Malati

Mathematics learning and teaching initiative

ALGEBRA

MODULE 1

ORDER OF OPERATIONS

GRADE 6

TEACHER DOCUMENT

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Module 1: Overview and Introduction

In this Module learners will use two kinds of calculators, the scientific and sequential calculators, to investigate the order of operations in number sentences.

Activities 1, 2, 4, 5, 13 and 14, which we will call the *foundation activities*, focus on *developing* the rules of the order of operations.

The other activities, which we will call the *fortification activities*, refocus on the rules, established in the *foundation activities.* There are five different types of fortification activities, one that involves calculations and the others that involve the recognition of structure:
1. Calculations
2. The possibility of left to right computation
3. Ordering of the operations in a string
4. Suggesting operations for a given order
5. The role of brackets

In order for the pupils to have *a fair chance* to cope with this Module (i.e. for *equity for all learners*)[1], it is necessary for the teacher to identify those pupils who have difficulty with simple numerical calculations and with the use of the calculator (these are *indispensable mathematical knowledge*,* i.e. necessary prerequisite knowledge to cope). Additional activities should be prepared for pupils with such difficulties. Discrete enrichment activities should be prepared for those pupils who do not have these difficulties.

Assessment activities will be integrated throughout the module. Based on the progress of the pupils as reflected in the assessment activities and the class activities, two kinds of activities for pupils at different levels are included:
1. Consolidation activities
2. Enrichment activities

The reader should also consult the following MALATI research papers, which form a background to the design of the material and the approach:

Liebenberg, R., Linchevski, L., Olivier, A. & Sasman, M. (1998). Laying the foundation for algebra: Developing an understanding of structure. **Proceedings of the Fourth Annual Congress of the Association for Mathematics Education of South Africa** (pp. 276-282). Pietersburg: University of the North.

Liebenberg, R.E., Linchevski, L., Sasman, M.C. & Olivier, A. (1999). Focusing on the structural aspects of numerical expressions. In J. Kuiper (Ed.), **Proceedings of the Seventh Annual Conference of the Southern African Association for Research in Mathematics and Science Education** (pp. 249-256). Harare, Zimbabwe.

Liebenberg, R., Sasman, M. & Olivier, A. (1999). From numerical equivalence to algebraic equivalence. **Proceedings of the Fifth Annual Congress of the Association for Mathematics Education of South Africa: Vol. 2**. (pp. 173-183). Port Elizabeth: Port Elizabeth Technikon.

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[1] For more information about MALATI's view on equity, about Indispensable Mathematical Knowledge, etc. see the paper **Assessment in support of equity.**

Multiplication and Addition

Clever Calculators!

> Some calculators can only do what we tell them to do. If we press the buttons in a certain order, the calculator does the operations in exactly the way we tell it to.
>
> These are called **sequential calculators.**

> Other calculators are very clever– they have been programmed with secret mathematical rules. If we press the buttons in a certain order, the calculator does the operations according to the rules that it has been programmed with.
>
> These are called **scientific calculators**.

First calculate the value of the number expression in the table below in your own way. Then enter the number expression into the calculators and write down the value of the expression given by each calculator.

	Number Expression	Your method of calculating	"Sequential" Calculator	Scientific Calculator
1	$4 + 5 \times 17$			
2	$12 \times 9 + 28$			
3	$14 \times 1 + 11$			
4	$4 \times 6 + 7 \times 9$			
5	$12 + 9 + 7$			
6	$15 + 8 \times 9 + 12$			
7	$4 \times 7 \times 9 + 5$			
8	$4 \times 7 \times 12 \times 9$			
9	$14 + 7 + 23 \times 7$			
10	$12 + 15 \times 3 \times 9$			

1. Use the data in the table to write down a rule for the order in which the operations in expressions involving both multiplication and addition should be carried out.

2. Create other number expressions to check your rule.

MALATI materials: Algebra, module 1

Teacher Notes: Activity 1

The broad aim of this activity is to get the pupils to analyse the data in the table and to establish a rule for the order of operations.

The series of activities (1, 2, 4, 5, …) in which pupils analyse the data in tables to establish the rule for the order of operation is also important in developing their thinking processes in generalisation. The pupils need to analsye the data, look for a pattern and then formulate a conjecture that must be validated and justified.

Note:
In this activity we are dealing with two kinds of expressions, additive expressions in which the controlling operation (the operation which is done last) is addition and multiplicative expressions in which the controlling operation is multiplication. All the expressions in the table except $4 \times 7 \times 12 \times 9$ are therefore *additive* expressions. In the additive expressions it is important to know what the **addends** are. This must be made explicit in the class discussion. It is not crucial that the terminology addend is used in the class at this point, we use it for the purposes of mathematical communication. Alternative ways of communicating this notion is described below.
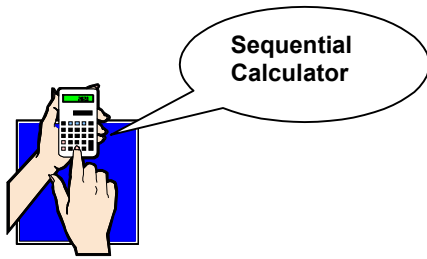
Whole class-discussion:
It might be necessary to assist pupils in the process of analysing the data in the table. Suggest to the pupils to mark those expressions where both calculators gave the same answers or where they did not give the same answers. Where the pupils got the same answers as the scientific calculator, get them to reflect on how they did their calculation as a way of determining the rule.

Once the pupils establish the rule, usually as, multiplication must be done first, check to see whether this is correctly understood by all the pupils. Several misinterpretations of the rule have been noted. For example, in doing the calculation $4 \times 6 + 7 \times 9$, some pupils misinterpret "multiply first" to mean, $4 \times 6 \times 9$ and then add 7. In addressing this problem we now need to focus on the number of addends in the expression. Start with an additive expression in which we have complete addends, for example, 12 + 9 + 7. Ask the pupils which numbers are being added. Now look at $4 \times 6 + 7 \times 9$ again. The pupils need to see that here we have two addends, 4×6 and 7×9. Another way of communicating the notion of the addend is to ask pupils if they can identify the multiplication unit. It needs to be made clear that **7 is not** an addend and that it a part of the addend. Emphasise that where the numbers that must be added are incomplete (still exist as a multiplication unit) they must be worked out first, a part of it cannot be added. Finally choose an example in which there are complete and incomplete addends, $14 + 7 + 23 \times 7$. Ask the pupils to identify the addends. Point out once again that **23 is not** an addend.

