Malati

Mathematics learning and teaching initiative

Introductory Calculus

Module 3

Grade 12

TEACHER DOCUMENT

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December 1999

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Note: **Error!**

You may find that some fractions in the Calculus materials on the CD are displayed as **Error!**

This happens because we have used the comma as list separator in the Equation field, and you have a different default setting. To have the fractions displayed correctly, you will have to (temporarily) change your default settings. In Windows 95 go Start | Settings | Control Panel | Regional Settings | Number
Now mark the decimal symbol as a period (point) and change the List separator to a comma.
(In Windows NT this is in the Number Format area of the International Control Panel).

1. How fast to travel

1. A man is travelling by car from Durban to Johannesburg. At 10:00 he is still 360 km from Johannesburg. At what speed should he travel to reach Johannesburg by 14:00?

2. Suppose the man in question 1 did manage to travel the remaining 360 km in exactly 4 hours. Do you believe that he travelled at exactly 90 km/h all the time? Discuss this with some classmates.

3. Abel and Mary are travelling with different cars from Johannesburg to Cape Town. They are travelling with big BMW's, both fitted with speed controllers. A speed controller is a device that keeps the speed of the car constant, and it is used by people who want to make sure that they do not exceed speed limits.

   Mary prefers not to drive at the same speed all the time. For the first two hours of her journey, she sets the speed at 110 km/h. For the next two hours she sets the speed at 100 km/h. For the next hour she sets the speed at 130 km/h.

   Abel wants to cover the same distance as Mary over the first five hours of his journey. But he prefers to travel at the same speed all the time. At what speed should he set his controller to achieve this?

> When we say that an object moves at an ***effective*** speed of 90 km/h for a period of time, we do not mean that the actual speed is 90 km/h during the period. We only mean that the same distance will be covered in the period of time if the object would move at a constant actual speed of 90 km/h.

4. Consider the situation in question 3 again.
   (a) Did Mary and Abel actually travel at the same speeds?
   (b) What was Mary's effective speed over the five hours?

5. A man travelled for 4 hours and covered a distance of 400 km in this time. Can one say that he travelled at 100 km/h for all this time?

Teacher's note*:*

Because of different interpretations associated with the term "average", this activity is aimed at clarifying the way in which "average" is used in calculus. In order to afford learners an opportunity to conceptualise the notion of "average" afresh, explicit use of the term is avoided; instead, an equally valid but less known term – "effective" – is used.

2. Slow and fast growing

1. In the following table the heights (in metres) of three children are given at different ages.

Age in years	1	2	3	4	5	6	7	8
John's height	0,34	0,47	0,77	0,98	1,15	1,32	1,46	1,57
Peter's height	0,42	0,51	0,76	0,93	1,09	1,26	1,31	1,42
Mary's height	0,36	0,44	0,75	0,99	1,20	1,32	1,41	1,51

(a) Who grows at the highest rate from age 1 to age 2?

(b) Who grows at the lowest rate from age 1 to age 2?

(c) Who grows at the highest rate from age 2 to age 3?

(d) Who grows at the lowest rate from age 2 to age 3?

(e) By how much does John grow over the whole period described in the table?

(f) How much, on the average, does John grow in one year?

The *effective growth rate over an age interval* (x_1, x_2) may be defined as the amount grown between age x_1 and age x_2, divided by the age difference $x_2 - x_1$.

This is the same as $\dfrac{\text{height at age } x_2 - \text{height at age } x_1}{x_2 - x_1}$

(g) Determine John's effective growth rate between ages 1 and 8.

(h) Determine John's effective growth rate between ages 2 and 8.

(i) Determine John's effective growth rate between ages 2 and 4.

(j) Determine John's effective growth rate between ages 4 and 8.

(k) Determine John's effective growth rate between ages 6 and 8.

(l) Determine John's effective growth rate between ages 7 and 8.

Teacher's note:

The activity is aimed at allowing learners to experience the relationship between the rate of change and the dependent variables (e.g. height) and to provide an opportunity for determining effective growth rates at specified intervals. A link between the terms "effective" and "average" is made possible in this activity.

3. Effective speeds over small intervals

1. An object starts moving at 09:00 (nine o'clock sharp) from a certain point A. In the first minute of its journey, i.e. from 09:00 till 09:01 it travels a distance of 7675 metres. Between 09:01 and 09:02 it travels a distance of 7725 metres. Between 09:02 and 09:03 it travels a distance of 7775 metres. Between 09:03 and 09:04 it travels a distance of 7825 metres. Between 09:04 and 09:05 it travels a distance of 7875 metres. Between 09:05 and 09:06 it travels a distance of 7925 metres. Between 09:06 and 09:07 it travels a distance of 7975 metres.

   (a) Represent this information in the following table:

Time interval	09:00-09:01	09:01-09:02	09:02-09:03	09:03-09:04	09:04-09:05	09:05-09:06	09:06-09:07
Distance travelled during time interval							

   (b) How far is the object from A at 09:01?

   (c) How far is the object from A at 09:02?

   (d) How far is the object from A at 09:03?

   (e) How far is the object from A at 09:04?

   (f) How far is the object from A at 09:05?

   (g) How far is the object from A at 09:06?

   (h) How far is the object from A at 09:07?

   (i) Represent your answers in the following table:

Time	09:00	09:01	09:02	09:03	09:04	09:05	09:06	09:07
Distance from A								

2. An object starts moving from a point A. The following table gives some information about the journey:

Time since start of journey (minutes)	0	1	2	3	4	5	6	7
Distance from A in metres	0	6533	13032	19497	25928	32325	38688	45017

Complete the following table:

Part of journey	1st min	2nd min	3rd min	4th min	5th min	6th min	7th min
Distance covered							

3. An object starts moving from a point A. The following table gives some information about the journey:

Part of journey	1st min	2nd min	3rd min	4th min	5th min	6th min	7th min
Distance covered	7907	7961	8015	8069	8123	8177	8231

Complete the following table:

Time since start of journey (minutes)	0	1	2	3	4	5	6	7
Distance from A in metres								

4. An object starts moving from a point A. After one minute it has covered a distance of 8964 metres. After 2 minutes it is 17983 metres from A.

   (a) What distance was covered in the second minute of the journey?

   (b) What distance was covered in the first two minutes of the journey?

   After 3 minutes the object is 27065 metres from A.

   (c) What distance was covered in the third minute of the journey?

   (d) What is the effective speed over the third minute of the journey?

   (e) What distance was covered over the first three minutes of the journey?

   (f) What is the effective speed over the first three minutes of the journey?

   (g) Compare your answers for questions (d) and (f). If they are the same, consider the following:

   Is the object travelling at a constant speed?

   If not, is it travelling faster all the time or is it travelling slower all the time?

The effective speed for the first three minutes of the journey, in metres per minute, is equal to the total distance covered in the first three minutes, divided by three, it is 27065 ÷ 3 = 9021,7 metres per minute.

After two minutes it had covered a distance of 17983 metres. After three minutes (one minute later) the object had covered a distance of 27065 metres. Hence during the third minute it had covered an additional 9082 metres. hence the effective speed during the third minute is *distance covered* ÷ *time* = 9082 ÷ 1 = 9082 metres per minute.

   (h) After 4 minutes the object is 36217 metres from A. What is the effective speed over the first 4 minutes?

   (i) What is the effective speed *during* the fourth minute of the journey?

5. The distance covered by a certain moving object after different periods of time (i.e. the distance from the starting point) is given in the table below.

Time in seconds	1	2	3	4	5	6	7	8
Distance in metres	16496	32984	49464	65936	82400	98856	115304	131744

(a) Is the object moving at a constant speed, or is it moving faster (or slower) as time goes on?

(b) What is the effective speed of the object over the first four seconds of the journey?

(c) What is the effective speed of the object over the last four seconds of the journey?

(d) The object hits a stationary object exactly eight seconds after the start of the journey. At what speed will it hit the stationary object? (Estimate, compare and discuss your estimates).

(e) Compare your answers for questions (c) and (d). If they are the same, consider the following:

Is the object travelling at a constant speed?

If not, is it travelling faster all the time or is it travelling slower all the time?

(f) What is the effective speed of the object over the last three seconds of the journey?

(g) What is the effective speed of the object over the last two seconds of the journey?

