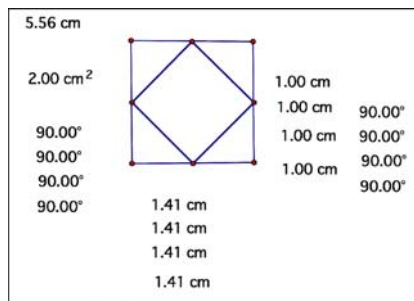

Fig. 3 Anil's initial diagram

Anil dragged the vertices of the quadrilateral, while keeping track of the measurements of its sides and its angles, until all the angles measured 90 degrees and all the sides were the same length. That is, Anil did not construct a diagram that would remain a square upon dragging. Rather, he measured the sides and angles of a generic quadrilateral and watched these measurements on the screen as he was dragging the diagram to make something "like a square." He stopped when he saw that all angles were 90 and sides were equal. At that point, Anil calculated the ratio of the perimeter of the outer square to a side of its midpoint quadrilateral and found that it was "like 5.66." Anil then changed the vertices of the original quadrilateral, to generate another square with different side length. Then, he calculated the ratio of the perimeter of the new square to the side of its dual quadrilateral. In several iterations of this process Anil noticed that this ratio appeared to be around 5.6, which he considered to be a significant discovery, even though he could not articulate why it was true.

Anil: You know, that's the perimeter and that's the a- the perimeter of the outer quad the area of the outer quad. And like each of the angles of the outer quad and the two segments of the outer quad. And on this side, these are the each of the segments of the inner quad.

Teacher: Okay.

Anil: And the angles of the inner…

Teacher: Okay and then what did you notice?

Anil: Yeah, and uh like I made a square. It was like 5.66 centimeters the perimeter of the outer quad. And the uh each of the segments were like 1.41 like each of them. And I just like took, wait, yeah, each of the inner quad, each of the segments are like 1. So I just took that and then I uh divided 5.6 no, divided one by 5.66. And I, uh… And like, yeah and then it was like 5.66 as a result. And then like, I did like, I just made the square a little smaller and the perimeter was 4.55 and each segment was like .8 centimeters. I did that and I divided and I got 5.68. And like, they all came near 5.6.

Teacher: Okay and why do you think that's true?

Anil: Um.

Teacher: You are dividing the perimeter of…

Anil: The outer quad by the um segments, of each of the segments of the inner quad, or of the square.

Teacher: Okay, here, let's go back to your picture.

Anil had moved out of Cabri to the calculator functions when finding the ratios. Ms. K. asked him to go back to the Cabri screen, to show his diagram. Her further questions led Anil to explain again what he did. Anil demonstrated that he confirmed his results by making a smaller square and still getting the same ratio. Ms. K. noticed that for that particular case, he was dividing by one. But Anil had illustrated cases where he divided by other values as well.

Teacher: Okay. Um. You divided. Let's look at, you divided the…

Anil: Perimeter.

Teacher: Perimeter of the outer thing…

Anil: Yeah by=

Teacher: = By the side of the inside one.

Anil: So, like whenever I make this smaller, like one to the left [and one up].

Teacher: Do you see you are still very close to dividing by one?

Anil: Oh, yeah. But then like the numbers don't, like, and then it just, first, I found that it's smaller. It's close to 5.66. [Talking to self: no, wait, I messed up.]

Teacher: Okay, so Anil looked at some things that some people probably didn't look at, the area and perimeter. I don't think I saw anybody looking at that in second hour. Did anybody else looked at area and perimeter? No.

Our analysis of this episode includes four different themes. First, we focus on the tools that Anil used, including calculator functions and language. We argue that semiotic tools were important mediators in his investigation. Secondly, we study Anil's hedges, using Rowland's (1995) classification of hedges to assess the soundness of his assertions. We posit that these hedges allowed him to present what seemed to be an interesting conjecture to class, even when he did not know the reasons why this conjecture held. Thirdly, we look at Anil's interaction with Ms. K. We argue that as she strived to understand Anil, he became more articulate about his procedures, opening possibilities to build shared knowledge in class. Finally, we discuss the underlying mathematical pattern in Anil's discovery and how the measuring conception of congruency describes behaviors that were crucial in enabling him to propose his conjecture.