It has also been noted that pupils adopt the rule rigidly to mean that multiplication must be done first, rather than looking at the nature of the addends in the expression. In order to develop the necessary flexibility of handling numerical expressions, it needs to be pointed out that where the addends are complete they may be added. Hence in the expression $14 + 7 + 23 \times 7$ we may add 14 and 7 first. Then we must multiply 23 by 7 before it is added to the sum of 14 and 7.

Multiplication and Subtraction

Clever Calculators!
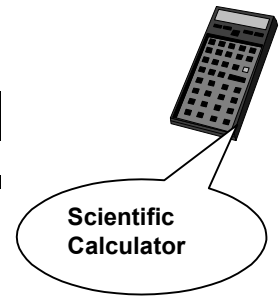
Sequential Calculator

Scientific Calculator

First calculate the value of the number expression in the table below in your own way. Then enter the number expression into the calculators and write down the value of the expression given by each calculator.

	Number Expressions	**Your method of calculating**	**"Sequential" Calculator**	**Scientific Calculator**
1	$35 - 4 \times 7$			
2	$12 \times 4 - 2$			
3	$14 \times 7 - 1$			
4	$24 - 1 \times 6$			
5	$12 \times 4 - 2 \times 15$			
6	$15 - 2 \times 6 - 3$			
7	$47 - 9 - 28 - 5$			
8	$4 \times 7 \times 9 - 2$			
9	$24 - 4 - 3 \times 2$			
10	$48 - 5 \times 2 \times 4$			

1. Use the data in the table to write down a rule for the order in which the operations in expressions involving both multiplication and subtraction should be carried out.

2. Create other number expressions to check your rule.

Teacher Notes: Activity 2

Refer to the teacher notes of Activity 1.

Note:
From a formal mathematical perspective, all the expressions in the table are in fact additive expressions, for example, in the expression $35 - 4 \times 7$, 35 and 28 are regarded as addends. We do not recommend that this level of discussion be entered into as it too abstract for the pupils.

Whole-class Discussion;
As in activity 1, focus the discussion on the numbers that are being subtracted. For example, in expression $12 \times 4 - 2 \times 15$, point out that 2×15 is subtracted from 12×4. Emphasise that **2 is only a part** of the multiplication unit and that is not the number which is subtracted. The expression $47 - 9 - 28 - 5$ in which all the numbers that are subtracted are complete, needs to be dealt with carefully. It is known that pupils use strategies as shown below:

$47 - 9 - 28 - 5$ [The pupil split the expression into two parts, $(49 - 7)$ and $(28 - 5)$]
$= 38 - 23$
$= 15$

Here we need to point out that although the numbers that are subtracted are complete we may not simply subtract them from each other. At this point the ambiguity can be resolved by the correct answers from the calculators. This discussion will be extended once the pupils have dealt with expressions involving both subtraction and addition. (See Activity 11)

Assessment:
After completing either Activity 2 or Activity 1 a short diagnostic assessment is suggested to identify specific difficulties that pupils have and that may interfere with their ability to cope with numerical expressions *structurally*.

The test overleaf is an example of a diagnostic test given after Activity 1. The items were carefully designed to see whether the nature of the *operations* (the number of + and × operations, and their position) in the expressions and the nature of the *numbers* chosen had any significant influence.[2]

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[2]   See also: Liebenberg, R. (1997). **The usefulness of an intensive diagnostic test**. In P. Kelsall & M. de Villiers (Eds.), Proceedings of the Third National Congress of the Association for Mathematics Education of South Africa: Vol. 2. (pp. 72-79). Durban: Natal University.

DIAGNOSTIC ASSESSMENT

Calculate the value of the following number expressions.

Show the *method* that you used. You may *not* use a calculator.

1. $4 \times 3 + 5 \times 2 \times 6$

2. $4 \times 3 + 5 + 2 \times 6$

3. $4 + 5 + 5 \times 2 \times 6 + 4$

4. $4 \times 5 + 5 \times 2 \times 6$

5. $4 \times 3 + 5 + 2 + 6$

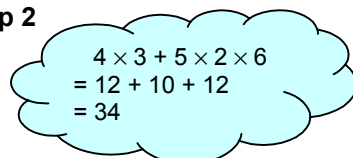6. $4 + 3 + 5 \times 2 + 6$

7. $4 + 5 + 5 \times 2 + 6$

8. $4 + 3 \times 5 + 2 + 6$

An analysis of our test results showed that pupils had difficulty in recognising multiplication as a *unit*. Four groups of pupils were identified on the basis of their ability to recognise the multiplication as a unit:

1. Recognises the unit and applies the rule correctly in all expressions
2. Recognises the unit but gives up the rule in certain expressions
3. Does not recognise the unit in all the expressions
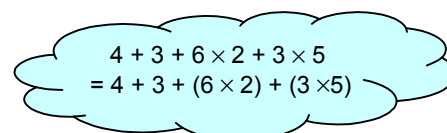4. Does not recognise the unit in expressions where there are two or more consecutive multiplication operations

Here are examples of pupil strategies in the above groups:

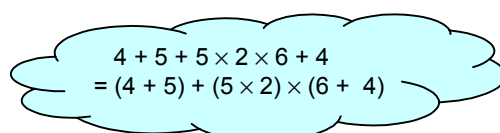**Group 2**

$4 \times 3 + 5 \times 2 \times 6$
$= 12 + 10 + 12$
$= 34$

The pupil multiplies $5 \times 2 \times 6$ incorrectly.

