Malati

Mathematics learning and teaching initiative

Statistics

Module 3: Systematic counting

Grades 8 and 9

Teacher document

Malati staff involved in developing these materials:

Kate Bennie
Kate Hudson
Karen Newstead

We acknowledge the valuable comments of Heleen Verhage and Donald Katz.

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Overview of Module 3: Systematic Counting

ACTIVITIES:

Making Numbers	Systematic recording of results, arrangements
Going to a Show	Elementary and more difficult arrangements
Arrangements	Permutation notation and terminology, working with factorials
Writing a Test	Arrangements with independent and dependent choices
Counting Counters	Arrangements in which some objects are repeated
Hit Parade	Informal introduction to combinations - selection of some objects from more objects: order important/not
Another Hit Parade	More difficult combinations – towards the mathematics. Followed by combination terminology and formula

Making Numbers

1. How many different three digit numbers can you make with these three digits?

3 ; 5 ; 2

2. How many different three digit numbers can you make without using any of the digits more than once?

3. Discuss and explain the difference between your answers in Questions 1 and 2.

Teacher Notes: 'Making Numbers'

This requires a systematic way of counting and recording results, in this case arrangements. Any pupils who obtain the answers using a mathematical expression (3 x 3 x 3 in the case of Question 1 and 3 x 2 x 1 in the case of Question 2) should be encouraged to share their methods, but at this stage the tree diagram or other systematic representations are sufficient.

Pupils should notice the large difference between the answers to Questions 1 and 2, in other words the difference made by the fact that one can/cannot repeat the digits.

Going to a Show

Noxolo and her friends are going to watch the DIZ dance show.

1. Use a tree diagram to find how many different arrangements you can make of the letters D, I and Z. Check whether you agree with your friend.

2. Now solve the same problem using the following reasoning:

   With how many different letters could the arrangement begin?

   And how many possible arrangements are there for EACH of these possibilities? How do you know?

   So, in how many different ways can the letters D, I and Z be arranged? Explain to your friend where you think this number (your answer) comes from, using your tree diagram in Question 1 if necessary.

3. How many different ways can you find to arrange the letters, D, I, Z and Y? Check whether you agree with your friend.

4. Noxolo and her friends Lungi, Busiswe, Sibongile, and Ziya have booked five seats for the dance show. W*ithout listing all the different ways*, determine in how many different ways Noxolo and her friends can sit (arrange themselves) in these five seats?

5. How do you know you have all the possible ways?

Teacher Notes: 'Going to a Show' and 'Arrangements'

This activity introduces permutations by looking at, firstly, the arrangement of three then four letters in different ways and, secondly, the possible arrangement of 5 people in 5 seats. Pupils are first asked to solve the problem using a tree diagram and then introduced to a reasoning process which may seem different but is in fact reflected in the tree diagram. In both cases the answer for arranging 3 objects is 3 x 2 x 1. They are then given practice in this concept with 4 and 5 objects, and encouraged to come up with a mathematical way of calculating the number of arrangements (Question 4). Systematic listing or a tree diagram can always be used to check whether one has all the possible arrangements.

The term 'permutation' and the formula and notation for calculating the number of permutations MUST NOT BE INTRODUCED UNTIL PUPILS HAVE SOLVED THE PROBLEMS AND DISCUSSED THEIR STRATEGIES. Following this, some purely mathematical activities on factorials are given. It is important that pupils understand the notation and discuss their findings with one another.

Arrangements

When we worked out how many words you can make with the letters D, I and Z and when we worked out in how many ways Noxolo and her friends could sit at the show, we were looking at how many different **arrangements** are possible.

For example, we found that for making a word with the letters D, I and Z, there were three possible ways to begin the arrangement, and for each of these there were 2 (or 2 x 1) possible arrangements. Thus the number of possible arrangements was:

$$3 \times 2 \times 1 = 6$$

Similarly, with four letters, the number of possible arrangements was:

$$4 \times 3 \times 2 \times 1 = 24$$

And the number of arrangements of seating for Noxolo, Lungi, Busiswe, Sibongile and Ziya was:

$$5 \times 4 \times 3 \times 2 \times 1 = 120$$

In mathematics we call the different arrangements **permutations**.