Anil's work required him to switch between different screens and function menus in Cabri and in the calculator. Anil used DGS tools to set the given quadrilateral as a square and generate other squares by measuring lengths. Anil also used other features of the calculator, to find the ratio. If he were doing this exploration by hand, he would have needed to draw, measure, and calculate, using different tools such as paper, pencil, protractor, ruler, and calculator. The TI-Voyage^{TM} 200 included all these functionalities in one artifact. Therefore, students could potentially perform different actions—measuring, dragging, and calculating—with just one artifact—the calculator. Analogous to Brett's case, we hypothesize that the availability of the calculator made some of these actions possible. Anil's use of the calculator exemplifies what it means to use the calculator as the basis of instrumented activity because he applied different motor skills, used different tools, and adapted the use of the artifact to the needs of solving a problem (Rabardel 1995; Vérillon and Rabardel 1995).

Anil also used language tools as mediators in his explanation. He used two kinds of technical language. On the one hand, the mathematical register included well-known terms such as square, perimeter, and area. On the other hand, he used special terms that had meaning in his class. While the expressions "outer-quad" and "inner-quad" are not part of the mathematical register, Ms. K. had introduced them when she introduced the problem on the 1st day of the unit. Ms. K. had not formally introduced area and perimeter in this class. These topics are usually covered in the second semester of the geometry course and this unit took place during the first semester. Similarly, squares had not been officially defined in this class yet. Thus, Anil relied on his prior knowledge of geometric terms when talking about squares, area, and perimeter. Anil's appropriation of the terms "outer-quad" and "inner-quad" allowed him to pay attention to special relationships between parts of the figures.

The activity required Anil to talk about properties of the quadrilaterals at issue. While he could not refer to these quadrilaterals by their names—a square, a rhombus, a parallelogram—the table of properties discouraged students from referring to special quadrilaterals by their names. Students had to talk about characteristics of the figures—such as congruent sides, congruent angles, parallel sides—instead of merely listing their names. Thus, while in different circumstances Anil might have ended his investigation saying,

"the inner quad is a square," the focus on properties pushed him to look more in depth for other features of the dual quadrilateral. Anil reversed some of the uses of the table by starting with a square (characterized by four congruent sides and four congruent angles) and focusing on the properties of the M-Quad (see Appendix 1). We argue that the requirement to complete the table with properties allowed Anil to focus on characteristics of the geometric figures that otherwise could have been taken for granted, ignored, or dismissed.

Anil used hedges to qualify his assertions. His use of "like" when he said, "I made like a square," could be described as an "Approximator" (Rowland 1995, p. 335). According to Rowland (1995, p. 335), "Approximator-hedges are located inside the proposition itself. The effect is to modify (as opposed to comment on) the proposition, making it more vague". In this case, Anil's use of "like" suits Rowland's (1995, p. 336) definition of an "Adaptor". Adaptors are a type of Approximator, which "suggest, but do not define" (p. 336). Anil could have hesitated to call his dynamic diagram a square because it does not stay as such when dragged. So by hedging, Anil was able to establish some distance about calling the figure a square.

Anil's use of "like" could also be attributed to the pervasiveness of innovative quotative markers in teen-talk (Bucholtz and Hall 2005). For example, when Anil said, "like I made a square" and when he said, "And like, they all came near 5.6," the use of "like" does not qualify the numerical values. However, in most cases, we found that Anil used hedges to describe the numerical values that resulted either from measuring or from his calculations. For example, he said that the perimeter of the outer quadrilateral "was like 5.66 centimeters" and that "each of the segments are like 1." In contrast, Anil did not use hedges to characterize numerical values in a few assertions as when he said, "I got 5.68." He reported the value of his calculation "5.68" without hedging. But when he said "near 5.6," hedging had the function of rounding to compare several values within the same range. Rowland calls these hedges "Rounders" (1995, p. 335), a type of Approximators. According to Rowland, "Rounders are common in the domain of measurements, of quantitative data" (1995, p. 335). Anil needed hedges to put forth his argument that there was something interesting in this data because numerical values of his calculations were not equivalent, but within the same range.

Anil's presentation of results happened within the context of a geometry class where Ms. K. had provided students some affordances by incorporating the two artifacts—the table of properties and the calculator—in the investigation during the 3rd and the 4th day of the unit. Ms. K. gave students the opportunity to make public presentation of the results. Her feedback during these presentations was essential for students to make sense of their work in public, through the moment-by-moment interactions with the teacher. We take the dialogue between Ms. K. and Anil as an opportunity for eliciting and constructing mathematical ideas through interactions with the teacher (Zack and Graves 2001).