Group 3

$4 + 3 + 6 \times 2 + 3 \times 5$
$= 4 + 3 + (6 \times 2) + (3 \times 5)$

Here the rule is misinterpreted to mean that once *a multiplication* is carried out the rule has been applied. The calculation is then done from left to right.

Group 4

$4 + 5 + 5 \times 2 \times 6 + 4$
$= (4 + 5) + (5 \times 2) \times (6 + 4)$

The numbers are paired off first.
The calculation is then done from left to right.

The class may now be diversified based on this analysis.

Recommended activities for the different groups:

Group 1: Enrichment

Group 2: Activity on multiplication (see next page).

Groups 3 and 4:
Give the pupils the correct answers to the test. Hand back their tests and let them reflect on their answers and try to establish their error (this is best done in pairs). Activity on identifying the unit of multiplication. A calculation activity can also be given. This consolidation can be done over two periods.

WE NEED TO MULTIPLY:

1. $4 \times 6 \times 3 =$

2. $3 \times 5 \times 7 =$

3. $2 \times 4 \times 6 =$

4. $3 \times 3 \times 3 =$

5. $1 \times 5 \times 1 =$

6. $4 \times 4 \times 4 =$

7. $1 \times 1 \times 1 =$

8. $3 \times 4 \times 10 \times 2 =$

9. $5 \times 4 \times 3 \times 5 =$

10. $2 \times 2 \times 2 \times 2 =$

11. $9 \times 1 \times 3 \times 4 =$

12. $6 \times 3 \times 2 \times 2 =$

13. $15 \times 4 \times 0 \times 8 =$

14. $12 \times 1 \times 10 \times 0 =$

ACTIVITY 3

Think of numbers to put in the following structure that gives the answer 124.

$$\Box + \Box \times \Box$$

Consider the number 124.

We can write it as $1 \times 100 + 2 \times 10 + 4 \times 1$.

Which of the two calculators will give the answer 124? Can you explain why?

Can you think of different numbers to put into the empty boxes to give the answer 124?

$$\Box \times \Box + \Box \times \Box + \Box \times \Box$$

Jane writes 56 as $5 \times 10 + 6 \times 1$

Write natural numbers between 0 and 10 in the structure $\Box \times 10 + \Box \times 1$ that gives:

1.	A number between 20 and 30	$\Box \times 10 + \Box \times 1$
2.	A number between 25 and 32	$\Box \times 10 + \Box \times 1$
3.	A number between 90 and 100	$\Box \times 10 + \Box \times 1$
4.	The biggest number	$\Box \times 10 + \Box \times 1$
5.	The smallest number	$\Box \times 10 + \Box \times 1$

Jane writes 156 as $1 \times 100 + 5 \times 10 + 6 \times 1$

Write numbers between 0 and 10 in the structure $\Box \times 100 + \Box \times 10 + \Box \times 1$ that gives a number between 215 and 355.

Teacher Notes: Activity 3

The purpose of this activity is to get the pupils to focus on the structure of the numerical expression.

Note:
The term structure can now be used and will again be used in later activities since the focus is on the structure of the number sentence instead of the calculation of the answer. Whilst the main focus is the structure of the numerical expression, this activity also reinforces the order of operations established in Activity 1. It is therefore extremely useful to revisit the teacher notes of Activity 1.

In this activity the pupils' knowledge of the base ten numbers is being tested. The structure of the base ten numbers are based on the order of operations hence we are trying to build on a structure with which the pupils are familiar even though they may not have focused on it.

In the context of the base ten numbers, it can be re-emphasised that mathematicians to remove any ambiguity imposed order. For example, the number 234 can be written as $2\times100+3\times10+4\times1$. If order is not imposed and we simply work from left to right then the value of the number expression is 2034.

Whole-class discussion:

Some pupils may choose numbers such as:

100	+	24	×	1
0	+	31	×	4
124	+	0	×	21

In these cases the order does not matter and it should be made explicit that these are special cases. These examples could be used to discuss the special properties of the numbers 1 and 0.

The discussion can also be extended to get the pupils to focus on the two numbers that are being added. For example, put a number in the second box and ask the pupils to find the numbers in the other two boxes:

| | + | 14 | × | |

The strategies used by the pupils to find the missing numbers can provide feedback as to how they used the structure. The pupils can also be encouraged to reflect on the number of possible combinations of numbers there are (it is an infinite number of course!). It might be interesting to suggest that a fraction be put in the box.

Another example can be given in which numbers are placed in two of the boxes:

| 7 | + | | × | 3 |

| 7 | + | 3 | × | |

The first structure requires the pupils to recognise the two addends. The second structure has a greater potential of seducing pupils to work sequentially. It might be interesting to note how many pupils at this point will recognise the two structures as being equivalent without doing a calculation.

Recommendation:

The following activities (see overleaf) could be used to further reinforce these special properties of 0 and 1:
1. Special Numbers
2. Special Tables

SPECIAL NUMBERS

1. It is given that:

$$30 \times ? = 17,5 \times \frac{18}{9} \times 0 \times 13$$

   What is the missing number?

   On what rule did you base your answer?

2. It is given that:

$$289,26 \times 1936,8 = ? \times 289,26$$

   What is the missing number?

   On what rule did you base your answer?

3. It is given that:

$$7 + ? + ? + ? = 7$$

   Put in the numbers that will make this equality correct.

   On what rule did you base your answer?

SPECIAL TABLES

Multiplication Table:

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

Addition Table:

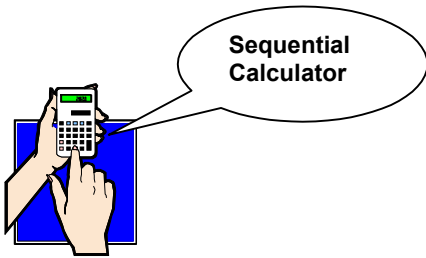| +	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	5
2	2	3	4	5	6
3	3	4	5	6	7
4	4	5	6	7	8

1. In the multiplication table the first row and the first column are identical. It is true for the addition table as well.
   What is the difference between the two cases?
   Which rule makes the first row and the first column identical in the multiplication table?
   What rule makes the first row and the first column identical in the additional table?