(h) What is the effective speed of the object over the last second of the journey?

(i) The object hits a stationary object exactly eight seconds after the start of the journey. At what speed will it hit the stationary object? (Estimate, compare and discuss your estimates).

6. Miriam arrived x minutes late for a lecture. Cynthia and Thabu also arrived late, but Cynthia arrived t minutes after Miriam, and Thabu arrived *h* minutes before Miriam.

(a) Write an expression which describes how late Miriam arrived.
(b) Write an expression which describes how late Thabu arrived.
(This may be tricky. It will solve you a lot of trouble later on if you sort it out now.)

***Teacher's note**:*

A distinction is drawn among different kinds of rate of change, namely; the constant rate of change, increasing rate of change and decreasing rate of change. The use of letter symbols is introduced and a common way of denoting intervals is introduced (e.g. 1st interval for interval [0-1]).

4. Approximating the effective speed at a certain moment

1. The distance d in metres covered by a moving object t seconds after it started to move is given by the formula $d = 1,6t^2$.

   (a) Determine the effective speed of the object over the first 5 seconds after it had started to move.

   (b) Determine the effective speed of the object for the period 1 second after it started to move up to 5 seconds after it had started to move.

   (c) The object hits a wall exactly 5 seconds after it had started to move. At what speed does it hit the wall? Approximate the speed as well as you can. (If you cannot do this now, first do questions (d) to (j) and then return to this question).

   (d) Determine the effective speed of the object over the last 3 seconds before it hit the wall.

   (e) Determine the effective speed of the object over the last 2 seconds before it hit the wall.

   (f) Determine the effective speed of the object over the last 1 second before it hit the wall.

   You are going to do a lot of calculations like these. It will help if you make a formula for the effective speed over the last h seconds before the object hits the wall. This effective speed is equal to

$$\frac{\text{distance after 5 seconds} - ?}{?}$$

   Hint: First write a formula for the distance covered over the last (5 – h) seconds.

   (g) Determine the effective speed of the object over the last 0,5 seconds before it hit the wall.

   (h) Determine the effective speed of the object over the last 0,1 seconds before it hit the wall.

   (i) Determine the effective speed of the object over the last 0,05 seconds before it hit the wall.

   (j) Determine the effective speed of the object over the last 0,01 seconds before it hit the wall.

2. (a) Expand $8(3 - x)^2$ and $6(x - h)^2$.

(b) Simplify $5x^2 - 5(x - h)^2$.

(c) Simplify $\dfrac{4(7)^2 - 4(7 - h)^2}{h}$

3. The distance d in metres covered by a moving object t seconds after it started to move is given by the formula $d = 4t^2$.

(a) Try to determine at what speed the object will hit a wall exactly 10 seconds after it had started to move. If you cannot do this now, first do question (b).

(b) Complete the following table:

h	3	2	1	0,5	0,1	0,05	0,01
Effective speed over last h seconds before wall is hit							

Hint: Make a formula for calculating these effective speeds and simplify it.

The distance d in metres covered by a moving object t seconds after it started to move is given by the formula $d = 5t^2$. It hits a wall exactly 4 seconds after it had started to move. We want to determine the effective speed over the last h seconds before it hits the wall. This effective speed is given by:

$$\frac{\text{distance after 4 seconds} - \text{distance after } (4 - h) \text{ seconds}}{h}$$

Since distance after t seconds is given by $5t^2$, this is

$$= \frac{5(4)^2 - 5(4 - h)^2}{h}$$

$$= \frac{5(4)^2 - 5(16 - 8h + h^2)}{h}$$

$$= \frac{5 \times 16 - 5 \times 16 + 40h - 5h^2}{h} \quad \text{for all values of } h \text{ except } h = 0$$

$$= \frac{40h - 5h^2}{h}$$

$$= 40 - 5h \qquad \text{for all values of } h, \text{ except } h = 0.$$

Hence the effective speed from $4 - h$ seconds until 4 seconds after the movement had started is given by the formula $40 - 5h$.

4. (a) Use this formula to complete the following table:

h	1	0,5	0,1	0,05	0,01	0,005	0,001
Effective speed over last *h* seconds before wall is hit							

(b) Which of the above effective speeds is closest to the actual speed after 4 seconds?

(c) How can an effective speed be calculated which is even closer to the actual speed after 4 seconds?

(d) Will the effective speed over any time interval before 4 seconds ever exceed 40 metres per second?

(e) What do you believe is the actual speed when the object hits the wall after 4 seconds?

(f) Use the same method to determine the speed at which the object would hit a wall after 5 seconds.

(g) Determine the formula by which the effective speed from 6 – h to 6 seconds can be calculated. Use this formula to determine the effective speed that is closer to the actual speed during the 6^{th} second.

(h) Determine the formula by which the effective speed from 7 – h to 7 seconds can be calculated. Use this formula to determine the effective speed that is closer to the actual speed during the 7^{th} second.

(i) Determine the formula by which the effective speed from 8 – h to 8 seconds can be calculated. Use this formula to determine the effective speed that is closer to the actual speed during the 8^{th} second.

(j) Determine the formula by which the effective speed from 9 – h to 9 seconds can be calculated. Use this formula to determine the effective speed that is closer to the actual speed during the 9^{th} second.

(k) Determine a formula by which the effective speed from x – h to x seconds can be calculated. Use this formula to determine the speed at which the object would hit a wall x seconds after it had started to move.

(l) (This will take quite some time)

Repeat questions (a) to (k) for a moving object whose distance d in metres after t seconds is given by the formula $d = 4t^2$.

(m) Repeat questions (a) to (k) for a moving object whose distance d in metres after t seconds is given by the formula $d = 3t^2$.

(n) Repeat questions (a) to (k) for a moving object whose distance d in metres after t seconds is given by the formula $d = 2t^2$.

(o) Try to write down a formula by which the speed after t seconds of a moving object can be determined, if the distance d in metres moved after t seconds is given by a formula of the form $d = kt^2$.

(p) Repeat as much of questions (a) to (n) as you need to determine a formula by which the speed after t seconds of a moving object can be determined, if the distance d in metres moved after t seconds is given by a formula of the form $d = kt^2 + mt$.

(q) Repeat as much of questions (a) to (n) as you need to determine a formula by which the speed after t seconds of a moving object can be determined, if the distance d in metres moved after t seconds is given by a formula of the form $d = at^2 + bt + c$.

5 (a) The speed of a moving object t seconds after it had started to move is given by the formula *speed* = $16t$. Find a formula by which the *distance* that the object has moved t seconds after it had started to move can be calculated.

(b) The speed of a moving object t seconds after it had started to move is given by the formula *speed* = $23t$. Find a formula by which the *distance* that the object has moved t seconds after it had started to move can be calculated.

6. A stone is thrown vertically upwards. Its height h in metres after t seconds is given by the formula $h = –4{,}9t^2 + 26{,}4t$.

(a) Complete the following table (be careful):

t (seconds)	1	2	3	4	5	6	7
h (metres)							

(b) When will it hit the ground again?

(c) When will its upward speed be zero?

(d) How high will it go?

***Teacher's note**:*
The use of a formula to describe the relationship between distance and time is introduced. The need for a general expression describing effective speed for functions e.g. $d = 1,6t^2$ and for a class of functions (e.g. $d = kt^2$) is provided. Conditions for which (1) the interval (h) needs to be made small and (2) what is actually achieved by that process is investigated. It is useful to make it explicit that the expressions $\frac{f(t)-f(t-h)}{h}$ and $\frac{f(t+h)-f(t)}{h}$ are equivalent.

5. Varying gradients

1. (a) Complete the table:

x	1	2	3	4	5	6	7	8
$g(x) = 3x + 5$								
$f(x) = 5x^2$								
$k(x) = 10 - 4x$								

(b) Which of the function(s) *increase(s)* as x is increased from 1 to 8?

(c) Which function(s) *decrease(s)?*

(d) Which function increases *the fastest?*

(e) Which function(s) change *at a constant rate?*

(f) Which function(s) change at a *varying rate*?

The value of a function normally varies as the value of the input variable varies.

For instance, when x is increased from 1 to 4, the value of the function g in the above table increases from 8 to 17 (i.e. an increase of 17 – 8 = 11), while the value of the function f increases from 5 to 80 (i.e. an increase of 80 – 5 = 75).