Ms. K.'s intervention during Anil's presentation fulfilled the dual goal of helping him to be more articulate about the process by which he formulated his conjecture and opening up his conjecture to public scrutiny. Ms. K. asked Anil to present his findings to the whole class. When she asked, "what did you notice?" Anil had to go beyond reporting what he did towards articulating his conjecture. By asking, "why do you think that's true?" Ms. K. pushed Anil to the phase of justifying his results. Anil did not give an answer and Ms. K. posed other questions leading Anil to describe what he did. In order for other students and for Ms. K. to have a shared understanding of Anil's conjecture, they needed him to explain what he did several times. Anil's active use of

the diagram through dragging and measuring while presenting his result was crucial in reaching a shared understanding. Even though there is no evidence in this excerpt about how other students reacted to Anil's conjecture, Ms. K.'s repetition of Anil's statements signals that she was following his explanation. This use of repetition in discourse either by revoicing (see O'Connor and Michaels 1996) or by completing the sentences started by the other interlocutor could be taken as a sign of making meaning through dialogue (Zack and Graves 2001, p. 263).

5.5 Underlying Mathematical Ideas in the Square Episode

A study of the underlying mathematical ideas in determining the ratio of the perimeter of a square to the length of the side of its dual quadrilateral may help in understanding the significance of Anil's conjecture for the observer and perhaps for the teacher. The constant ratio that Anil noticed derives from the similarity between a square and its dual. The scale factor of the square to its dual is $\sqrt{2} : 1$. Since the given quadrilateral was a square, then the segments connecting the midpoints of the square determine four isosceles triangles. Because of the Pythagorean theorem, each side of the dual quadrilateral have a length equal to square root of 2 times half of the side of the original square. That is, if the side of the original square is $2x$, then the side of the inner figure (which is also a square) is $x\sqrt{2}$. Anil found the ratio of the perimeter ($8x$) of the outer quadrilateral to the length of the inner quadrilateral $\left(x\sqrt{2}\right)$. This ratio is $\frac{8x}{x\sqrt{2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2} \approx 5.66$. That is, the ratio he chose to calculate is, as he thought, independent of the dimensions of the square (see Fig. 4).

The variations in Anil's calculations could have been the consequence of the fact that Anil did not construct a square. But it is also possible that those calculations varied because of limitations of the artifacts in the measurements and in approximations of radicals. Anil overlooked variations in the numbers and his interpretation was similar to what Olivero and Robutti (2001) claim to be needed when reading numerical values, which results from the use of DGS tools—the understanding of intervals or range of values.

5.6 The MeaP Conception of Congruency as Seen Through Anil's Work

We conclude the analysis of the perimeter episode describing the conception of congruency ascribable to Anil's presentation.
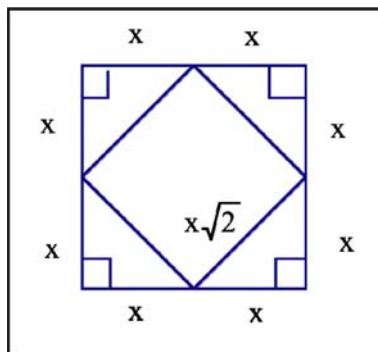

Fig. 4 Underlying mathematical ideas in Anil's presentation

P_{MeaP}	Problem	To make a figure with specified metric properties.
R_{MeaP}	Operations	Make a connected quadrilateral using Cabri tools
		Measure the sides and store the measures on the screen
		Measure the angles and store the measures on the screen
		Drag the vertices of the quadrilateralss
L_{MeaP}	Semiotic register	The calculator is the medium for the problem and its solution
		Dragging (hand) tool to generate other squares
		Cabri tool for constructing segments
		Numerical values for the measurement of lengths (in cm)
		Cabri tool for measuring the angles
		Numerical values for the measurements of angles (in degrees)
		Mathematical terms such as square, segment, and angles
$\sum_{\text{MeaP}}$	Control structure	If two angles or two sides have the same measure they are congruent
		A square has congruent sides
		A square has four right angles

The MeaP conception of congruency was useful to make sense of Anil's observed actions as he set the initial conditions of his investigation. The measure-preserving conception of congruency (MeaP) describes how he set up a square, with all sides and all angles congruent. The solution using a dynamic diagram requires dragging the vertices of the square to get new squares with different dimensions and repeating the process of finding the ratio several times. Anil's use of the MeaP to set the initial conditions of the problem contrasts with Brett's, who depended upon the MeaP to decide on the correctness of his claim. Anil's solution was a step further than using visual perception for finding a constant value and based upon the comparison of different numerical values.