2. In the multiplication table the second row is identical to the heading line.
   In the addition table the first row is identical to the heading line.
   What is the reason for this?

3. In the multiplication table the first row is identical to the first column, the second row to the second column and so on. This is true to the addition table as well. Check! What is the reason for this?

> Here is another way to say this:
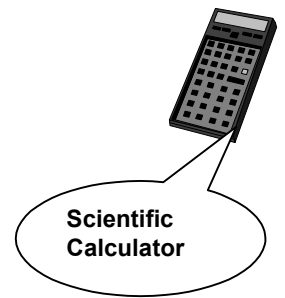> **The diagonal is a line of symmetry of the table.**
> What does this *mean*?
> *Why* is there a line of symmetry?

×	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	6	8
3	0	3	6	9	12
4	0	4	8	12	16

Division and Addition

Clever Calculators!

Sequential Calculator

Scientific Calculator

First calculate the value of the number expression in the table below in your own way. Thereafter enter the number expression into the calculators and write down the value of the expression given by each calculator.

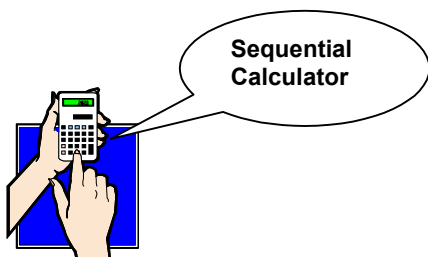|	Number Expression	Your method of calculating	"Sequential" Calculator	Scientific Calculator
1	12 ÷ 4 + 2			
2	12 + 4 ÷ 2			
3	14 ÷ 1 + 6			
4	36 ÷ 3 ÷ 4			
5	6 + 18 ÷ 6 + 3			
6	13 + 8 + 12 ÷ 3			
7	4 + 16 ÷ 4 ÷ 2			
8	24 + 12 + 18			
9	6 ÷ 3 + 8 ÷ 2			
10	24 ÷ 4 ÷ 4 + 8			

1. Use the data in the table to write down a rule for the order in which the operations in expressions involving both division and addition should be carried out.

2. Create other number expressions to check your rule.
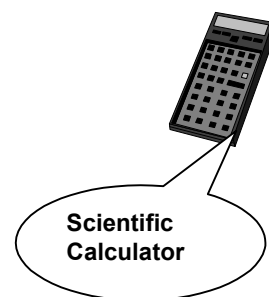
Teacher Notes: Activity 4 and Activity 5

In a number expression like 32 ÷ 4 ÷ 4 + 8, some pupils may use the following method of calculating:

> 32 ÷ 4 ÷ 4 + 8
> = 32 ÷ 1 + 8   [the pupil first divides 4 ÷ 4]
> = 32 + 8
> = 40

This conflict can be resolved by the answers given by the calculators. The analogue situation in the case of subtraction, for example 72 − 30 − 15 can be revisited to draw comparisons. For now the pupils need to accept that we may work from left to right in the structures $a \div b \div c$ and $a - b - c$ (note the letters are used here for the purpose of communication with the teacher only – it is not intended for pupils!). In the next unit we will revisit these structures and explore their equivalent structures such as $a \div (b \times c)$ and $a - (b + c)$.

ACTIVITY 5   Division and Subtraction

Sequential Calculator

Clever Calculators!

Scientific Calculator

First calculate the value of the number expression in the table below in your own way. Thereafter enter the number expression into the calculators and write down the value of the expression given by each calculator.

	Number Expression	Your method of calculating	"Sequential" Calculator	Scientific Calculator
1	12 ÷ 4 – 2			
2	12 – 4 ÷ 2			
3	14 ÷ 1 – 6			
4	48 ÷ 4 ÷ 6			
5	18 – 6 ÷ 2 – 3			
6	23 –8 – 12 ÷ 3			
7	16 – 8 ÷ 4 ÷ 2			
8	36 ÷ 3 – 8 ÷ 2			
9	72 – 30 – 15			
10	24 ÷ 4 ÷ 3 – 2			

1. Use the data in the table to write down a rule for the order in which the operations in expressions involving both division and subtraction should be carried out.

2. Create other number expressions to check your rule.

MALATI materials: Algebra, module 1   16

ACTIVITY 6

Put numbers in the following structure that gives the answer 124.

$$\boxed{} \; + \; \boxed{} \; \div \; \boxed{}$$

The number 124 can be written as 8 + 196 ÷2 + 18.

Which of the two calculators will give the answer 124? Can you explain why?

Put numbers into the structure below to give the answer 124:

$$\boxed{} \; + \; \boxed{} \; \div \; \boxed{} \; + \; \boxed{}$$

Teacher Note: Activity 6

Refer to the notes of Activity 3.

ACTIVITY 7

Find an order in which the operations have to be done in the following number sentences.

In each case find two ways in which it can be done if possible.

Fill in numbers in the empty boxes to show the order in which the operations have to be done.

EXAMPLE:

3 + 8 × 7 + 5

| 3 | 1 | 2 |

3 + 8 × 7 + 5

| 2 | 1 | 3 |

3 + 8 × 7 + 5

1a $4 \times 5 + 6 \times 2$

1b $4 \times 5 + 6 \times 2$

2a $20 \div 5 - 4 \div 2$

2b $20 \div 5 - 4 \div 2$

3a $4 + 4 + 4 \times 4$

3b $4 + 4 + 4 \times 4$

4a $8 - 5 \div 5 \div 2 - 6$

4b $8 - 5 \div 5 \div 2 - 6$

5a $5 \times 4 + 7 \times 4 + 3$

5b $5 \times 4 + 7 \times 4 + 3$

Teacher Notes: Activity 7

The aim of this activity is to focus on the structure of the expression by exploring the different computational procedures that are permissible for the *structure*. In the whole-class discussion, check whether the pupils recognise the addends in the structures.

A pupil who writes down the correct order of operation need not necessarily recognise the addends in the structure. For example, pupils will fill in the order as follows:

   1   3   2

$4 \times 5 + 6 \times 2$ but carry out the calculation as: $4 \times 5 \times 2 + 6$.