Over the same interval (x = 1 to x = 4), the value of the function k ***decreases*** from 6 to –30 (i.e. a decrease of –30 – 6 = –36).

It is clear that the value of the function f above changes ***faster*** (at a higher ***rate***) than the value of the function g as x is increased from 1 to 4. We can express this by saying:

The function f above has a higher ***effective rate of change*** over the interval 1 to 4 than the function g. Another word for "rate of change" is "gradient", so we can also say that f has a ***higher gradient*** than g.

The effective rate of change (gradient) over an interval (x_1, x_2) of a function is defined as *the difference between the function values at* x_2 *and* x_1, divided by $x_2 - x_1$.

If the function is labelled f, this can be expressed as $\dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

2. This question refers to the functions in question 1.

   (a) Determine the effective gradient of f over the interval (2; 7).

   (b) Determine the effective gradient of f over the interval (2; 5).

   (c) Determine the effective gradient of f over the interval (1; 3).

   (d) Determine the effective gradient of f over any other interval.

   (e) Determine the effective gradient of g over the interval (2; 7).

   (f) Determine the effective gradient of g over the interval (2; 5).

   (g) Determine the effective gradient of g over the interval (1; 3).

   (h) Determine the effective gradient of g over any other interval.

When a function describes some practical situation, the effective gradient of a function over an interval normally describes some important aspect of the practical situation.

If, for instance, the function $f(x) = 5x^2$ describes the relationship between the displacement $f(x)$ of a moving body and the time x, the gradient $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the effective speed of the body for the period x_1 to x_2.

In such a situation one often wishes to know what the speed at a specific point in time will be. One may for instance wish to know at what speed the object will hit another object 4 seconds after it had started to move.

3. Determine, as precisely as you can, utilising what you have learnt in the previous tasks, the actual gradient of each of the functions in question 1 at $x = 4$.

Teacher's note*:*

The activity introduces the "gradient" as a form of "rate of change". The way in which gradients of different types of functions behave is investigated and provision is made for determining the gradient at a point (the derivative).

6. Making sure

In the previous tasks, you have made a certain assumption when determining the actual gradient of a function at a specific value of the variable. In this exercise you will do some investigations to gain more clarity about the validity of this assumption.

1. Expand:

   (a) $(x + y)^2$

   (b) $(x + y)(x^2 + 2xy + y^2)$

   (c) $(x + y)^3$

   (d) $(x - y)^2$

   (e) $(x - y)^3$

2. Check all your answers for question 1 by first substituting 5 for x and 3 for y, and then substituting 10 for x and 5 for y, into both the original expressions and the expressions you have produced.

3. The quantity $\dfrac{f(x) - f(x-h)}{h}$ is called the effective gradient over the interval $(x - h;\ x)$ of the function f.

   (a) If $f(x) = x^3$, determine the effective gradient for each of the following intervals:

   (First make a formula with which the effective gradient of f over an interval $(5 - h, 5)$ can be calculated, and simplify this formula.)

   (1) (3; 5)
   (2) (4; 5)
   (3) (4,5; 5)
   (4) (4,9; 5)
   (5) (4,99; 5)
   (6) (4,999; 5)

   (b) Specify an interval over which the effective gradient of f will differ from 75 by less than 0,1.

   (c) Specify an interval over which the effective gradient of f will differ from 75 by less than 0,01.

   (d) Specify an interval over which the effective gradient of f will differ from 75 by less than 0,0001.

   (e) Specify an interval over which the effective gradient of f will differ from 75 by less than 0,00001.

Since the effective gradient of f over the interval $(5 - h; 5)$ can be made as close to 75 as one wishes, simply by taking a small enough value of h, we may assume that *the gradient of f at 5* is 75.

If $f(x)$ was the distance in metres covered by a moving object in x seconds after it started to move, this would mean that the speed of the object after 5 seconds is assumed to be exactly 75 metres per second.

(f) Determine the gradient of f at 3.

4. (a) Make a formula with which the effective gradient of the function $g(x) = x^2$ over an interval $(x - h; x)$ can be determined. Simplify this formula.

(b) Use your formula to determine the effective gradients of g over the following intervals:

(2; 4), (3; 4), (3,5; 4), (3,9; 4), (3,99; 4), (3,999; 4), (3,999; 4)

(c) What is the gradient of g at 4?

(d) Determine the gradient of g at 2.

(e) Determine the gradient of g at 5.

(f) Determine the gradient of g at 6.

(g) Determine the gradient of g at 7.

(h) Determine the gradient of g at 8.

(i) Determine the gradient of g at 3.

(i) Determine the gradient of g at 3,4.

(j) Can you suggest a simple formula with which the gradient of g at any value of x can be determined?

(k) Check your formula for x = 2,3 and x = 3,7.

5. Complete the table (this is quite a lot of work) for the function $r(x) = 3x^2$:

x	10	9	8	7	6	5	4	3
gradient of r at x								

6. Try to find a formula with which the gradient of any function of the form $f(x) = ax^2$ at any value of x can be determined. You may wish to try to also find "gradient formulae" for functions of the form $g(x) = ax^3$.

***Teacher's note**:*

The activity focuses on providing an opportunity to ascertain that the value of a gradient at a point is assumed from making the interval as small as we please. There is also an opportunity to ascertain that making the interval negligibly small does not significantly change the value of the gradient at a point. Letter symbol manipulation of algebraic expressions that pupils often encounter in calculus sections is introduced.

7. The gradient of functions of the form ax^3

The effective gradient over the interval $(x - h, x)$ of a function of the form $f(x) = \text{a}x^3$ is given by $\dfrac{f(x) - f(x-h)}{h}$ where h is the "width" of the interval.

Since $f(x) = \text{a}x^3$ in this case,

$$\frac{f(x) - f(x-h)}{h} = \frac{\text{a}x^3 - \text{a}(x-h)^3}{h}$$

$$= \frac{ax^3 - a(x-h)(x-h)^2}{h}$$

$$= \frac{ax^3 - a(x-h)(x^2 - 2xh + h^2)}{h}$$

$$= \frac{ax^3 - a(x^3 - 3x^2 + 3xh^2 - h^2)}{h}$$

$$= \frac{ax^3 - ax^3 + 3ax^2h - 3axh^2 + ah^3}{h}$$

$$= \frac{3ax^2h - 3axh^2 + ah^3}{h}$$

$$= 3ax^2 - 3axh + ah^2$$

1. (a) Use the above to write down (in simplest form) a formula for the effective gradient over the interval $(5 - h, 5)$ of the function $f(x) = 4x^3$

   (b) Suppose someone calculates the effective gradients for h = 2; 1; 0,5; 0,1; 0,05; 0,01; 0,001; 0,001; 0,0001. Will the answers come closer and closer to some number without ever reaching or exceeding it? If so, what is this number? If you can answer this question with confidence, go to question 2. If you have difficulty with this question, calculate the gradients (use the programming facility on your calculator) and do the other parts of question 1.

   (c) Find a value of h for which the effective gradient of $f(x) = 4x^3$ over the interval $(5 - h, 5)$ differs from 300 by less than 1.

   (d) Find a value of h for which the effective gradient of $f(x) = 4x^3$ over the interval $(5 - h, 5)$ differs from 300 by less than 0,1.

(e) Find a value of h for which the effective gradient of $f(x) = 4x^3$ over the interval $(5 - h, 5)$ differs from 300 by less than 0,001.

(f) Find a value of h for which the effective gradient of $f(x) = 4x^3$ over the interval $(5 - h, 5)$ differs from 300 by less than 0,0001.

(g) Find a value of h for which the effective gradient of $f(x) = 4x^3$ over the interval $(5 - h, 5)$ differs from 300 by less than 0,000000001.

(h) How would you defend the claim (in writing) that the gradient of $f(x) = 4x^3$ at $x = 5$ is 300?

The fact that $300 - 60h + 4h^2$ can be made as close to 300 as one wishes by taking a small enough value of h is normally expressed by saying that

"*the limit of* ($300 - 60h + 4h^2$) as h tends to 0 is 300".

