A MOTIVATION FOR DEVELOPING MORE FLEXIBLE MEANINGS AND STRATEGIES FOR DIVISION IN PRIMARY SCHOOL

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This study provides a motivation for exposing young pupils to a wider variety of division problems than is currently the case. Year 5 and 6 English pupils' strategies for solving six written symbolic division problems were classified and analysed in terms of success. A flexible choice of a range of strategies was found to be associated with success on these tasks. Problems which challenged pupils' own limited models, derived from their initial experiences of division, were solved less successfully.

Introduction and Background

Children's first model of division is based on sharing activities at home and at school (Anghileri, 1995). Tirosh and Graeber (1991) refer to the interpretation of a division problem as a sharing procedure as the partitive model of division. For example, the problem '96÷4' is interpreted in the partitive model as '96 shared between (shared by) 4'. However, this problem can also be interpreted as 'how many fours in 96'; the so-called measurement (quotative) model of division (Tirosh & Graeber, 1991). For example, when problems are extended to include fractions, division is most readily comprehensible in terms of this latter model (Kouba, 1989; Murray, Human & Olivier, 1992; Tirosh & Graeber, 1991).

Sharing relates to one interpretation of and one solution procedure for division, not appropriate for all division problems (Durkin & Shire, 1991). Close adherence to such limited interpretations inhibits children in their solution procedures for solving more complex division problems as they do not have access to a variety of different procedures necessary for successful problem solving. Real understanding of division enables pupils to select from a variety of available interpretations and solution strategies according to the type of problem and the numbers involved (Anghileri, 1995).

Informal introduction of division as sharing and initial school presentation of division as simplified examples may lead a working model of division in which division always 'makes smaller' (Tirosh & Graeber, 1991). Fischbein, Deri, Nello and Marino (1985) refer to two other "primitive, intuitive beliefs" about division, namely that that the divisor is always less than the dividend, and that the divisor must be a whole number. During the process of generalisation, pupils often increase the range of examples covered by such rules to include *all* problems which include the '÷' sign. Even after pupils have been exposed to wider, more correct mathematical notions about division, the 'sharing' working model persists and inhibits the solution of problems whose numerical data violates the constraints of this model (Fischbein *et al*., 1985; Tirosh & Graeber, 1991).

It appears that a certain degree of flexibility is called for in dealing with different kinds of division problems even when no context is provided for these problems. The tasks for the study reported here were designed with this in mind.

Aims and Expectations

This study aimed to investigate pupils' concept of division by considering their strategies for solving symbolic division problems. This paper focuses on the frequency of use of strategies by Year 5 and 6 pupils, and the success and failure of these pupils on the problems given. Given the range of problem types (see below), it was expected that without flexibility the pupils would be unlikely to achieve success in all of the problems. Better performance was also expected when problems conform to the limited model of division than when, for example, problems have fraction divisors or divisors larger than the dividend, or when the belief that 'division makes smaller' is challenged.

The tasks

The tasks given were 96÷4, 34÷7, 6000÷6, 4÷½, 6÷12 and 68÷17. The problems were presented in symbols on individual cards. Although no context was provided for these problems and they could thus not be classified as 'measurement' or 'partitive' problems, their successful solution requires a variety of strategies. For example, while sharing might lead to the successful solution of 6÷12, it is not an efficient strategy for solving 4÷½. Similarly, counting up in groups of the divisor may lead to success in solving 34÷7 and 68÷17, but is not an efficient strategy for solving 6000÷6. The problem 4÷½ also challenges the belief that the divisor is always a whole number and that division makes smaller, while 6÷12 challenges the belief that the divisor is always smaller than the dividend.

Sample and Methodology

The sample for this part of the study consisted of 54 pupils, 27 girls and 27 boys, from 3 schools in or near an academic town in the U.K.: a city primary school (14 pupils), a rural primary school (14 pupils) and a large town middle school (26 pupils). Eight of these pupils were in Year 6, and the rest were in Year 5.

Video-taped interviews were conducted with pairs of pupils. Each pupil started by working independently on each problem enabling individual strategies to be recorded. After they had completed each problem, the pupils were asked to explain their solution procedures to the interviewer, who highlighted for discussion any discrepancies between their solutions. The videotapes were subsequently viewed and transcribed, and the strategies categorised and coded. Written recordings were also considered in this categorisation, as pencil and paper were available to the pupils at all times.

Results

The pattern of response for the Year 6 pupils did not appear to differ significantly from that of the Year 5 pupils. This may be partly owing to the fact that five of the eight Year 6 pupils were in the rural school, in which Year 5 and 6 pupils were taught Mathematics together. Thus the data from the Year 5 and 6 pupils is presented together.

Table I shows the number of pupils who were successful on each problem.