1. What do you notice about the number of different permutations of 3 objects, 4 objects and 5 objects? Discuss this with your friends.

2. How many permutations are there of 6 objects?

3. On your calculator, find the button that looks like this: $\boxed{x!}$

Now press 3 followed by this button, 4 followed by this button and 5 followed by this button. What do you notice?

> In mathematics, we say the number of permutations (arrangements) of *n* objects is given by *n!* or *n factorial*.
>
> *n!* means to multiply the whole numbers from *n* all the way down to 1.

4. Find the value of 7! without using your calculator.

5. Find the value of $\frac{25!}{23!}$ without using a calculator.

6. Discuss with your friend which of the following are true and which are not:

   (a) 3! + 3! = 6!

   (b) 10! = 10 × 9!

   (c) $\frac{8!}{4!}$ = 2!

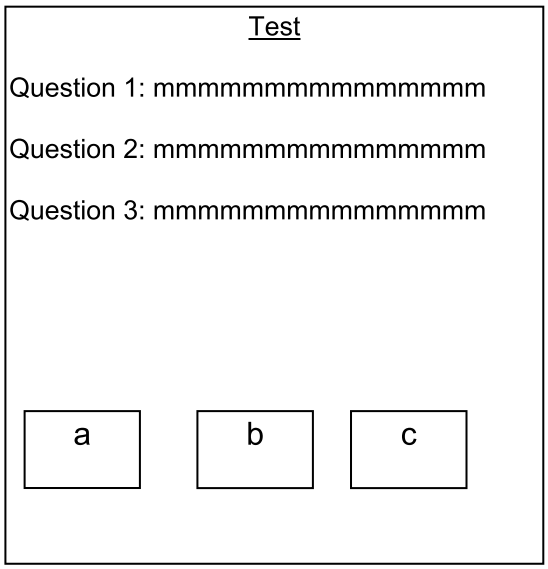
   (d) 6! × 7! = 10!

   (e) 10 × 9 × 8 = $\frac{10!}{7!}$

7. Is it true that 1! + 4! + 5! = 145? Is this always the case? Explain!

Writing a Test

1. A test has 3 questions. The answers are given in three boxes at the bottom of the tests, and the answers must be matched to the questions (one per question). One cannot give the same answer for more than one question.



   In how many different ways can this test be answered? (Hint: What are the choices for the first question? And the second question?)

2. Another test also has three questions, but this time one can answer EACH question by writing one of the options 'true', 'false' or 'Can't answer with the given information'. In how many different ways can this test be answered?

3. Discuss with your friend the difference between the two situations above (the matching-answers situation and the choice-of-three situation). Why are there more possible ways in the second situation?

When answers are matched to questions, an answer is 'used up' and the number of choices for the next question is one less.

Thus for number 1, there are 3 choices for the first question, two for the second and only one for the third. The number of different ways of answering is

$$3 \times 2 \times 1 = 6$$

Refresh your memory: …why do we multiply the numbers? Draw a tree diagram!

The choice for the second question *depends* on what you have already chosen for the first question. Similarly, the choice for the third question *depends* on what has already been chosen for the first two questions.

We say the choices are *dependent*.

When we can repeat the same choices for different questions, the answers are not 'used up'. Thus for number 2, there are 3 choices for each question. The number of different ways of answering is $3 \times 3 \times 3 = 27$.

The choice for any question *does not depend* on what answers have been chosen for the previous questions.

We say the choices are *independent*.

4. (a) Now imagine a test in which answers are matched to the questions as in number 1, but this time there are 8 possible answers, but still only 3 questions. In how many different ways can the test be answered?

   (b) Are the choices dependent or independent? Explain why you think so.