The following is a shorthand way to express this: $\lim_{h \to 0} 300 - 60h + 4h^2 = 300$

It is not difficult to understand why $300 - 60h + 4h^2$ can be made as close to 300 as one wishes by taking a small enough value of h:

Clearly, the smaller one makes h, the smaller (closer to 0) $60h$ and $4h^2$ becomes, but 300 remains 300..

We can say: $\lim_{h \to 0} 300 - 60h + 4h^2 = 300$ because $\lim_{h \to 0} 60h = 0$ and $\lim_{h \to 0} 4h^2 = 0$.

2. (a) Write down a formula with which the effective gradient over the interval $(3 - h, 3)$ of the function $f(x) = 4x^3$ can be calculated.

   (b) What is the limit of this effective gradient as h tends to 0?

   (c) At what rate is $f(x)$ changing when $x = 3$?

3. At what rate is $f(x)$ changing when $x = 2$? This is the same as asking "What is the gradient of $f(x)$ at $x = 3$". You may have to do some work to answer this question.

4. At what rate is $f(x)$ changing when $x = 3,4$?

For any values of a and x, the closer h is made to 0, the closer $3ax^2 - 3axh + ah^2$ will be to $3ax^2$", or more precisely, $3ax^2 - 3axh + ah^2$ can be made as close to $3ax^2$ as we wish, simply by making h close enough to 0.

Stated differently, $\lim_{h \to 0} (3ax^2 - 3axh + ah^2) = 3ax^2$ for any value of a.

Hence the gradient of ax^3 at any specific value of x is given by 3ax^2.

We say that $3ax^2$ is the *derivative* or *derived function* of $f(x) = ax^3$.
In a previous exercise we also used the term *gradient formula* for the same idea.

5. (a) Write down the formulae for the derived functions of the functions $f(x) = 5x^3$ and $f(x) = 1{,}6x^3$.

   (b) Determine the gradients of $f(x) = 1{,}6x^3$ at $x = 3$; $x = 4$; $x = 5$; $x = 6$; $x = 7$ and $x = 5$.

6. Water is flowing into a dam at an increasing rate during a storm. The volume (in litres) of water in the dam (which was empty when the storm started) is given by the formula $V = 0{,}7t^3$ where t is the time in minutes since water started to flow into the dam.

   (a) At what rate is water flowing into the dam 15 minutes after water had started to flow into the dam?

   (b) How much water flowed into the dam in the second half hour after water had started to flow into the dam?

Teacher's note:

This activity aims at developing the concept of a derivative by first giving exercises in which learners may note the effect of 'shrinking h'. The formal definition is deliberately held back until the ideas underlying the concept are reasonably developed.

8. Looking at the gradient of ax^3 from the other side, and finding the derivatives of some other functions.

1. (a) Derive a simple formula for the effective gradient over the interval $[x, x + h]$ for functions of the form $f(x) = ax^3$.

   (b) Compare the limits, as h tends to zero, of the effective gradients over the intervals [3, (3 + h] and [3 – h, 3] of the function $f(x) = 2x^3$.

   (c) Repeat question (b) for any function of the form $f(x) = ax^3$ and any value of x. Compare your finding with those of classmates.

2. Determine the derived function of $g(x) = 3x^2 + 5x$.

> The derived function of a function $f(x)$ is indicated by the symbol $f'(x)$.
>
> $f'(x)$ is read "f prime x".
>
> We write: If $f(x) = ax^3$, then $f'(x) = 3ax^2$.

3. Prove that if $f(x) = ax^2$, then $f'(x) = 2ax$.

4. Prove that if $f(x) = ax^4$, then $f'(x) = 4ax^3$.

5. Suggest a formula for finding $f'(x)$ for any function of the form $f(x) = ax^n$. You may wish to test your formula for ax^5.

> The part of mathematics that deals with finding the gradients of functions is called the **"differential calculus".**

Instead of the symbol $f(x)$ (or $g(x)$ or using some other letter as a name for the function) to indicate the values of a function, a single letter, for instance y may be used.

For instance, instead of writing $f(x) = 3x^2 + 5x + 3$, one may write
$y = 3x^2 + 5x + 3$.

When the "y-notation" is used instead of the $f(x)$ -notation, the derived function is sometimes indicated by $\frac{dy}{dx}$ instead of by $f'(x)$.

So for the function defined by $y = 3x^2$, we may write $\frac{dy}{dx} = 6x$.

6. If $q(x) = 3x^2 - 5x$, determine $q'(4)$ without using a formula. This means that you have to go through the whole argument concerning the gradient of $q(x)$ over the interval $(4 - h, h)$ or $(4, 4 + h)$.

   This process is referred to as "differentiation from first principles".

7. (a) If $y = 5x^2 + 7x$ and $z = -5x^2 + 7x$, determine $\frac{dy}{dx}$ for $x = 3$ and $\frac{dz}{dx}$ for $x = 3$ by differentiating from first principles

   (b) Find formulae with which $\frac{dy}{dx}$ and $\frac{dz}{dx}$ can be determined for any value of x.

8. (a) If $y = 7x + 5$ and $z = 7x + 3$, determine $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

   (b) Do all functions with formulae of the form $y = 7x + k$ have the same gradient?

   (c) If $y = 5x^2$ and $z = 7x$, what is $\frac{dy}{dx} + \frac{dz}{dx}$?

   (d) Compare your answers to questions 7(b) and 8(c).

   (e) Have you now learnt something which can make it easy to find the derivative of the function $f(x) = 3x^2 + 5x^3$? Take time to make quite sure about this.

9. (a) Determine the gradient of $f(x) = 5x^3$ for $x = 3$. You may simply use the formula for the derivative of $f(x) = 5x^3$.

   (b) Determine the derivative of $g(x) = 5x^3 + 3x^2$.

10. (a) Determine the derivative of $f(x) = x^2 + 5$. If you have difficulties, determine it from first principles.

   (b) Determine the derivatives of $g(x) = x^2 - 5$ and $h(x) = x^2 + 2$.

   (c) Can you explain why the functions f, g and h all have the same derivative?

11. If $g(x) = 2x^4 - 3x^2 - 5$, determine $g'(1)$, $g'(2)$, $g'(-1)$ and $g'(0)$.

Teacher's note*:*

Gradients of cubic functions and gradients of polynomials of degree 4 are investigated. The first exercise is aimed at highlighting that if the limit of a function exists, then it has the same value irrespective of the side from which x (the independent variable) is approached. Some widely used notations for the derivative, f`(x) and dy/dx), are introduced. There are also tasks which lead to the general observation that $dy/dx=anx^{n-1}$ *for* $y=ax^n$

9. Things about graphs

1. (a) In what respect would the graphs of the following two functions differ? (Make a prediction and then plot the graphs to check your prediction):

x	1	2	3	4	5	6	7	8
$f(x)$	10	12	14	16	18	20	22	24

x	1	2	3	4	5	6	7	8
$g(x)$	10	8	6	4	2	0	–2	–4

(b) Determine the gradients of the functions f and g.

2. (a) In what respect would the graphs of the following two functions differ? (Make a prediction and then plot the graphs to check your prediction):

x	1	2	3	4	5	6	7	8
$f(x)$	10	12	14	16	18	20	22	24

x	1	2	3	4	5	6	7	8
$k(x)$	10	13	16	19	22	25	28	31

(b) Determine the gradients of the functions f and k.

3. Which of the following functions will have the "steepest" graph, i.e. which graph will "rise quickest" as the value of x is increased? (Plot the graphs to investigate).

$f(x) = 3x + 1$ $g(x) = 2x + 1$

$k(x) = 0,5x + 1$ $t(x) = 3,5x + 1$

$p(x) = 1,5x + 1$ $q(x) = 2,5x + 1$

The "steepness" of a straight line graph may be expressed as the ratio

$$\frac{\text{amount by which the graph "rises" from } x_1 \text{ to } x_2}{x_2 - x_1}$$

This amount is called the *slope* of the graph.

4. Determine the slope of the graph of each of the functions in question 1.

5. If the graph of a linear function does not "rise" from left to right, but "falls", what can you say about the slope?

6. Draw the graph of any function of the form $f(x)$ = 2,4x + k. (You have to choose a value of k). Determine the slope of the graph. Compare results in the group.