5.7 Comparing the Parallelogram and the Square Episodes

We contend that constraints and possibilities of the artifacts, the table of properties and the calculator, allowed Brett and Anil to pursue this investigation differently than when studying static diagrams. They used the resources at hand to generate new ways of interacting with geometric objects and to present mathematically interesting conjectures. Brett and Anil exemplify what Erickson (2004) calls the "bricoleur" (p. 165) by doing, "some combination of rule following and situated adaptation, which can be thought of as a kind of moment-by-moment bricolage" (p. 196). Here, we explain some of the common characteristics of Brett's and Anil's presentation.

As bricoleurs, Brett and Anil took advantage of geometrical language—terms that they had previously studied in this and other mathematics classes and also new terms defined by the teacher during the replacement unit on quadrilaterals. Brett's presentation included other features of verbal language (i.e. indexicals) and non-verbal signs (i.e. pointing) as part of the semiotic register to represent the problem and its solution. Anil's presentation relied on numerical values to propose his conjecture, and his use of hedges about numerical values attests to his interpretation of measurements and calculations in Cabri as approximations (Olivero and Robutti 2001). In addition, the teacher prompted Brett and Anil to make sense of their results by asking questions and commenting on their work. The teacher gave students direct feedback, which they incorporated in their talk. So the dialogue was

another tool for students to show evidence of learning through their participation in discourse (Zack and Graves 2001).

Another common feature of Brett's and Anil's presentation is the combination of dragging and measuring. By measuring, Brett found that the quadrilateral was a parallelogram. By dragging, Brett was able to generate quickly many examples. By combining measuring and dragging, Brett could test his conjecture. His conjecture resisted that test, even when changing the quadrilateral by dragging. Anil used dragging and measuring to make sure that he started with a square. Anil dragged the vertices of the original quadrilateral carefully enough so that all the sides and all the angles remained congruent. By dragging, measuring, and calculating Anil verified that his conjecture was true for other squares as well. Using Hollebrands' (2007) distinction, we could say that Brett and Anil used reactive and proactive strategies because they used the results of combining measuring and dragging to formulate and also to test conjectures. Anil used a proactive strategy when he set up a "square" by combining measuring and dragging.

Brett's and Anil's interactions with diagrams could be characterized as *generative* (Herbst 2004). Brett added diagonals to the original quadrilateral. Adding these diagonals is a key step towards noticing that the "medial-line theorem" might be of use to prove that the quadrilateral connecting the midpoints of any quadrilateral is a parallelogram. Anil used the diagram generatively in choosing what feature to pay attention to: Of all the things that he could have measured he chose to investigate a ratio that connected a square and its dual. However, beyond adding new elements to find relationships (Brett's case) or narrowing down the set of possibilities for investigation (Anil), when each of them started dragging their diagrams their interactions with diagrams went beyond the generative mode of interaction.

Brett's and Anil's use of diagrams exemplify what we propose to call the *functional* mode of interaction with diagrams. Within this mode, students relate inputs and outputs, looking at invariants when introducing changes to a dynamic diagram. Students establish relationships between a diagram, D_k, and measurements (or sets of measurements), $f(D_k)$, which describe features of that diagram. The study of relationships between measurements may reveal invariants. That is, a student makes measurements of features of a diagram and maps that diagram to a set of measurements (or numerical calculations based upon those measurements), denoted by $f: D \rightarrow M$. Elements of M result from measuring features of a diagram. Given D_k, $f(D_k) = M_k = (m_{k1}, m_{k2}, m_{k3} \dots m_{kn})$. By dragging, students could illustrate quickly many examples of a figure where the same relationship between a diagram and its measurements hold. Within the functional mode of interaction with diagrams, students find invariants in measurements and establish an equivalence relation between different diagrams, $D_1, D_2, \dots D_n$.

For example, Brett's measurements mapped a diagram that appeared to be a parallelogram to the quadruplet of its angle measurements, $f_{\text{Brett}}(D_1) = (128.48°, 51.52°, 128.48°, 51.52°)$. He noticed that opposite angles had the same measurements. By dragging D_1 into D_2 he could generate another set of measurements $f_{\text{Brett}}(D_2)$ where the measurements where not the same but they were in the same relationship to each other, since the first and the third angle measures and the second and fourth angle measures were pairwise equal. This result could lead him to conclude that each of the different diagrams, $D_1, D_2, \dots D_n$, were parallelograms.

Similarly, Anil established a mapping between a diagram (of a square and its dual quadrilateral) and a ratio. He calculated the ratio of the perimeter of the square to the length of one of the segments connecting the midpoints of the square. That is, Anil associated the diagram to the ratio, $f_{\text{Anil}}(D_1) = 5.66$. By dragging, he generated other

squares, D_1, D_2, …D_n, finding an invariant in the ratios $f_{\text{Anil}}(D_k) \approx 5.66$—interpreting these ratios to be within the same range. Anil found the equivalence class of squares with the same ratio between the perimeter and the length of the segment connecting consecutive midpoints of a square.