Teacher Notes: 'Writing a Test'

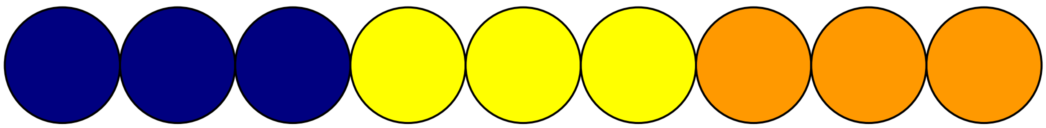
This activity introduces pupils to the number of possible arrangements in the case where the choices are dependent (get 'used up') and independent (do not get 'used up', can be used again). After 'Going to a Show', the pupils should be able to reason the answers as follows: In Question 1, there are three possibilities for the first response, and for each of these there are two possibilities for the two responses to follow, so the number of arrangements is 3 x 2 x 1. In Question 2, however, the number of possibilities for each question is 3, regardless of which response is given first, so the number of possibilities is 3 x 3 x 3.

The difference in answers is explained after the pupils have been given a chance to solve the problems.

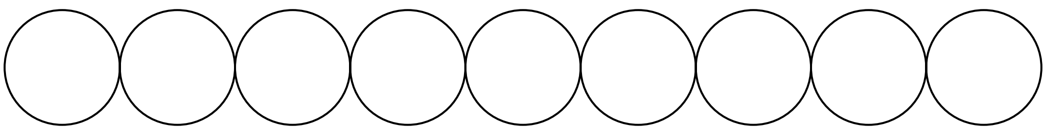
In Question 4, pupils should be encouraged to use similar reasoning – how many possibilities are there for the first question? And for each of these, how many possibilities are there for the second question? And then for the third? If they are struggling to understand that the answer is 8 x 7 x 6, they should be encouraged to start drawing a tree diagram. It might be tedious but pupils may not need to complete the tree diagram to understand the reasoning.

COUNTING COUNTERS !

Back:



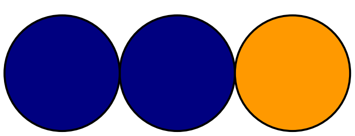
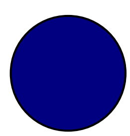
Front:



1 2 3 4 5 6 7 8 9

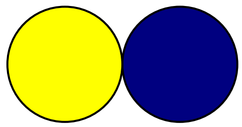
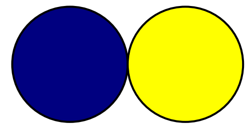
Make nine counters and number them from 1 to 9. Turn the counters upside down and colour the back of the first three counters blue, the next three yellow and the last three red.

In the following questions you will be required to arrange the counters you have made. It is important that when you arrange the counters you arrange them in a row (in these exercise circles, columns etc are not allowed).

Allowed:  Not Allowed: 

In arrangements the order is important, that is, a yellow counter followed by a blue counter is different from a blue counter followed by a yellow counter.

In the diagram below Arrangement 1 is different from Arrangement 2.

Arrangement 1:  Arrangement 2: 

1. Take counters 1 and 4.

   (a) How many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

2. Take counters 1 and 2.

   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

3. In Question 1 the counters have different colours, in Question 2 the counters have the same colour.
Explain why 1(b) and 2(b) have different answers.

4. Take counters 1, 4 and 7.

   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

5. Take counters 1, 2 and 4.

   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

6. Take counters 1, 2 and 3.

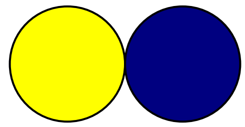
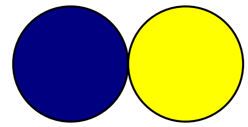
   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

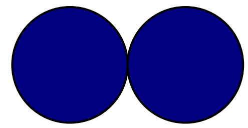
7. **In Question 3 all the counters have different colours, in Question 4 two counters have the same colour and in Question 5 all the counters have the same colour. Explain why 4(b), 5(b) and 6(b) have different answers.**

In the first two questions there were two counters. The number of different ways of arranging two objects is 2 x 1 or 2!