Problem	**Number successful**	**Number attempted**	**% Successful**
96÷4	28	54	52%
34÷7	6	54	11%
6000÷6	33	54	61%
4÷½	20	54	35%
6÷12	10	52	19%
68÷17	14	48	29%

Table I: Success on Each Problem

As expected the success rate on the problems 4÷½ and 6÷12 which challenge common experience-based ideas - that division makes smaller, that the divisor is always a whole number and that the divisor is always smaller than the dividend - was relatively low. Perhaps more surprisingly, success on 34÷7 was rare, mainly owing to a lack of understanding of 'remainder'. A common answer for this problem was '5 remainder 1'.

Common strategies included counting up (usually in groups of the divisor); partitioning the dividend, and algorithmic approaches involving dealing separately with the digits of the dividend. The strategies used by the pupils were categorised as listed in Table II.

Strategy		**Example**		
CU Counting up	using facts	Lauren (Year 6)	4÷½	Half and half, I counted two halves is one, another two halves is one....
	using tables	Danny (Year 6)	34÷7	I tried to do 7 into 34; the 7 times table up to 28...no more sevens would fit
	using tallies	Stephanie (Year 5)	68÷17	(drew 68 lines and crossed out groups of 17) I just did the way that I did those. But I found that there was a...it was remainder 2, it wasn't an exact, that made it quite tricky.
PD Partitioning the dividend	using facts	Naomi (Year 5)	96÷4	80 divided by 4...the answer was 20. 16 divided by four, so the answer was 4.
	using tallies	Kate (Year 5)	96÷4	Well I've made 96 lines and I'm going to try and divide them into 4.
A Algorithmic approach	AW written algorithm	Matthew (Year 5):	96÷4	
	AD dealing with digits	Daisy (Year 5)	6000÷6	There's one 6 in 6, no 6's in nought, so you just put three 0's at the top.
	LE logical error	Melanie (Year 5)	68÷17	10 divided by 60 is 10. Put the 8 divided by 7 is 1...one remainder. 11 remainder 1.
AF Associated number fact		Melanie (Year 5)	96÷4	100 divided by 4, take away 6...I mean take away 4.
M Mental strategy or image		Naomi (Year 5)	6÷12	I knew that 12 was double 6, so the only way to fit 12 into 6...would be 12 halves.
O Other	Oa approximation	Daisy (Year 5)	68÷17	Two seventeens are about thirty-something, so it's about 4.
	Ot trial and improve	Elizabeth (Year 5)	45÷5	I took a guess, I like said te...11 or 12, then added 12 five times, and the answer was...well, it wasn't right, so then I took...it was more than the answer, so I like had 11 and 10....then I wrote down 9 and it worked out the right answer.
	Or rule	Johanna (Year 5)	5÷1	It's hard to explain but you try to divide something by 1, you end up with the number you got to start with.
	Om memory	Hayley (Year 5)	6÷12	Well, I've heard the sum before...so I didn't exactly work it out, I just remembered, and I wasn't really sure about the answer. So...and I thought, when I looked back into one of my other lessons I thought they said the answer was 2.
	Op pattern	Laurence (Year 5)	4÷½	Well, because two... if it was divided by two they'd get 2, if it was divided by 1 they'd get four, if it was divided by half they got 6 or 8....if it went into 2 times table.

I/? Indecipherable strategy, guess or strategy unclear		Sammy (Year 5)	68÷17	I divided 17 into quarters and that's 7 and 7 and 7 and 7. And 14...7 and 7 makes 14 and then another 7 and 7 makes 14. So you put the 14's together and that makes 28 and you add the 10 from the 17 on and that makes 38.
MP Misreading problem	SN Switching the numbers	David (Year 5)	6÷12	(It's 2) because there's two 6's in 12's.
	DS Doing a different sum	Michael (Year 5)	34÷7	I crossed out the (3) and gave a 1 to the 4 so it's 14, then I had...take away 7, and I had 7 and then I just put the 2 from up here down here.
	E Error in reading problem	Neil (Year 5)	4÷½	4 divided by one and a half, so one person has one then the other one has one, then there's 2 left, thenyou give that one to the other person and to the otherso it's 2 and a half of one.

Table II: Coding of Strategies

It is clear from the above table that some pupils used relatively sophisticated and powerful strategies, for example changing the dividend to a more convenient number - a known multiple of the divisor - and then adjusting the answer (AF). Some pupils used a variety of ways of representing the problem, including drawing pictures and creating a context or story, for example Ellie (Year 5), solving 6000÷6: "You've got 6 people, give them each 1000".

Table III shows the frequency of use of the various strategies solution attempts for the six problems. In some cases, more than one strategy is indicated for the same solution attempt by the same pupil. For example, written algorithms would often accompany one of the other strategies. It is possible that the pupils used an alternative strategy to *explain* since they found it unnecessary to explain an algorithmic approach.