The functional mode of interaction with diagrams builds on the empirical mode of interaction with diagrams (Herbst 2004) in that it relies on proximal experiences with attributes of a diagram. However, these two modes of interaction with diagrams are different in that in the functional mode the student maps constructed relationships between variable measurements to constructed relationships between variable diagrams resulting from dragging. Brett and Anil found invariants in the numerical values. Brett found that opposite angles of a quadrilateral were congruent. Anil found that the ratio of the perimeter of a square to the length of the segment connecting the midpoints tended to be within the same range.

Tools within dynamic geometry provided new affordances for students to make discoveries. In the parallelogram and the square episodes, the dragging and the measuring tool opened the possibility of investigating many drawings instead of one and then seeing what things changed and which remained constant. More importantly, students could only "see" what remained constant because they knew what to look at by acts they made on diagrams such as choosing what to measure. Thus, the functional mode of interaction with diagrams also builds on the generative mode of interaction with diagrams. However, the study of invariants in measurements and calculations involves a different stance towards diagrams. Students' discoveries, through the use of measuring and dragging, were supported by a measuring conception of congruency where they took as congruent objects that had equal measure.

6 Instructional Significance of the Two Episodes

In reference to tasks that require using dynamic geometry, Laborde has used the expression "black box situation" to talk about problems (Laborde 2001, p. 294 and 311) where students have to identify geometric properties that do not change—invariants—in spite of manipulations of a diagram by dragging. According to Laborde, it takes time for teachers who use DGS to design tasks that involve black box problems. The management of tasks involving black box problems, where there is a high level of uncertainty for the teacher in anticipating students' work, adds to the complexity of incorporating technology in mathematics classrooms.

The tasks in which students could get involved while working on a problem is an important element of instructional design. At times, teachers' modifications of the mathematical tasks respond to students' difficulties in working with the assigned problem (Herbst 2006). According to Herbert Simon (1975), "Discovery of what subjects learn can be approached experimentally, but important preliminary insights can be gained by analyzing the structure of the task itself to determine the possible alternative ways of performing it" (p. 268). Mathematical tasks involving DGS tend to include elements of visual perception, where students make discoveries by noticing observable changes or invariants in a dynamic diagram by dragging geometric objects. The availability of other tools provides affordances for students to go beyond visual perception. For example, by measuring students can use the numerical values and gather data about relationships between geometric objects. Data collection is especially useful when combining dragging and measuring, because of the availability of many examples of the same figure.

Gravemeijer (2004) proposes ways of using students' thinking in designing mathematical tasks based upon Martin Simon's (1995, 2004) construct of *hypothetical learning trajectory* (HLT). The HLT involves a teacher's anticipation of the learning that students could have: "It is hypothetical because the actual learning trajectory is not knowable in advance. It characterizes an expected tendency. Individual students' learning proceeds along idiosyncratic, although often similar paths" (Simon 1995, p. 135). The analysis of Brett's and Anil's presentations could be used as a base for developing a HLT. Their way of combining dragging and measuring tools served different purposes in their work towards solving the mathematical task, but their actions have something in common: Anil's and Brett's presentations show an example of students' transition from a perceptual conception of congruency towards a measure-preserving conception of congruency.

Anil and Brett used the measuring capabilities of DGS to achieve two different goals. In the parallelogram episode, Brett used the measuring tool to make a conjecture about the resulting figure. His conjecture can be described using the MeaP conception of congruency. Brett made a "reasoned conjecture" because he purposefully gathered numerical evidence about relationships between angles in a parallelogram guided by known properties of parallelograms. Brett's explanation to the whole class in the parallelogram episode illustrates how through dragging and measuring he could determine features of the resulting figure. At the teacher's request, Brett dragged a vertex of the original quadrilateral. By dragging he made apparent to all that the figures continued to be parallelograms even if they were different ones and different from each other.

In the square episode, Anil used the measuring tool to control that he had a square to begin with. Once Anil found the ratio of the perimeter of the square and a side of its dual quadrilateral, he dragged the vertices to get other squares with different lengths. The MeaP conception of congruency supported Anil's investigation in setting the initial conditions of the quadrilateral. Anil's square had congruent angles because they all had the same measure and congruent sides because they all had the same length. Anil's action was similar to the episode described by Healy and Hoyles (2001) where Tim and Richard used dragging and measuring to make a quadrilateral with a set of consecutive perpendicular angle bisectors. In the square episode, Anil combined dragging and measuring to make several squares.