A possible way of getting the pupils to focus on the addends is to ask them to use the "sequential" calculator to work out the value of the number expressions.
For example, in the expression $3 + 8 \times 7 + 5$, the numbers can be entered into the "sequential" calculator in two ways:
1. $8 \times 7 + 3 + 5$
2. $8 \times 7 + 5 + 3$

In the expression $4 \times 5 + 2 \times 6$, the pupils need to use the memory function on the "sequential" calculator to put the answers of 4×5 and 2×6 into memory. This physical process of putting the multiplicative unit into memory may serve as way of reflecting on the addends in the structure. Getting the pupils to write down the computer keys used and the screen displayed may assist the pupils in reflecting on this process, for example:

Key	4	×	5	M+	2	×	6	+	MRC	=
Screen	4	$4^\times$	$5^\times$	20	2	$2^\times$	$6^\times$	12_+	20_+	32

This process reinforces the fact that in the computational process, the *addends have to be worked out first* before they are added.

In the expression $8 - 5 \div 5 \div 2 - 6$ the key and screen display for the two ways in which the computation can be carried out is:

       3   1   2   4   4   1   2   3

4a. $8 - 5 \div 5 \div 2 - 6$ 4b. $8 - 5 \div 5 \div 2 - 6$

Key	5	÷	5	÷	2	M+	8	−	MRC	−	6	=
Screen	5	$5^\div$	$5^\div$	$1^\div$	$2^\div$	0.5	8	$8-$	$0.5-$	$7.5-$	$6-$	1.5

Key	5	÷	5	÷	2	M+	8	−	6	−	MRC	=
Screen	5	$5^\div$	$5^\div$	$1^\div$	$2^\div$	0.5	8	$8-$	$6-$	$2-$	$0.5-$	1.5

The shaded part of the table highlights the addends in the structures:
$$8 - 0.5 - 6$$
$$8 - 6 - 0.5$$
Note: The two expressions are numerically equivalent.

Note: The notation introduced in this activity (the small numbers in the boxes) is new to the pupils. The meaning of this notation should be made explicit in the discussion.

ACTIVITY 8

The small number above each box gives the order in which the operation was done.

In the empty boxes write down possible operations.

In each case give two possibilities and give the answer:

Example: 10   3   5   4

10 $\boxed{\times}^1$ 3 $\boxed{\times}^2$ 5 $\boxed{+}^3$ 4 = 154   10 $\boxed{\times}^1$ 3 $\boxed{+}^2$ 5 $\boxed{+}^3$ 4 = 39

1a.   5 $\boxed{}^1$ 3 $\boxed{}^2$ 1   1b.   5 $\boxed{}^1$ 3 $\boxed{}^2$ 1

2a.   7 $\boxed{}^3$ 3 $\boxed{}^2$ 5 $\boxed{}^1$ 4   2b.   7 $\boxed{}^3$ 3 $\boxed{}^2$ 5 $\boxed{}^1$ 4

3a.   2 $\boxed{}^3$ 8 $\boxed{}^1$ 3 $\boxed{}^4$ 6 $\boxed{}^2$ 2   3b.   2 $\boxed{}^3$ 8 $\boxed{}^1$ 3 $\boxed{}^4$ 6 $\boxed{}^2$ 2

4a.   2 $\boxed{}^3$ 8 $\boxed{}^1$ 3 $\boxed{}^2$ 6 $\boxed{}^4$ 2   4b.   2 $\boxed{}^3$ 8 $\boxed{}^1$ 3 $\boxed{}^2$ 6 $\boxed{}^4$ 2

5a.   7 $\boxed{}^2$ 3 $\boxed{}^3$ 9 $\boxed{}^1$ 10   5b.   7 $\boxed{}^2$ 3 $\boxed{}^3$ 9 $\boxed{}^1$ 10

6a.   84 $\boxed{}^2$ 12 $\boxed{}^3$ 15 $\boxed{}^1$ 6   6b.   84 $\boxed{}^2$ 12 $\boxed{}^3$ 15 $\boxed{}^1$ 6

ACTIVITY 9

Zaida, Vusi and Marvin's Expressions

Zaida, Vusi and Marvin are asked to build number expressions.

They are given the structure of the number expression that shows only the *numbers* and their *positions*.

They are also given the *order* in which the operations must be done. The small number above each box indicates the order in which the operation is done.

They have to use the operations + and × in the number expressions.

<div align="center">

	2		**3**		**1**	
7	☐	3	☐	9	☐	10

</div>

Zaida builds the following number expressions:

$$\overset{2}{7 \ + \ 3} \ \overset{3}{+} \ \overset{1}{9 \ \times \ 10}$$

$$\overset{2}{7 \ \times \ 3} \ \overset{3}{+} \ \overset{1}{9 \ \times \ 10}$$

Vusi built the following expression:

$$\overset{2}{7 \ \times \ 3} \ \overset{3}{+} \ \overset{1}{9 \ + \ 10}$$

Marvin built the following expression:

$$\overset{2}{7 \ + \ 3} \ \overset{3}{\times} \ \overset{1}{9 \ + \ 10}$$

Do you agree with them? Explain.

ACTIVITY 10

Without calculating, indicate with a ✓ if a left to right computation will give the correct numerical value. Indicate with a ✗ if it does not.

Example: <u>✗</u> $36 - 8 \div 2$

     <u>✓</u> $2 \times 6 + 12$

1.   __ $54 + 8 \div 3$

2.   __ $14 - 2 \times 3$

3.   __ $8 \times 5 + 7$

4.   __ $45 - 7 \times 4 - 3$

5.   __ $3 \times 2 + 5 \times 4$

6.   __ $3 \times 12 \times 5 + 11$

7.   __ $19 + 5 - 4 - 2$

8.   __ $3 + 10 \times 4 - 9$

9.   __ $25 \div 5 \div 5$

10.   __ $36 - 6 \div 5$

11.   __ $28 \div 4 + 9 \div 3$

12.   __ $36 \div 6 + 3 \times 5 - 2$

Teacher Notes: Activity 10

The purpose of this activity is to consolidate the idea of focusing on the *structure* of a numerical expression.