Problem	**CU**	**PD**	**A**	**AF**	**M**	**I/?**	**Oa**	**Og**	**Ot**	**Or**	**Om**	**Op**
96÷4	33	30	31	2	2	7			6			
34÷7	72	11	32		0	11		2	4			
6000÷6	17	15	37		35	9	2	4		4	6	
4÷½	35		13		13	7				2		2
6÷12	4	0	14		19	15		2		4	2	
68÷17	42	10	27	6		10	2		4			

Table III: Use of Strategies by Problem (in Percentages)

As expected, the different problems required different strategies for solution. It is also interesting to note that in the solution of 96÷4, 32% of *successful* solutions involved an algorithmic approach (A), 32% partitioning the dividend (PD) and 36% counting up in groups of the divisor (CU). In the solution of 34÷7, all of the six successful solutions involved counting up in groups of the divisor (CU). In the solution of 6000÷6, 55% of successful solutions involved mental strategies or images (M) and 39% an algorithmic approach (A). In the solution of 4÷½, 75% of successful solutions involved counting up in groups of the divisor (CU). All of the successful solutions for 6÷12 involved mental strategies or images (M). In the solution of 68÷17, 93% of successful strategies involve counting up in groups of the divisor (CU).

As predicted, flexibility in the selection of solution strategy was associated with greater success in solving the problems: The correlation between the number of strategies which each pupil attempted in the course of solving the six problems and the number of problems on which they were successful was significant ($r=0.444$; $p<0.01$).

Investigation of individual pupils' responses also supported the view that pupils who were flexible in their solution strategies achieved more success than those who were limited to only one approach. For example, Hollie (Year 5) was successful for 5 out of the problems, using counting up in groups of the divisor for 4÷½ and 68÷17, partitioning the dividend for 96÷4 and mental strategies for 6000÷6 and 6÷12. Her strategy for 34÷7 was indecipherable and she did not reach an answer. On the other hand, Lucy (Year 5) had a limited choice of strategies available to her and attempted a strategy of counting up in groups of the divisor for all the problems except 6÷12, although she did attempt one additional strategy (written algorithm) in the case of 68÷17. She was not successful in any of the problems.

Few pupils were able to use pencil and paper as a successful aid to their memory. An interesting phenomenon observed was the inability of some pupils to keep track of the (sometimes relatively sophisticated) strategies which they began. A good example of this is Lauren (Year 5) solving 96÷4. "Well what I did was I, um, um, sorry...I did 4 divided by 10 first, which was, um, no 10...I know what I did. I did 4 divided by 40 which was 10...no, sorry, I meant 40 divided by 4 which was 10, and then, so I put 10 down, and then I did 4, 4 divided by 80, no 80 divided by 4 which was 20, and then, and then., then I added 16 on, and that's what I got. 46". Lauren was attempting to partition the dividend but lost track firstly while splitting up 80, and then by simply adding 16 on at the end. The question arises as to whether carrying out a strategy which places too great a load on working memory should be regarded as an inappropriate strategy choice or whether this is, more positively, part of the process of strategy development. In any case, pupils may need to be taught or encouraged to make effective use of written recordings and visual representations.

Within the constraints of this paper, it is not possible to report fully on error analysis. Briefly, as expected, there was evidence of some common ideas based on a limited model of division. What is interesting is the pupils' way of dealing with this challenge. In the case of 4÷½, doing a different sum (DS) (in this case ½ OF 4) occurred in 38% of unsuccessful attempts at this problem. Reversing the two numbers of the problem (SN) occurred in 83% of unsuccessful attempts at 6÷12, for example Danny (Year 6): "Can't do 6 divided by 12, so I did 12 divided by 6. I switched things around...did 12 divided by 6". In this case, one way of dealing with numerical data which challenges pupils' derived models of division appears to be changing the *roles* of the divisor and dividend.

Discussion

Although counting up in groups of the divisor and written algorithms were common choices, the richness and diversity of strategies, mental, written and verbal, which the pupils in this study used is impressive, in line with the findings of Murray *et al*. (1991; 1994).

An important finding of this study concerned the fact that *flexibility* in the choice of strategies enables greater success in the solution of a variety of challenging division problems. For this, interpretation of division as both sharing and measurement is necessary. At a later stage, pupils may then become even *more* flexible and able to choose a general, most efficient strategy (in line with the measurement interpretation) for all division problems (Murray *et al*., 1992).

Teachers need to encourage the development of both interpretations of division and a variety of strategies for solving division problems. One way of doing this may be to expose pupils to both partitive (sharing) and quotative (grouping) problems in context, at an early age.

The problems 6÷12 and 4÷½, for example, are difficult to interpret if one is constrained by a model of division in which the divisor is always a whole number and the quotient and divisor are always smaller than the dividend. There was evidence of such ideas in the pupils who participated in this study. Tirosh and Graeber (1991) suggest that it is particularly important to challenge and hold discussions about common misconceptions about division and the meaning of the division symbol.

In general, pupils should be exposed at an earlier age to a wider range of division problems, which present a more realistic view than the simplified examples which conform to the sharing model from which they induce not only a general pattern which limits their further understandings of division.

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