Brett's and Anil's conjectures could be taken as different developments of the mathematical task anticipated by the teacher. In both cases the measure preserving conception of congruency played an important role in the making of conjectures. This conception of congruency in turn enabled them to focus on properties as opposed to on shapes; moreover it enabled them to control how they use those properties to make claims (in Brett's case) and to bound decisions as to the range of objects for which a claim could be made (in Anil's case). Those behaviors were instructionally significant in that they presented opportunities. The parallelogram episode could have been an opportunity to start the study of the medial-line theorem. The teacher could have seized Brett's initiative to draw a diagonal as an auxiliary line and make visible the geometric relationships of the inner figures embedded in the diagram. For example, the teacher could have taken Brett's diagram and asked about possible ways to justify that the opposite angles in the inner figure are congruent. The teacher could have used Brett's conjecture as an opportunity for students to appreciate the relevance of the "medial-line" theorem. The square episode could have provoked discussions about similarity, right triangles, and applications of the Pythagorean theorem. The teacher could have helped students in proving that by connecting successive midpoints of a square one gets another square. Then the teacher could have introduced similarity to find the constant ratio between the two squares.

7 Conclusions

New technological artifacts bring about new ways of interacting with geometric diagrams. Seymour Papert (1980/1993, p. 23), in his book *Mindstorms,* gave a new vision of learning with computers when he suggested using computers as an "object-to-think-with". He proposed changing the culture of educational settings towards a more "personal, less alienating relationship with knowledge" (Papert 1980/1993, p. 177), where students and teachers create bonds learning about learning.

Our research community is still aligning its questions with changes in the use of technology in the classrooms. Some mathematics teachers see the potential of using dynamic geometry to explore new mathematical terrain and to grant more agency to students. Nevertheless, most teachers refer to improvements in the classroom atmosphere, students' increased motivation, and the efficiency of showing many examples at once as some of the reasons for incorporating dynamic geometry in their classrooms (Lampert 1993; Ruthven, et al. 2005). The analyses of the parallelogram and the square episodes show how the use of technological artifacts could bring about new conceptions of mathematical ideas in students' work. By incorporating new actions—dragging and measuring—the teacher and the students in this study developed new knowledge, with the use of new tools.

One of the lessons from the enterprise of putting computers in classrooms is that the availability of new technology does not necessarily imply changes in instructional practices (Cuban 1986, 2001). According to Cuban, there is a mismatch between reformers' expectations for change and teachers' ability to change. This mismatch results from reformers' lack of awareness about instructional issues regarding the use of technology in classrooms. Particularly in the case of DGS, Laborde (2001) notices that teachers need to develop some expertise in designing instructional tasks that would incorporate new technological tools without compromising the depth of students' mathematical experience.

In this study, measuring, which is usually perceived as an undesirable activity in the high school geometry class, allowed students to relate to geometric objects in a different way: By their properties rather than by their shapes. Measuring with DGS tools allowed students to go deeper in their investigation. Students talked about geometric relationships that they had not discovered when using paper and pencil diagrams. In spite of the fact that students did not make constructions, the interplay between speech acts, gestures and use of tools (e.g. dragging and measuring) allowed students to act upon diagrams and participate in classroom discourse. Moreover, by measuring students made statements that were not merely based upon visual perception.

There was evidence for students' learning in their shift from a visual perception conception of congruency to a measure-preserving conception of congruency. Affordances and constraints of artifacts enabled students to make new discoveries by measuring. However, these discoveries were also possible because of the work of the teacher who helped students to internalize these artifacts by modeling their use and by having students present their work in class. A measuring-preserving conception of congruency supported students' findings when combining measuring and dragging tools of DGS. While measuring still does not promote the transformation-conception of congruency needed for a theoretical approach to geometric figures, its capacity to shift attention from shapes to properties could be the seed for students to discover new mathematical ideas and to go beyond visual perception.

Appendix 1: Tables of Properties[7]

See Tables 1 and 2.