The formula

$$\frac{\text{amount by which the graph "rises" from } x_1 \text{ to } x_2}{x_2 - x_1}$$

for the slope of a straight line graph can also be expressed as

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}$$ for any linear function f.

If the linear function is described in the form y = mx + c, the formula can be given as

$$\frac{y_2 - y_1}{x_2 - x_1}$$

Clearly, the slope of a graph of a linear function is equal to the gradient of the function.

7. In each case decide whether the statement is true or false and illustrate your answer with an example:

   (a) If the gradient of a linear function is positive, the graph will go "upwards" from left to right.

   (b) If the gradient of a linear function g is bigger than the gradient of another linear function h, and both the gradients are positive, then the graph of g will be "steeper" than the graph of h, i.e. it will go "sharper upwards" to the right.

8. The gradient of a certain linear function g is 2,4 and $g(0) = 1$. Draw a graph of g without working out any function values.

9. (a) The graph of the linear function f passes through the points (–2; 10) and (5; –3). Plot this graph over the domain –7 to 7.

   (b) Determine the gradient of f *from the graph.*

   (c) Find the formula for f.

   (d) Explain the graphical meaning of the *gradient of a linear function*.

10. The graph of a linear function g is shown on the next page. Take readings from the graph and find the formula for the function.

11. Draw an accurate graph, on graph paper, of the function $f(x) = x^2$ over the interval (–4; 4). Such a curve is called a parabola. (Graph paper is provided on page 34.)

    (a) What is the slope of this graph?

    (b) You may find it impossible to answer question (a). Maybe it is easier to produce some answer to the question

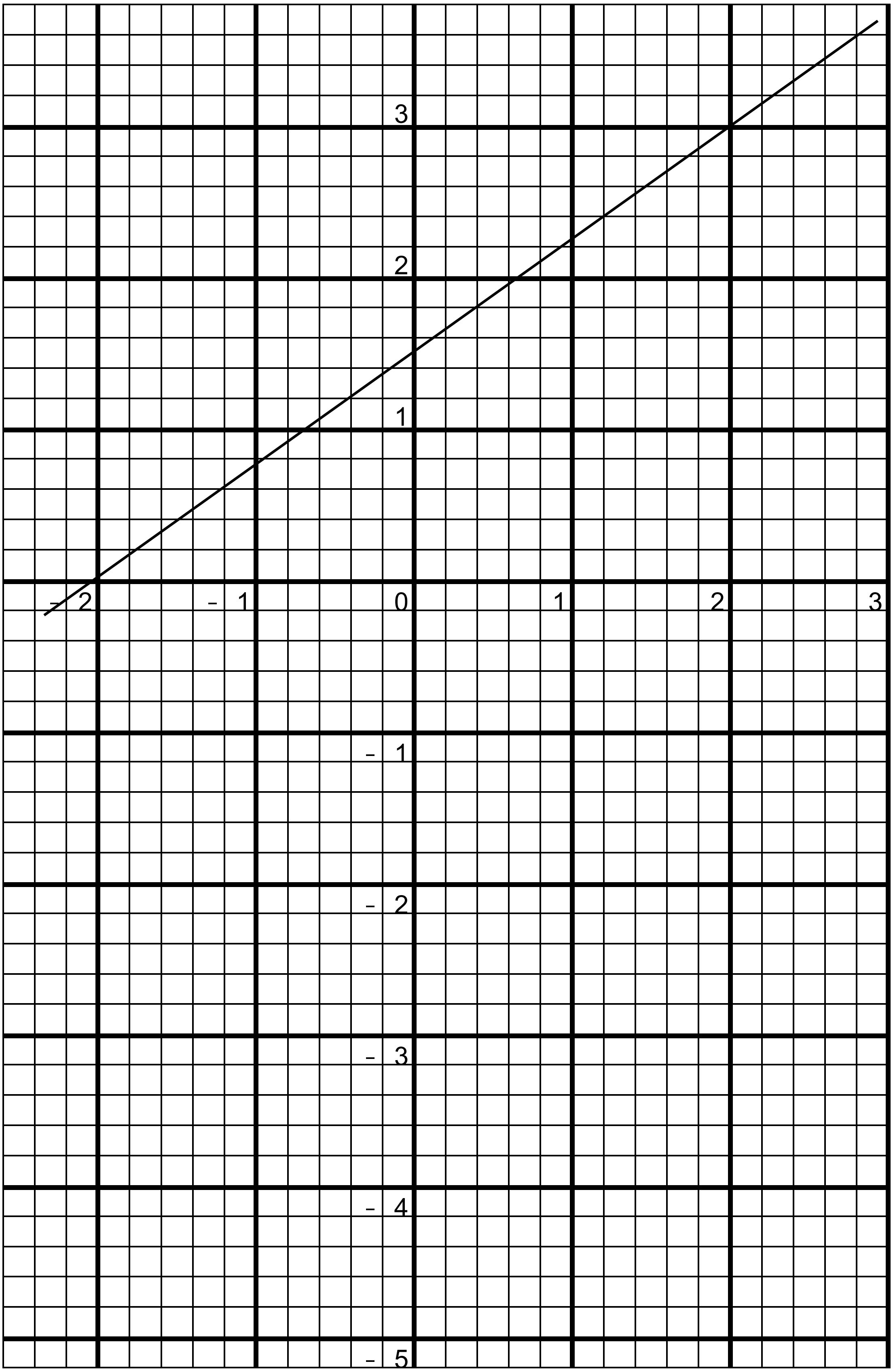
    "what is the slope of this graph over the interval (2; 4)?"

    and to the following questions:

    (c) What is the slope of this graph over the interval (2; 3)?

    (d) What is the slope of this graph over the interval (3; 4)?

The following Activity (10) will help you to get some clarity about the slopes of curved graphs.


3
2
1
- 2
- 1
0
1
2
3
- 1
- 2
- 3
- 4
- 5

Teacher's note:

The activity aims at investigating the influence of gradients on (1) the shape of graph (straight or curved) and (2) the "steepness" of the graph. The term "slope" is introduced as a geometric property of a graph.

10. Tangents and the slope of curves

1. (a) Draw a circle accurately (use compasses).

   (b) Mark any point A outside the circle and about 15 cm away from the circle. Draw a straight line from the point A through the circle so that it cuts the circle at two points. Such a line is called a *secant* to the circle. Draw another secant from the point A so that it cuts the circle at two points less than 1 cm apart.

   (c) Draw a straight line from A so that it touches the circle at one point only. Mark this point B. Such a line is called a *tangent* to the circle. Draw another tangent from point A to the circle, and mark the point where this tangent touches the circle C.

   (d) Measure AB and AC.

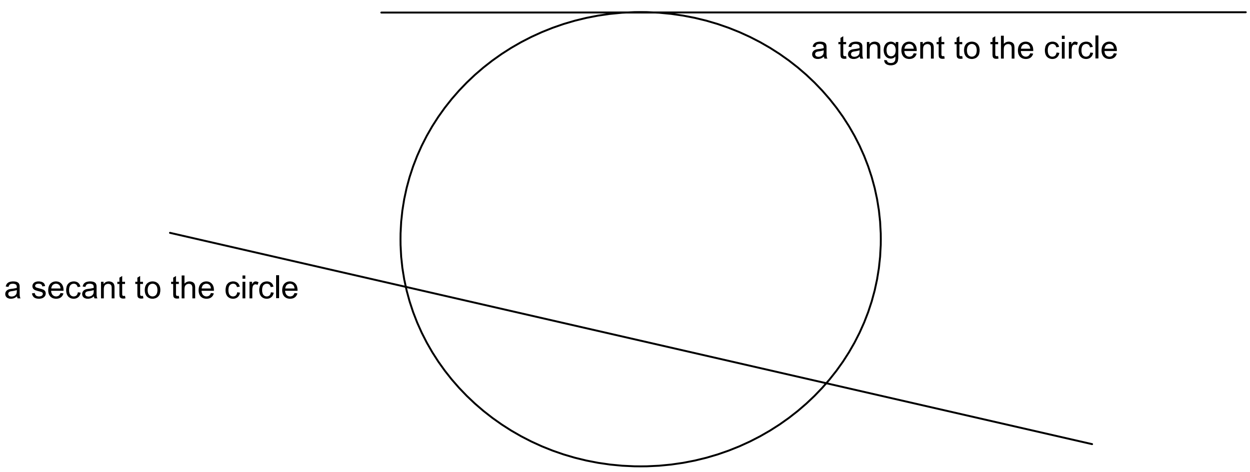
   (e) Join A with the midpoint M of the circle. Also join B to M and C to M.