Note:

It should be stressed that in this activity we are not interested in the numerical value of the number expression. The idea is to get the pupil to know how the calculation should be done according to the order of operations. In a way we want them to use the numbers as generalised numbers. The pupils must be clearly aware what in the structure is *invariant*.

To make this more explicit teachers could perhaps draw an analogy between the number expression and a casino-like machine. For example, if the following was displayed on the machine:
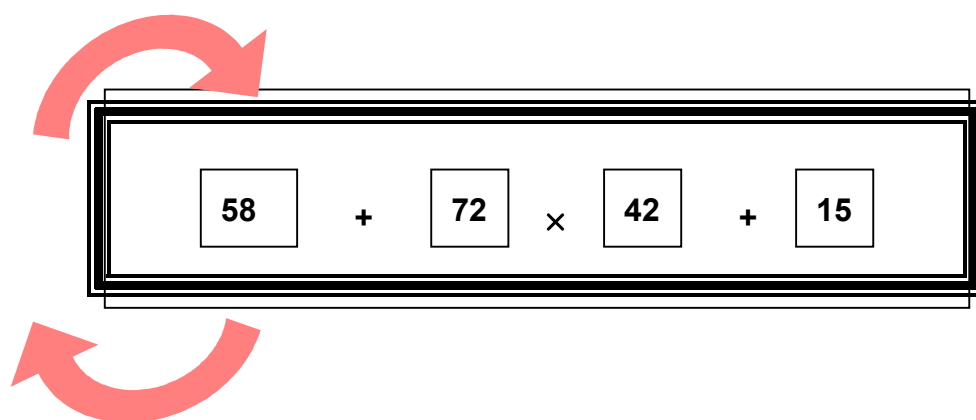
where the numbers in the machine may change (i.e. they are variables) every time the machine is turned but the operations remain the same (i.e. they are invariant). It must be made explicit that no matter what the numbers are, the same order of operations still applies.

ASSESSMENT 1

1. Calculate the value of the following expressions:
    [Note: you are not allowed to use a calculator]

    (a) $47 - 7 \times 5 =$

    (b) $5 \times 3 + 3 \times 2 =$

    (c) $112 - 12 \times 3 + 7 =$

    (d) $30 + 15 \div 5 =$

    (e) $240 \div 15 \times 2 =$

    (f) $130 - 10 + 10 + 10 =$

    (g) $136 - 36 \div 3 + 8 \times 7 =$

    (h) $64 \div 8 \div 4 =$

2. Replace $\boxed{}$ with numbers to make each of the following sentences true:

    (a) $2 + \boxed{} \times 4 = 10$

    (b) $2 \times \boxed{} - 4 = 10$

3. Replace $\boxed{}$ with numbers to make each of the following sentences true:

    (a) $\boxed{} + \boxed{} \times \boxed{} = 96$

    (b) $\boxed{} + \boxed{} \times \boxed{} + \boxed{} = 96$

4. Replace $\boxed{}$ with numbers and ∗ with "÷" and "+" to make the following true:

    $\boxed{} * \boxed{} * \boxed{} * \boxed{} = 24$

Teacher Notes: Assessment

Calculation activities can be given to those learners who need it as a form of consolidation at the end of this assessment.

The following activities,1-2-3-4 Puzzle, Making Numbers and the Four-Fours Problem can be given to the other learners as a form of enrichment.

1 - 2 - 3 - 4 PUZZLE
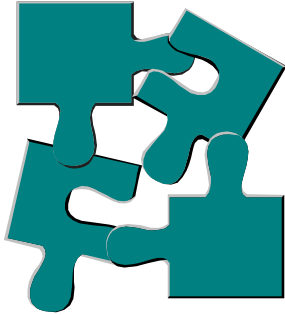
This problem is about using the digits 1, 2, 3 and 4, in *any order* you choose, to create number expressions with different numerical values, according to the rules for the *order of operations*.

A 1-2-3-4 expression is any expression which uses each of these digits *exactly once,* according to the following rule:

You may use any one of the four basic arithmetic operations – addition,

multiplication, subtraction, division (according to the order of operations rules).

For example:

2 + 1 × 3 − 4 is a 1-2-3-4 expression for the number 1.

(because 2 + 1 × 3 − 4 = 1)

Your task is to create as many 1-2-3-4 expressions as possible for the numbers from 1 to 25.

MAKING NUMBERS

1. Use all the numbers 1, 3, 5, 6, 7 once each, one pair of brackets, and each of the operations +, −, ×, ÷ once and only once to make the following numbers:

   8; 1; 0; 7; 9

2. Use each of the numbers 2, 3, 5, 7, 11 once, any brackets you need, the operations +, −, ×, ÷ as often as you like, to make the largest prime number that you can.

   (Use a calculator)

3. Put in +, −, ×, ÷ as needed to make the following true. (No brackets)

   (a) 12   5   2   1 = 3

   (b) 18   12   3   2 = 12

   (c) 24   4   2   3 = 7

   (d) 12   4   6   2 = 11

   (e) 6   2 = 5   2

4. Put in brackets if needed and +, −, ×, ÷ as needed to make the following statements true:

   (a) 12   3   1 = 5

   (b) 20   5   2   3 = 5

   (c) 24   6   2   2   4 = 2

   (d) 8   3   8   3 = 8   8   3   3`

5. Using the digits 1, 9, 8 and 6, in that order, we can make successive natural numbers as follows:

$1 = 1 \times \sqrt{9} - 8 + 6$

$2 = 1 + \sqrt{9} - 8 + 6$ or $(1 + \sqrt{9} + 8) \div 6$

$3 = (1 + 9 + 8) \div 6$

$4 = 1 + 9 + 8 - 6$ or $1 \times \sqrt{9} \times 8 \div 6$

$5 = (1 + 9) \div (8 - 6)$ or $1 \times \sqrt{9} + 8 - 6$ or $1 + \sqrt{9} \times 8 \div 6$

$6 = 1 - (9 - 8) + 6$ or $1 \times (9 - 8) \times 6$ or $(1 \times 9 - 8) \times 6$ or $1 + \sqrt{9} + 8 - 6$

Check that the above examples are correct.