So in question 1 the numbers can be

1 4 or **4 1** and the colour patterns  or 

In Question 2 there are still 2! different number patterns

1 2 or **2 1** but the colour pattern can only be 

Because the colours are the same we do not know whether the counter with number 1 on it is in the first or second position and, similarly, we do not know which counter has the number 2 on it. From the back **1 2** and **2 1** look the same. Thus although there are 2! ways of arranging 2 objects because they look the same we divide by 2! so the number of patterns is

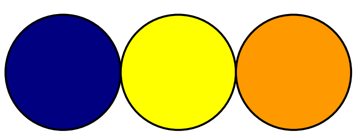
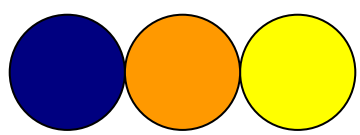
$$2! \div 2! = 1$$

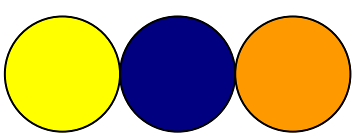
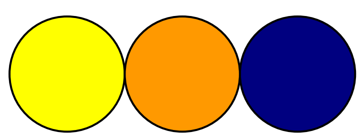
In Questions 4, 5 and 6 there are three counters. The number of different ways of arranging three objects is 3 x 2 x 1 or 3!.

So in Question 4 the numbers could be

1 4 7 or **1 7 4** or **4 1 7** or **4 7 1** or **7 1 4** or **7 4 1**

and the colours

 or  or

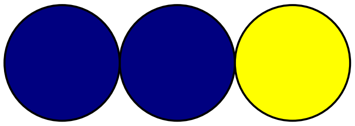
 or  or

or

In Question 5 there are still 3! different number patterns

1 2 4 or **1 4 2** or **2 1 4** or **2 4 1** or **4 1 2** or **4 2 1**

but the colours can only be



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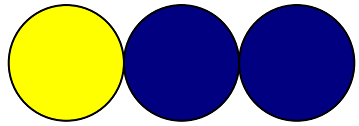

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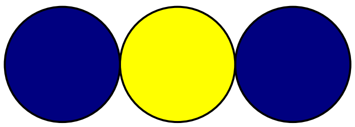


Because the two blue counters look the same we do not know whether the blue counter with number 1 on it comes before or after the blue counter with the number 2 on it. From the back **1 2 4** and **2 1 4** look the same. Similarly **4 1 2** and **4 2 1** look the same and **1 4 2** and **2 4 1** look the same. There are 3! ways of arranging the 3 counters but because two of the counters look the same (and there are 2! ways of arranging two counters) we have to divide 3! by 2!

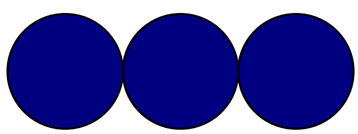
Thus the number of patterns is only

$$3! \div 2! = 3$$

In Question 6 there are still 3! different number patterns

1 2 3 or **1 3 2** or **2 1 3** or **2 3 1** or **3 1 2** or **3 2 1**

but the colours can only be



Because the three blue counters look the same we do not know which counter is in which position. From the back **1 2 3; 1 3 2; 2 1 3; 2 3 1; 3 1 2; 3 2 1** look the same. There are 3! ways of arranging 3 counters but because all 3 of them look the same (and there are 3! ways of arranging 3 counters) we have to divide 3! by 3! Thus the number of patterns is only

$$3! \div 3! = 1$$

8. Take counters 1, 2, 4 and 7.

   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

9. Take counters 1, 2, 4 and 5.

   (a) In how many ways can you arrange them to get different numbers?

   (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

10. Take counters 1, 2, 3 and 4.

    (a) In how many ways can you arrange them to get different numbers?

    (b) Now turn the counters upside down so that you can see only the colours and not the numbers. In how many ways can you arrange them to get different patterns?

11. In Question 8 two counters have the same colours, in Question 9 two pairs of counters have the same colour and in Question 10 three counters have the same colour. Explain why 8(b), 9(b) and 10(b) have different answers.

12. Without using the counters 1, 2, 3, 4 and 7, predict how many different colour patterns you could make. (Then use the counters to check!)