   (f) (For homework): Produce a logical explanation of the fact that AB = AC.

   (g) Mark a point E directly above A and some 2cm away from A. Draw the two tangents from E to the circle.

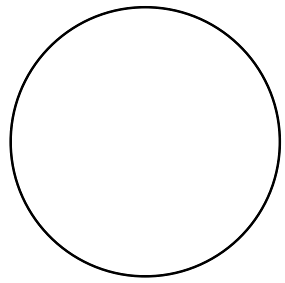
   (h) Can a tangent from A and a tangent from E touch the circle at the same point?

   (i) Draw another circle and mark a point F on its circumference. Draw a tangent which touches the circle at F. Can a different line through F than the one you have drawn also be a tangent to the circle?



2. (a) Draw a circle with a radius of about 10 cm.

(b) At some distance from the circle, mark 4 points lying in straight line, about 3 mm apart, more or less as shown in the sketch below, but with a bigger circle.



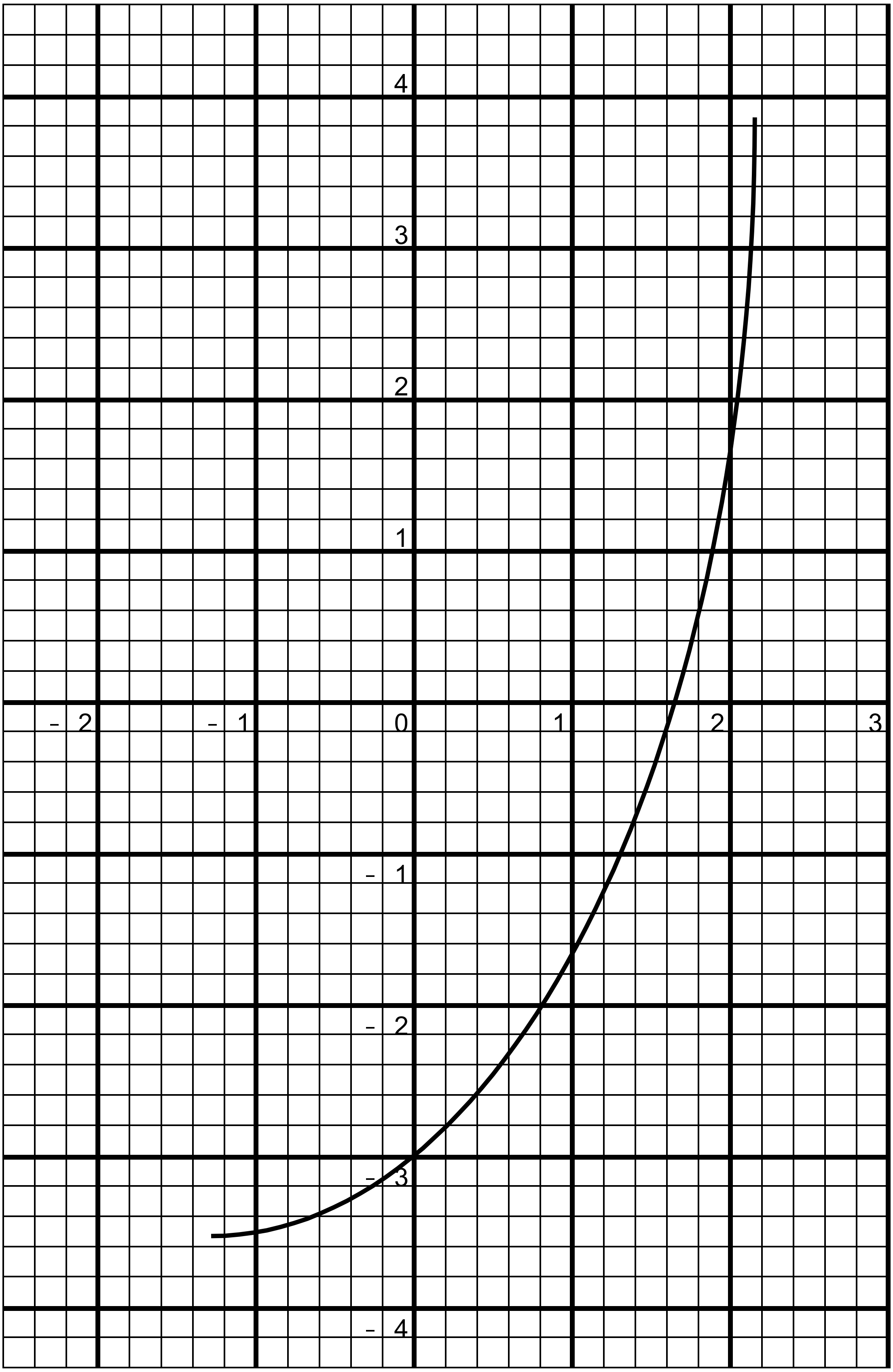
(c) From each of these points, draw a tangent to the upper half of the circle.

(d) Do you think that there could be more than one tangent at a specific point on the perimeter of a circle? Discuss with some classmates.

3. (a) The graph of a function f is given on the next page. Draw tangents to the graph at $x = -1$, $x = 0$, $x = 1$ and $x = 2$.

(b) Take readings from the graph and determine the slope of each tangent.

(c) What happens to the slopes of the tangents as you move from left to right?


4
3
2
1
- 2
- 1
0
1
2
3
- 1
- 2
- 3
- 4

4. (a) Complete the following table for the function $f(x) = x^2$. Use your calculator and round the function values off to two figures after the decimal sign.

x	0	0,5	1,0	1,5	2,0	2,5	3,0	3,5	4,0	4,5	5,0	5,5	6,0	6,5
$f(x)$														

(b) On graph paper, plot an accurate graph of f over the interval (0; 6,5).

(c) Draw, as well as you can, tangents to the curve at the points (6; 36), (4; 16), and (2; 4).

(d) Determine the slopes of the four tangent lines.

(e) What do you observe with respect to the slopes as you go from left to right?

(f) Will this tendency that you observed also hold for the tangent lines at (0; 0), (1; 1), (3; 9), (5; 25)?

(g) Calculate $f'(6)$, $f'(4)$ and $f'(2)$. Compare these values with the slopes of the tangent lines at the points (6: 36), (4; 16), and (2; 4).

5. (a) Plot an accurate graph of $f(x) = x^2$ for x ranging from –4 to 4. Such a curve is called a *parabola*.