How far can you extend this list?

You may only use the symbols above.

FOUR-FOURS PROBLEM

What is the highest number you can reach beginning at 1, 2, 3, 4 etc. by using

 4, 4, 4, 4?

You must make use of the four figures each time.

Example:

$$\frac{4 \times 4}{4 \times 4} = 1$$

$$\frac{4 \times 4}{4 + 4} = 2$$

ACTIVITY 11

1. Think of four different ways in which the numbers 15, 8 and 4 and the operations

   + and × can be used once only so that the answer 47 is obtained.

2. How many different ways can the numbers 9, 6, 8 and 4 and the operations +

   and × be used so that the answer 47 is obtained?

   The numbers are to be used once only.

> **Numerical expressions that have the same numerical *value* (have the same answers) are called *equivalent numerical expressions*.**

3. Use the numbers 15, 8 and 4 and the operations − and × to create equivalent

   expressions.

4. How many different equivalent expressions can you write using the numbers 15,

   8 and 4 and the operations + and ÷ once only?

Teacher Notes: Activity 11

The aim of this activity is to develop the notion of equivalence in a numerical context. It has to be made very clear to the pupils that one can use different kinds of manipulations to transform one numerical expression into another without having to determine the numerical value of the expression. For example, we can use the numbers 15, 8 and 4 to build equivalent numerical expressions

$$15 + 8 \times 4; \quad 8 \times 4 + 15; \quad 15 + 4 \times 8 \quad \text{and} \quad 4 \times 8 + 15$$

based on the commutative properties of addition and multiplication. When the number 15 is decomposed into $9 + 6$ we can create another expression

$$9 + 6 + 8 \times 4$$

that is numerically equivalent to the four expressions above.

It is necessary to point out in the discussion that the transformation is not always obvious and might involve several manipulations. For example, it is not obvious that the expressions $3 + 2 + 3 \times 5$ and $22 \div 2$ are numerically equivalent. We can go through different kinds of manipulations to show their equivalence without knowing its numerical value, for example:

$$22 \div 2$$
$$= 5 + 6 \times 2 \div 2$$
$$= 5 + 6$$
$$= 3 + 2 + 3 \times 2$$

It is precisely this inability to *recognise* the necessary manipulations that make it difficult for pupils, for example, to accept that the two equations $x^2 = 1$ and $|x| = 1$ are equivalent. Pupils will readily accept that $x^2 = 1$ and $x^2 - 1 = 0$ are equivalent equations as they can easily recognise the necessary manipulation to transform the one into the other.

DON'T CALCULATE

Which of the following number expressions has the same answer as:

$$367 + 68 \times 214 \times 1966 + 814 \times 45$$

1. $367 + 68 \times 1966 \times 214 + 814 \times 45$

2. $214 \times 68 \times 1966 + 367 + 814 \times 45$

3. $45 \times 814 + 367 + 214 \times 68 \times 1966$

4. $68 + 367 \times 214 \times 1966 + 814 \times 45$

5. $1966 \times 214 \times 68 + 45 \times 814 + 367$
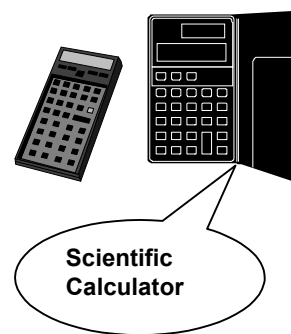
6. $367 + 68 \times 214 \times 814 + 1966 \times 45$

Explain how you decided whether the number expressions were equivalent.

Teacher Notes: Activity 12

The aim of this activity is to encourage learners to use the *structure* of the expressions to judge whether they are equivalent, hence the choice of the large numbers.

ACTIVITY 13   Addition and Subtraction
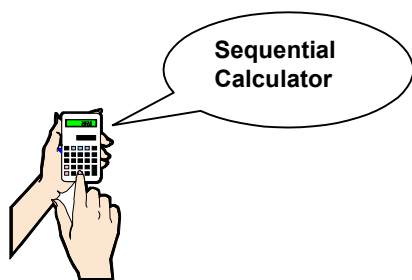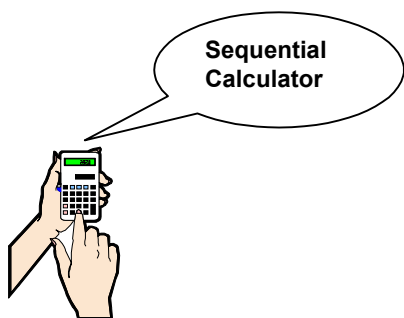
Sequential Calculator

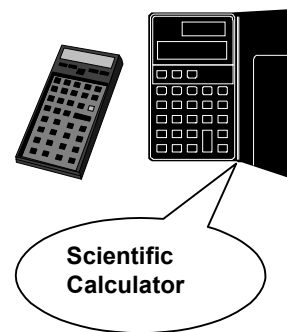Clever Calculators!

Scientific Calculator

First calculate the value of the number expression in the table below in your own way. Then enter the number expression into the calculators and write down the value of the expression given by each calculator.

	Number Expressions	Your method of calculating	"Sequential" Calculator	Scientific Calculator
1	5 + 3 − 4			
2	5 + 3 − 4 − 2			
3	5 − 3 + 4 − 2			
4	15 − 3 − 4 + 2			
5	20 + 3 − 5 + 6 − 12			
6	10 − 3 + 2			
7	20 + 8 + 16 + 9			
8	35 − 8 − 7 − 12			

1. Use the data in the table to write down a rule for the order in which the operations in expressions involving both addition and subtraction should be carried out.

2. Create other number expressions to check your rule.

ACTIVITY 14   Multiplication and Division

Sequential Calculator

Clever Calculators!

Scientific Calculator

First calculate the value of the number expression in the table below in your own way. Then enter the number expression into the calculators and write down the value of the expression given by each calculator.