Teacher Notes: 'Counting Counters'

The purpose of this activity is to enable pupils to experimentally derive the formula for the number of distinct arrangements of n objects when p are indistinguishable i.e. n! ÷ p! Pupils are encouraged to reflect on their findings after each group of questions, and the formal mathematical reasoning is only introduced in the second half of the activity.

The numbers on the counters are all distinct, thus pupils should be able to use their previous knowledge of systematic counting to determine all the different patterns.

Question 1 has two counters with different colours so there are simply 2! patterns. Question 2 has two counters of the same colour so, although there are 2 number patterns i.e. 1 4 and 4 1, both counters look the same from the back so there is only 1 pattern (or 2! patterns divided by 2! indistinguishable patterns i.e. 2! ÷ 2! = 1).

Questions 4, 5 and 6 have 3 counters. In Question 3 the colours are different so there are 3! patterns. In Question 4 two counters have the same colour: it is impossible to distinguish between these two counters when only their colours are showing. The pupils have already determined that there are 2! ways of arranging two counters and, because 2! of them look the same, they must divide by 2!. The number of distinguishable patterns is thus 3! ÷ 2!. In Question 6 the colours are the same so, although there are 3! arrangements, only 1 colour pattern can be seen. Mathematically this can be expressed as 3! ÷ 3!.

Questions 8, 9 and 10 have four counters and the same process as in Questions 4 to 6 is followed. Question 7 is interesting as here there are two blue and two yellow counters. There are 2! ways of arranging the blue counters although the colour patterns look the same. Similarly, there are 2! ways of arranging the yellow counters but the colour pattern looks the same. Thus although there are 4! arrangements of four counters the pupils must divide by 2! for the indistinguishable arrangements of blue counters and 2! for indistinguishable arrangements the yellow counters i.e. 4! ÷ (2! x 2!) = 6.

In Question 12 pupils should be encouraged to make a prediction and explain their method of calculation before working with the counters. From their predictions and explanations, the teacher should be able to assess their understanding of the principle of dividing the total number of arrangements by the number of indistinguishable arrangements.

HIT PARADE

Mr Jive's best-selling singles (in no particular order) are as follows:

Mr Jive's Top 4
Save my Soul (Madonna)
Born with a Broken Heart (Bruce Springsteen)
Dream Dream Dream (REM)
Carnation Salad (Madness)

1. The single which sells the most during the week will get a gold medal, and the runner-up will get a silver medal. Given that there are four singles, how many different possibilities are there for awarding a gold and a silver medal?

   (Hint: Use a tree diagram if you are stuck)

2. Mr Jive has been informed by the record industry that the system is changing…now the TWO singles that sell the most during the week will each get a gold medal. Now, how many groups of two singles could win gold medals? (It is not important which one sells most and which one is the runner-up; they both get gold medals).

   (Hint: You might want to start by drawing a tree diagram. Then see if you can calculate the answer in a different way)

3. Compare your answers to Questions 1 and 2. What do you notice? Can you explain this? Discuss with your friends.

4. The top-selling three singles at the end of the week all get medals.

   (a) How many different groups of singles could win medals if

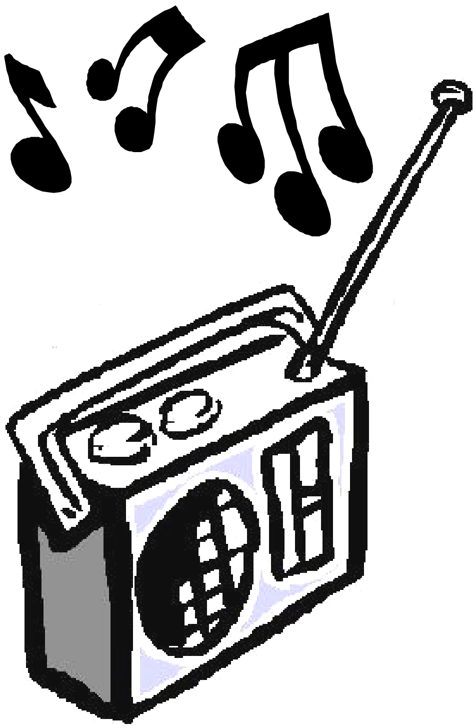
      (i) there are gold, silver and bronze medals for the top seller, second and third place respectively?