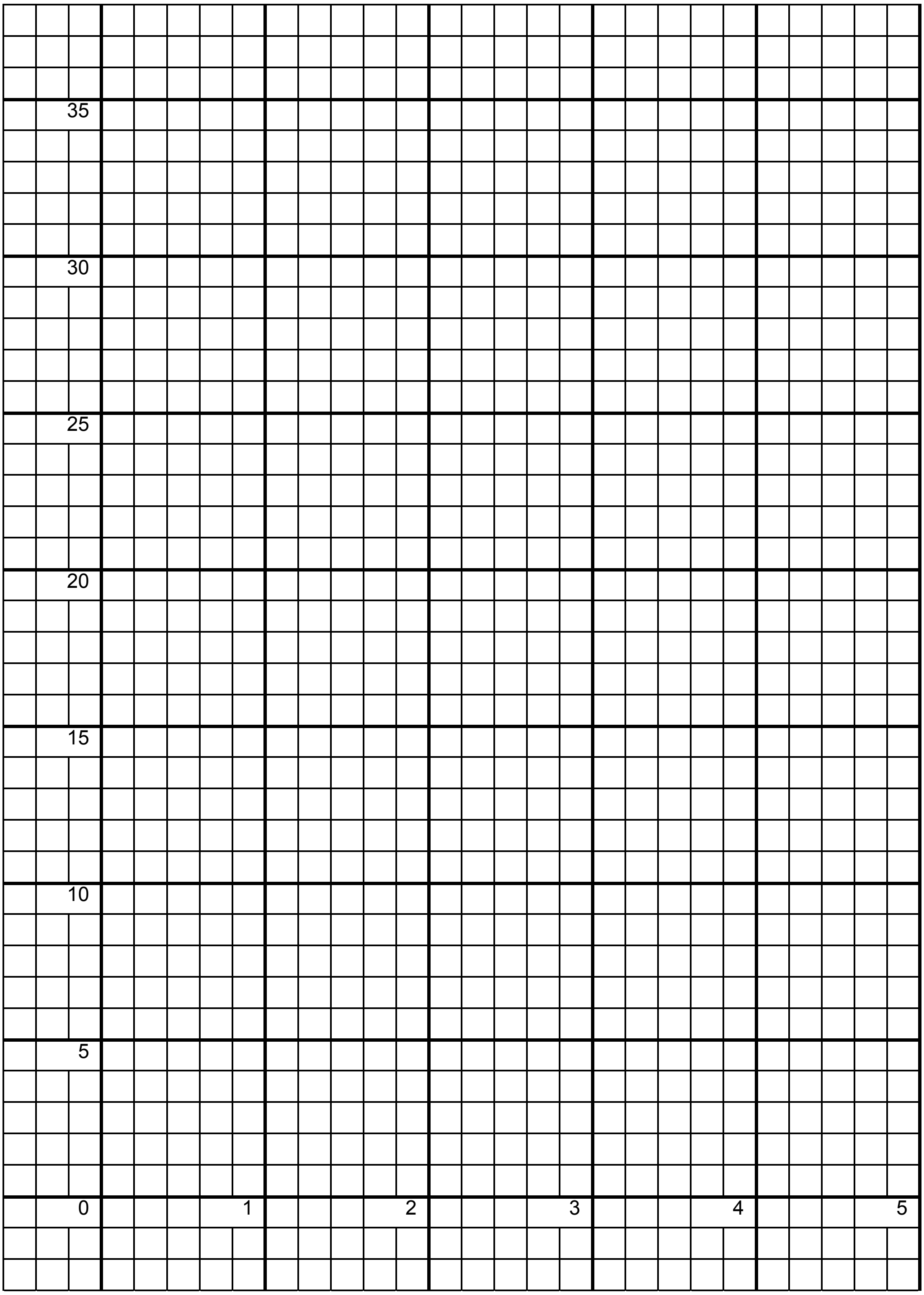
(b) Calculate $f'(-3)$ and $f'(2)$.

(c) Find a place on the graph where the tangent will have a slope equal to $f'(2)$ and draw this tangent. What are the co-ordinates of the point where it touches the graph of f?

(d) Find a place on the graph where the tangent will have a slope equal to $f'(-3)$ and draw this tangent. What are the co-ordinates of the point where it touches the graph of f?

(e) Draw the tangent to the graph of f which touches the graph at (0; 0).

(f) What is the value of $f'(x)$ when $x = 0$?


35
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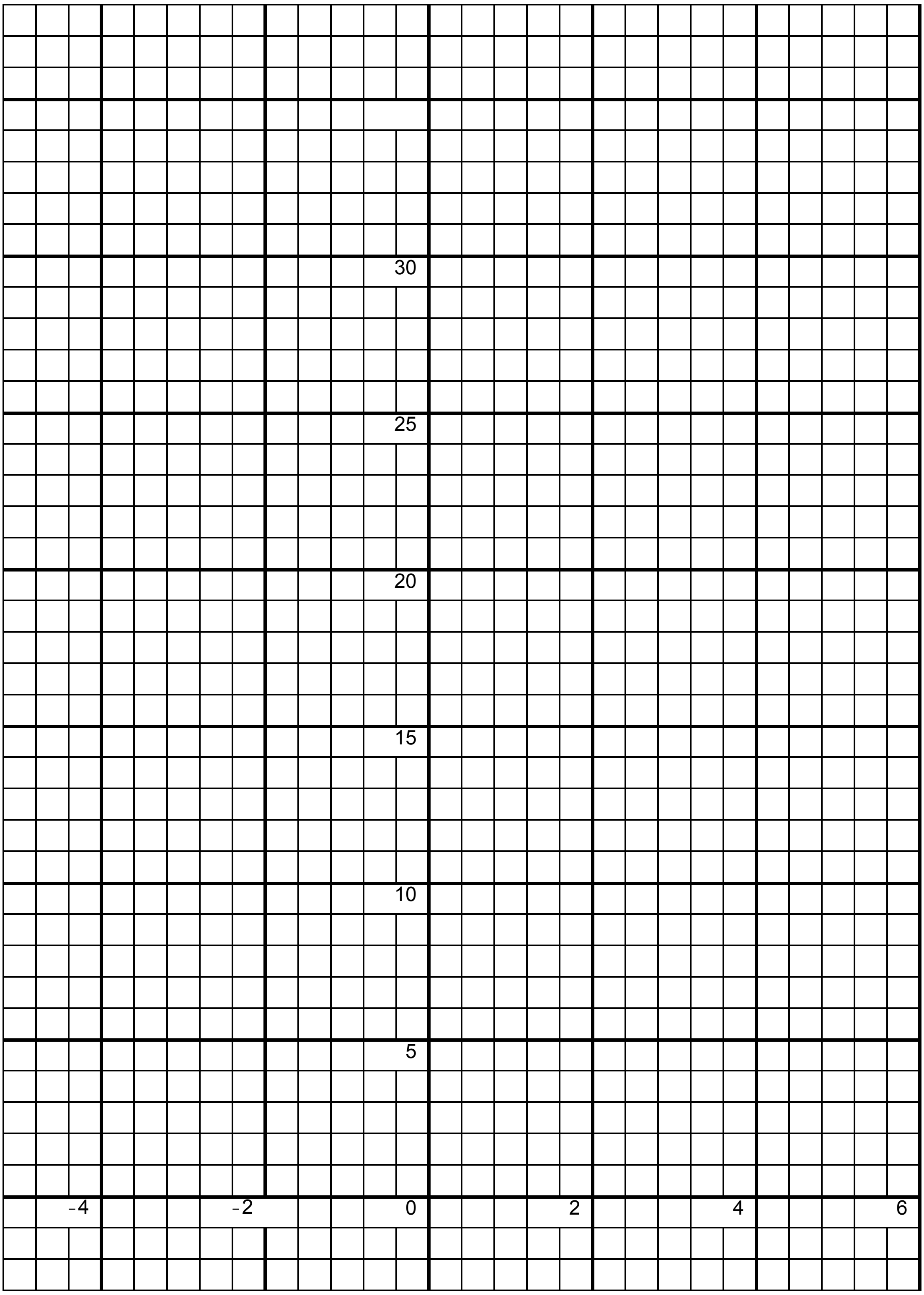
30
25
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10
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-4
-2
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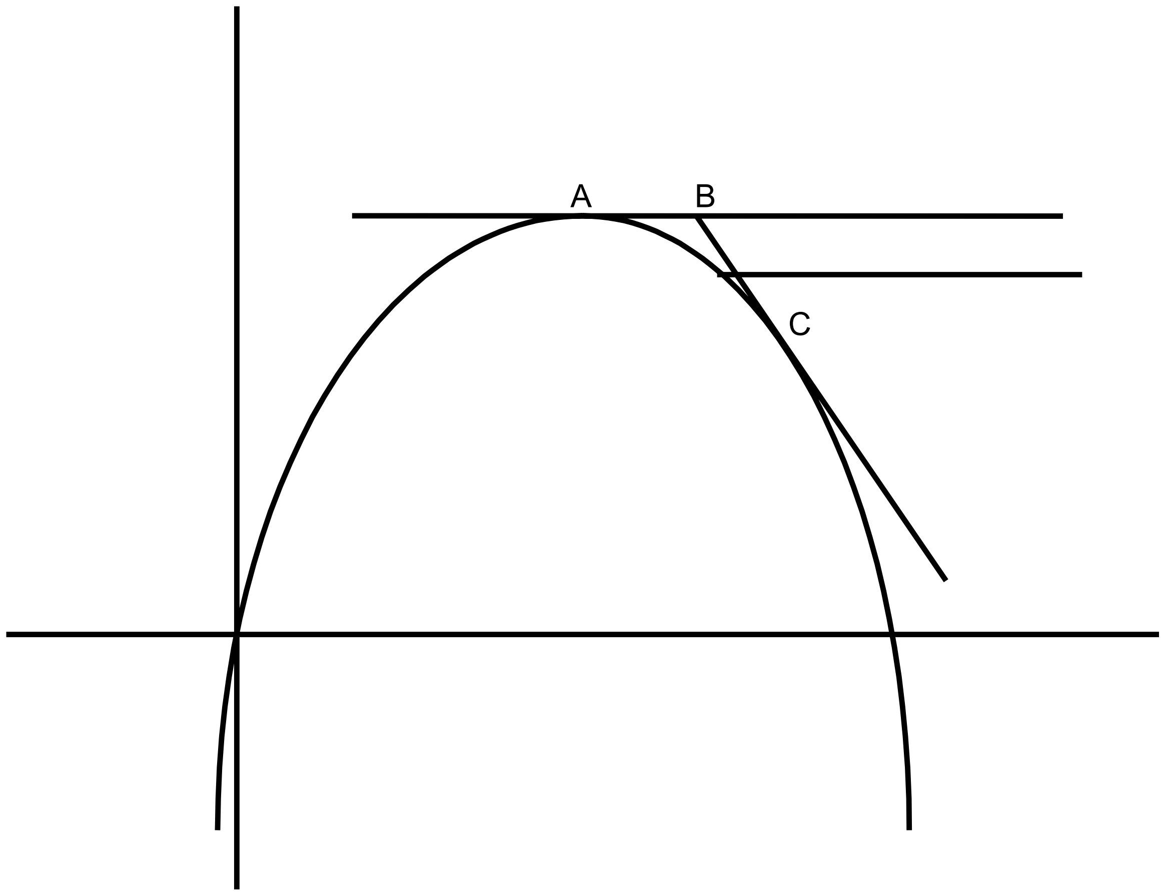
2
4
6