Table 1 Properties by students in the 2nd period

Properties of O-Quad	M-Quad
All angles 90°	Rhombus
No sides or angles congruent	Parallelogram
Two pairs of congruent opposite sides	Parallelogram
Congruent sides and angles	Square
All congruent sides	Rectangle
Two pairs of congruent opposite angles	Parallelogram
One pair of parallel sides and another pair of congruent sides	Rhombus
Two pairs of congruent sides (not opposite each other)	Rectangle
Diagonals bisect each other	Parallelogram
Diagonals perpendicular	Rectangle

Table 2 Properties by students in the 7th period

Properties of O-Quad	M-Quad
Two pairs of congruent opposite sides	Parallelogram
Sides and angles are not congruent	Parallelogram
Four 90° angles	Rhombus
Four 90° angles and four congruent sides	Square
Two pairs of adjacent congruent sides	Rectangle
Perpendicular diagonals	Rectangle
Two pairs of congruent opposite angles	Parallelogram
No parallel sides	Parallelogram
Four congruent sides	Rectangle

---

[7] The teacher used this table to summarize conjectures gathered by students in the first two days of the unit. We translated symbolic notation to denote congruent angles, parallel sides, and parallelograms to make the table more readable.

References

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2002). A cognitive analysis of dragging practises in Cabri environments. *Zentralblatt für Didaktik der Mathematik, 34*(3), 66–72.

Arzarello, F., Olivero, F., Paola, D., & Robutti, O. (2007). The transition to formal proof in geometry. In P. Boero (Ed.), *Theorems in school: From history, epistemology and cognition to classroom practice* (pp. 305–323). Rotterdam: Sense.

Balacheff, N., & Gaudin, N. (2003). Conceptual framework. In S. Soury-Lavergne (Ed.), *Baghera assessment project: Designing a hybrid and emergent educational society. Les Cahiers du Laboratoire Leibniz, 81* (pp. 3–22). Grenoble, France: Laboratoire Leibniz-IMAG.

Balacheff, N., & Gaudin, N. (2009). *Modeling students' conceptions: The case of function. CBMS Issues in Mathematics Education. 16*. American Mathematical Society.

Balacheff, N., & Kaput, J. (1996). Computer-based learning environments in mathematics. In A. Bishop, et al. (Eds.), *International handbook of mathematics education* (pp. 69–501). The Netherlands: Kluwer.

Bourdieu, P. (1990). *The logic of practice*. Stanford, CA: Stanford University Press.

Bucholtz, M., & Hall, K. (2005). Identity and interaction: A sociocultural linguistic approach. *Discourse Studies, 7*(4–5), 585–614.

Chazan, D. (1993). High school geometry students' justification for their views of empirical evidence and mathematical proof. *Educational Studies in Mathematics, 24*, 359–387.

Chazan, D., & Houde, R. (1989). *How to use conjecturing and microcomputers to teach geometry*. Reston, VA: NCTM.

Chazan, D., & Yerushalmy, M. (1998). Charting a course for secondary geometry. In R. Lehrer, & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 67–90). Mahwah, NJ: Erlbaum.

Cuban, L. (1986). *Teachers and machines: The classroom use of technology since 1920*. New York, NY: Teachers College Press.

Cuban, L. (2001). *Oversold and underused: Computers in the classroom*. Cambridge, MA: Harvard University Press.

De Villiers, M. (1998). An alternative approach to proof in dynamic geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 369–393). Mahwah, NJ: Lawrence Erlbaum.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167–180.

Erickson, F. (2004). *Talk and social theory*. Cambridge, UK: Polity.

Eves, H. (1990). *An introduction to the history of mathematics*. Fort Worth, TX: Saunders.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*, 139–162.

Goldenberg, E., & Cuoco, A. (1998). What is dynamic geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 351–367). Mahwah, NJ: Lawrence Erlbaum.

Gravemeijer, K. (2004). Local instruction theories as means of support for teachers in reform mathematics education. *Mathematical Thinking and Learning, 6*(2), 105–128.

Halliday, M. A. K., & Hasan, R. (1976). *Cohesion in English*. London: Longman.

Halliday, M. A. K., & Matthiessen, C. (2004). *An introduction to functional grammar* (3rd ed.). London: Hodder Arnold.

Healy, L., & Hoyles, C. (2001). Software tools for geometrical problem solving: Potential pitfalls. *International Journal of Computers for Mathematical Learning, 6*, 235–256.

Herbst, P. (2004). Interaction with diagrams and the making of reasoned conjectures in geometry. *Zentralblatt für Didaktik der Mathematik, 36*(5), 129–139.

Herbst, P. (2005). Knowing about "equal area" while proving a claim about equal areas. *Recherches en Didactique des Mathématiques, 25*(1), 11–56.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*, 313–347.

Herbst, P., & Balacheff, N. (2009). Proving and knowing in public: What counts as proof in a classroom. In M. Blanton, D. Stylianou, & E. Knuth (Eds.), *Teaching and learning of proof across the grades: A K16 perspective* (pp. 40–63). New York: Routledge.