      (ii) all three best-sellers will get gold medals, regardless of the order?

   (b) What do you notice about your answers to (i) and (ii)?

5. Write expressions for the answers to number 4 in terms of factorials. See if you agree with your friend.

ANOTHER HIT PARADE



Below is a list of the 10 singles of which Mr Jive sells the most (in no particular order).

Mr Jive's Top 10

Save my Soul (Madonna)
Born with a Broken Heart (Bruce Springsteen)
Dream Dream Dream (REM)
Carnation Salad (Madness)
Garbage (Hotstix)
Dirty Windows (Tina Turner)
Just Another Day on the Funny Farm (Mal Ati)
Stuck on the Ceiling (Lionel Richie)
Bee Mine (Honey Queen)
Losing Face (Michael Jackson)

1. The 3 singles that sell the most by the end of the week will each get a gold medal. How many different groups of 3 singles could win gold?
2. The top seven at the end of the week all get medals. How many different groups of singles could win medals?
3. What do you notice about the answers to numbers 1 and 2? Can you explain this?
4. Write expressions for the answers to 1 and 2 in terms of factorials.

In order to select 3 objects from 10 objects, we reason as follows:

If the order DOES matter (for example if there is an award for the first, second and third place), then

there are 10 choices for the first one, 9 choices for the second,

and 8 choices for the third.

Thus the number of choices is 10 x 9 x 8

(Do you remember why we multiply? Refresh your memory on permutations if you don't!)

BUT if the order DOESN'T matter (if we are just selecting a group of 3 objects from 10 objects), then

the number of choices that we have calculated (10 x 9 x 8) is too big.

In fact, it is 6 times too big…can you see why?

(If not, use a tree diagram to help you!)

In mathematics, we call a selection of things in which the order does not matter a ***combination*****.**

In order to work out how many *combinations* there are of 3 objects from 10 objects,

if the order *had* mattered,

we divide the number of possible choices of 3 objects from 10

because the order does not matter!

by the number of different arrangements of these 3 objects

$$\text{thus } \frac{10 \times 9 \times 8}{3 \times 2 \times 1}$$

Thus we divide 10 x 9 x 8 by 3!

But we have seen that 10 x 9 x 8 = $\frac{10!}{7!}$

The number of different combinations of 3 objects from 10 objects is thus given by:

$$\frac{10 \times 9 \times 8}{3!} = \frac{10!}{7! \times 3!}$$

Teacher Notes: 'Hit Parade' and 'Another Hit Parade'

This activity introduces the selection of p objects from n objects, in a context in which order does matter and in a context in which order is not important (combination). The formulae are $\frac{n!}{(n-p)!}$ *and* $\frac{n!}{p!(n-p)!}$ *respectively.*

Pupils are first asked to solve the problems without guidance. One way of doing this would be tree diagrams on which, in the case of Question 2, one eliminates all the options that are 'doubles' or repeats. From the tree diagram, it should be evident that there are 4 x 3 possibilities for Question 1 but only 6 possibilities (combinations) for Question 2.

Similarly, the number of possibilities in Question 4(a) is (i) 4 x 3 x 2 and (ii) 4. In terms of factorials, these can be written as 4! or $\frac{4!}{1!}$ *and* $\frac{4!}{3!}$ *or* $\frac{4!}{3!\times 1!}$ *respectively. Pupils should recall from 'Counting Counters' the reason for dividing by 3! (the number of different arrangements of three objects which are considered the same – the order is not important).*

In 'Another Hit Parade', pupils are asked to compare what happens when one calculates the number of combinations of 3 objects from 10, and the number of combinations of 7 objects from 10. It is important that they come to realise that these are the same…if 3 are selected then 7 are not selected. The formula also shows why they are same: $\frac{10!}{3!\times 7!}$ *is the same as* $\frac{10!}{7!\times 3!}$.

Once again the formula and the terminology must only be introduced AFTER pupils have grasped the concept.