	Number Expressions	Your method of calculating	"Sequential" Calculator	Scientific Calculator
1	$15 \div 3 \times 2$			
2	$15 \times 2 \div 3$			
3	$16 \div 2 \times 3 \div 6$			
4	$6 \times 3 \div 2 \div 3$			
5	$2 \times 3 \times 4 \div 8$			
6	$12 \times 5 \times 7 \times 4$			
7	$56 \div 2 \div 2 \div 7$			

1. Use the data in the table to write down a rule for the order in which operations in expressions involving both addition and subtraction should be carried out.

2. Create other number expressions to check your rule.

ACTIVITY 15

1. John wants to end up with the number 100.

   Fill in numbers in the empty boxes which makes this possible.

   (a)   100 − ☐ + ☐ = 100

   (b)   100 + ☐ − ☐ = 100

   (c)   ☐ + 100 − ☐ = 100

   (d)   ☐ − 100 + ☐ = 100

   (e)   ☐ − ☐ + 100 = 100

   (f)   ☐ + ☐ − 100 = 100

   Can you formulate rules about the numbers in the empty boxes?

2. John wants to end up with the number 50.

   Fill in the operations in the empty boxes that make this possible.

   Write down all the possibilities in each case.

   (a)   50 ☐ 20 ☐ 20 = 50

   (b)   50 ☐ 10 ☐ 20 ☐ 10 ☐ 20 = 50

   Can you formulate rules about the operations in the boxes?

ACTIVITY 16

Without calculating, insert **=** or ≠ between the number expressions:

1. 28 + (2 × 5)   28 + 2 × 5

2. 18 + 54 − 4 + 25   18 + (54 − 4) + 25

3. (1254 + 2973) × 7   1254 + 2973 × 7

4. 288 ÷ 32 ÷ 8   288 ÷ (32 ÷ 8)

5. (124 + 16) × 41 × 9   124 + 16 × 41 × 9

6. 124 + 16 × 41 × 9   124 + (16 × 41 × 9)

7. 488 ÷ 8 − 6 ÷ 3   488 ÷ 8 − (6 ÷ 3)

8. 488 ÷ 8 − 6 ÷ 3   488 ÷ (8 − 6) ÷ 3

9. (21 × 13) × (42 ÷ 6) 21 × 13 × 42 ÷ 6

10. 21 × 13 × 42 ÷ 6   21 × (13 × 42 ÷ 6)

ACTIVITY 17

In the empty boxes write down the order in which the operations should be done.

Write down *two ways* in which it can be done, if possible.

1. ☐ ☐ ☐   ☐ ☐ ☐
    $7 - 4 + 18 \times 9$ $7 - 4 + 18 \times 9$

2. ☐ ☐ ☐   ☐ ☐ ☐
    $24 \div 3 + 7 \times 2$ $24 \div 3 + 7 \times 2$

3. ☐ ☐ ☐ ☐   ☐ ☐ ☐ ☐
    $7 + 3 \times 4 - 15 \div 5$ $7 + 3 \times 4 - 15 \div 5$

4. ☐ ☐ ☐ ☐   ☐ ☐ ☐ ☐
    $16 \times 14 \div 7 - 3 \times 4$ $16 \times 14 \div 7 - 3 \times 4$

5. ☐ ☐ ☐   ☐ ☐ ☐
    $35 \div 5 \times 12 + 8$ $35 \div 5 \times 12 + 8$

6. ☐ ☐ ☐   ☐ ☐ ☐
    $45 \times 4 - 35 \div 7$ $45 \times 4 - 35 \div 7$

7. ☐ ☐ ☐ ☐ ☐   ☐ ☐ ☐ ☐ ☐
    $7 \times (3 + 7) + 14 \div (5 + 2)$ $7 \times (3 + 7) + 14 \div (5 + 2)$

8. ☐ ☐ ☐   ☐ ☐ ☐
    $5 \times (18 + 4) \times 6$ $5 \times (18 + 4) \times 6$

ACTIVITY 18

Without calculating, indicate with a ✓ whether it is possible to do a left to right computation. Indicate with a ✗ if it is impossible.

1. _____ $(15 - 9) \div 2 - 5 \times 5$

2. _____ $(56 - 16) \times (79 - 29)$

3. _____ $(4 + 3) \times 2 - 9 + 4$

4. _____ $18 \div 15 - 10 - 3 \times 2$

5. _____ $18 \div (15 - 10) - (3 \times 2)$

6. _____ $18 \div (15 - 10) - 3 \times 2$

7. _____ $18 \div (15 - 10 - 3) \times 2$

8. _____ $18 \div 15 - (10 - 3) \times 2$

9. _____ $36 - 8 \div (23 - 21) + 7$

10. _____ $(36 - 8) \div (23 - 21) + 7$

ACTIVITY 19

Evaluate the following numerical expressions. Show two methods if possible.

Example:

Method 1

> 25 + 72 − 16 − 25
>
> = 25 − 25 + 72 − 16   (25 − 25 = 0)
>
> = 72 − 16
>
> = 56

Method 2

> 25 + 72 − 16 − 25
>
> = 97 − 16 − 25
>
> = 81 − 25   (97 − 16 = 81)
>
> = 56

1. 32 + 16 − 6 − 16

2. 8000 + 900 − 200 − 3 + 60 − 1000 + 1 − 50

3. 8000 + 3 − 1 + 50 − 1000 − 60 + 200 − 900

4. 150 − 25 + 25 + 72

5. 87 − 23 + 29 + 13 + 72

6. $180 \div 27 \div 3$

7. $2000 \times 27 \div 5 \div 3 \div 3$

8. $25 \div 8 \times 4 \times 12 - 3$

9. $28 - 8 \div 4 + 3 \times 2$

10. $60 \div 10 - 2 \times 3$

Teacher Notes: Activity 19

The aim of this activity is to revisit the rules that were established in Activities 13 and 14.

The learners are encouraged to find more than one method to re-enforce the structural view.

An assessment can be given that covers the work in this Module.

More calculation activities can be designed for those learners who need it.