Hohenwarter, M. (2001). *GeoGebra [Computer software]*. FCR-STEM, Learning Institute, Florida State University. www.geogebra.org.

Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education, 38*(2), 164–192.

Hölzl, R. (1996). How does "dragging" affect the learning of geometry. *International Journal of Computers for Mathematical Learning, 1*(2), 169–187.

Hoyles, C., & Noss, R. (1994). Dynamic geometry environments: What's the point? …". *Mathematics Teacher, 87*(9), 716–717.

Jackiw, N. (1991). *The Geometer's Sketchpad [Computer software]*. Berkeley, CA: Key Curriculum.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretation when using dynamic geometry software and their evolving mathematical explanation. *Educational Studies in Mathematics, 44*, 55–85.

Laborde, C. (2001). Integration of technology in the design of geometry tasks with cabri-geometry. *International Journal of Computers for Mathematical Learning, 6*, 283–317.

Laborde, C. (2005). The hidden role of diagrams in students' construction of meaning in geometry. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 159–179). Dordrecht, The Netherlands: Kluwer.

Laborde, J.-M., & Bellemain, F. (1993). *Cabri-Geometry II [Computer software]*. Texas Instruments and Université Joseph Fourier.

Lampert, M. (1993). Teacher's thinking about students' thinking about geometry: The effects of new teaching tools. In J. L. Schwartz, M. Yerushalmy, & B. Wilson (Eds.), *The geometric supposer: What is it a case of?* (pp. 143–177). Hillsdale, NJ: Erlbaum.

Mariotti, M. (2000). Introduction to proof: The mediation of a dynamic software environment. *Educational Studies in Mathematics, 44*, 25–53.

Mariotti, M. (2001). Justifying and proving in the Cabri environment. *International Journal of Computers for Mathematics Learning, 6*, 257–281.

Mesa, V. (2004). Characterizing practices associated with functions in middle school textbooks: An empirical approach. *Educational Studies in Mathematics, 56*, 255–286.

Noss, R., & Hoyles, C. (1996). *Windows on mathematical meanings*. Dordrecht, The Netherlands: Kluwer.

O'Connor, M. C., & Michaels, S. (1996). Shifting participant frameworks: Orchestrating thinking practices in group discussion. In D. Hicks (Ed.), *Child discourse and social learning* (pp. 63–102). Cambridge, U.K.: Cambridge University Press.

Olivero, F., & Robutti, O. (2001). *Measuring and proving in open geometry problems with Cabri*. Quaderni del Dipartimento di Matematica, Università di Torino, N.10/2001.

Olivero, F., & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning, 12*(2), 135–156.

Papert, S. (1980/1993). *Mindstorms: Children, computers, and powerful ideas* (2nd ed.)*.* New York: Basic Books.

Parzysz, B. (1988). Knowing" vs. "seeing. Problems of the plane representation of space geometry figures. *Educational Studies in Mathematics, 19*(7), 9–92.

Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Routledge.

Rabardel, P. (1995). *Les hommes et les technologies*. Paris: Armand Colin.

Rowland, T. (1995). Hedges in mathematics talk: Linguistic pointers to uncertainty. *Educational Studies in Mathematics, 29*, 327–353.

Rowland, T. (1999). Pronouns in mathematics talk: Power, vagueness and generalisation. *For the Learning of Mathematics, 19*(2), 19–26.

Ruthven, K., Hennessy, S., & Deaney, R. (2005). Current practice in using dynamic geometry to teach angle properties. *Micromath, 21*(1), 9–13.

Simon, H. (1975). The functional equivalence of problem solving skills. *Cognitive Psychology, 7*(2), 268–288.

Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114–145.

Simon, M., & Tzur, R. (2004). Explicating the role of mathematical tasks in conceptual learning: An elaboration of the hypothetical learning trajectory. *Mathematical Thinking and Learning, 6*(2), 91–104.

Vérillon, P. (2000). Revisiting Piaget and Vygotsky: In search of a learning model for technology education. *The Journal of Technology Studies, 26*(1), 3–10.

Vérillon, P., & Rabardel, P. (1995). Cognition and artifacts: A contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology of Education, 10*(1), 77–101.

Zack, V., & Graves, B. (2001). Making mathematical meaning through dialogue: "once you think of it, the Z minus three seems pretty weird". *Educational Studies in Mathematics, 46*, 229–271.

Zbiek, R. M., Heid, M. K., Blume, G. W., & Dick, T. P. (2007). Research on technology in mathematics education: A perspective of constructs. In F. K. Lester Jr (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1169–1207). Reston, VA: National Council of Teachers of Mathematics.