Teacher's note

This activity introduces the tangent line and the logic of using the tangent to determine the gradients of curves at specific points.

11. More about derivatives and graphs of functions

1. A rough sketch of the graph of $y = -x^2 + 6x$ is given the figure below. AB and BC are tangents to the curve at A and C respectively. AB is parallel to the x-axis and BC meets the curve at C(4; 8).



(a) Determine $\frac{dy}{dx}$.

(b) Determine the gradient of BC, and the equation of BC.

(c) Solve the equation $-x^2 + 6x = 0$.

(d) At what x-values does the graph of $y = -x^2 + 6x$ cut the x-axis?

(e) Determine the co-ordinates of A.

(f) Determine the co-ordinates of B.

2. The graph of a function $f(x)$ cuts the x-axis at –4 and 5. What does this information tell you about the roots of the equation $f(x) = 0$?

3. If $g(x)$ is a quadratic function and the graph of g touches the x-axis at one point only, what does this tell you about the root(s) of the equation $g(x) = 0$?

4. (a) A certain function g has the following properties:

   $g(-1) = g(2) = g(3) = 0$; $g'(-1) = -10$; $g'(2) = 14$ and $g'(3) = -15$.

   Make a rough sketch of what the graph of g could possibly look like.

   (b) A certain function h has the following properties:

   $h(-1) = h(2) = h(3) = 0$; $h'(-1) = 10$; $h'(2) = -14$ and $h'(3) = 15$.

   Make a rough sketch of what the graph of h could possibly look like.

Teacher's note*:*

The activity is aimed at using information about graphs to determine the derivative and equation of a tangent. Using information about certain properties of a function (e.g. gradient) to sketch or predict the shape of the graph.

12. Differentiating functions of the form $\frac{k}{x}$

1. Calculate by hand:

   (a) $\frac{3}{5}+\frac{6}{7}$

   (b) $\frac{6}{7}-\frac{3}{5}$

   (c) $\frac{17}{13}-\frac{3}{5}$

2. (a) Complete the table:

x	2	3	4	5	6	7	8	9
$\frac{3}{x}+\frac{3}{x+1}$								
$\frac{6x+3}{x^2+x}$								

   (b) Describe what you observe and why it is the case.

3. In each case write as a single fraction (check with some values of x that your expressions are actually equivalent):

   (a) $\frac{5}{x-2}+\frac{3}{x}$

   (b) $\frac{3}{3}+\frac{3}{5-x}$

   (c) $\frac{x}{1}-\frac{x}{1-x}$

   (d) $\frac{3}{2^2}-\frac{3}{(2-x)^2}$

4. Determine, by differentiating from first principles:

(a) $f'(2)$ if $f(x) = \frac{3}{x}$

(b) $g'(2)$ if $g(x) = \frac{3}{x^2}$

(c) a formula for $f'(x)$ if $f(x) = \frac{3}{x}$

(d) a formula for $k'(x)$ if $k(x) = \frac{a}{x}$

(e) a formula for $g'(x)$ if $g(x) = \frac{3}{x^2}$

(f) a formula for $q'(x)$ if $q(x) = \frac{a}{x^2}$

5. (a) Determine $\frac{dy}{dx}$ if $y = \frac{5}{x^2} + 4x^3 - 5x + 2$

(b) Determine the gradients of the function defined by $\frac{5}{x^2} + 4x^3 - 5x + 2$ at the following values of x: 3,2 5,5 –4

Teacher's note*:*

The activity provides an opportunity for investigating the convenience of using equivalent expressions of functions of the form $\frac{k}{x}$ *and the extent to which equivalent expressions may be used.*

13. Extreme values of functions, and turning points on graphs

1. (a) What is the minimum value of $5(x - 3)^2 + 2$, and for what value of x is this attained? (Make a table of values if you have difficulty).

   (b) Does $5(x - 3)^2 + 2$ have a maximum value?

   (c) What is the range of $f(x) = 5(x - 3)^2 + 2$? By the range of a function we mean all the values that the function has.

   (d) Plot a graph of $5(x - 3)^2 + 2$ with the extreme value (maximum or minimum value) approximately in the middle of the sheet.

   (e) Explain why $5(x - 3)^2 + 2$ has a minimum value of 2. Also explain why this minimum value is attained at $x = 3$.

2. (a) Does $-5(x - 3)^2 + 2$ have a minimum value or a maximum value? What is this value and at what value of x is it attained?

   (b) Show that $-5(x - 3)^2 + 2 = -5x^2 + 30x - 43$ for all x.

   (c) Does $-5x^2 + 30x - 43$ have a minimum value or a maximum value? What is this value and at what value of x is it attained?

   (d) Plot a graph of $-5x^2 + 30x - 43$ with the extreme value roughly in the middle of the sheet.

   (e) Explain why $-5(x - 3)^2 + 2$ has a maximum value of 2. Also explain why this maximum value is attained at $x = 3$.

3. (a) Show that $a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$ is equivalent to $ax^2 + bx + c$.

(b) You have shown in (a) that $ax^2 + bx + c$ can be written in the form

$a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$

Write $3x^2 + 12x + 16$ in the form $a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a})$

(c) Determine the extreme value of $3x^2 + 12x + 16$ and the value of x at which this is attained.

(d) Compare your answers to (c) carefully to your answer to (b). What do you observe.

(e) Plot a graph of $3x^2 + 12x + 16$ with the extreme value roughly in the middle of the sheet.

The point on the graph of a quadratic function which indicates the extreme value of the function is called the **turning point** of the graph (function).

4. Determine the co-ordinates (the x- and y-values) of the turning points of the graphs of each of the following quadratic functions:

(a) $y = 3x^2 – 5x + 12$

(b) $y = –3x^2 – 5x + 12$

5. A certain quadratic function $f(x) = ax^2 + bx + c$ has a maximum value of 5½ when $x = –3$. Further, $f(2) = –7$. Determine the values of a, b and c.

6. $f(x) = ax^2 + bx + c$. $f(2) = 0$, $f(5) = 0$ and $f(0) = 6$.

(a) Determine a, b and c.
(b) Determine the range of f.

7. g is a quadratic function. The graph of g cuts the x-axis at –2 and +5, and it cuts the y-axis at –4. Find the co-ordinates of the turning point.

Teacher's note*:*

Graphs are used to distinguish between turning points as maximum or minimum points and the term "turning point" (for graphs of quadratic functions) is defined.