Malati

Mathematics learning and teaching initiative

ALGEBRA

Module 2

Brackets and dripping letters

Grades 6 and 7

TEACHER DOCUMENT

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Module 2: Overview and Introduction

In this Module we introduce additional roles of brackets. The role of brackets as "do it first" is well known. The additional roles of brackets are:

(1) Brackets that only indicate the order of operations – they don't change it. We call it *redundant* brackets. For example, in writing $a + (b + c)$ we are showing that in this case we are adding the b and c first, which will give the same answer if we calculate $a + b + c$ according to the left-to-right rule, i.e. if we calculate $(a + b) + c$.

(2) Brackets which change the order of operations but don't change the result of the expression (e.g. when using the distributive law).

(3) Brackets which change the order and the results.

The structure of Module 2 is:

- The first set of expressions will focus on addition and subtraction
- The second set on multiplication and division
- Then follows an overview
- The third set will focus on multiplication, addition and subtraction
- The distributive law
- The fourth set will focus on division, addition and subtraction

In this Module we also develop an understanding of the use of letters in algebraic language, namely to develop an understanding of the letter as a *number*.

ACTIVITY 1

Addition and Subtraction

	A	B	C	D
1	25 + 7 + 5	25 − 7 + 5	25 + 7 − 5	25 − 7 − 5
2	(25 + 7) + 5	(25 − 7) + 5	(25 + 7) − 5	(25 − 7) – 5
3	25 + (7 + 5)	25 − (7 + 5)	25 + (7 − 5)	25 − (7 − 5)

1. Evaluate the expressions in the table.

2. In each column mark the equivalent expressions.

3. Use the data in the table to write down a rule about expressions with brackets that are equivalent to expressions without brackets.

4. Use the numbers 37, 17 and 7 and the operations + and – to create expressions according to the structures in the table above. Write your expressions in a table as shown below.

	A	**B**	**C**	**D**
1				
2				
3				

   Now evaluate the expressions and check the rule that you wrote down in question 3.

5. Choose three numbers and use the operations + and – to create expressions as in the previous examples. Complete a table as shown and mark the expressions that will give the same answers in each column.

ACTIVITY 2

What will happen if we have more than three numbers?
Let us check with the four numbers 72; 50; 23; 7 and the operations + and – :

72 + 50 + 23 + 7	72 + 50 – 23 + 7	72 – 50 – 23 + 7
(72 + 50) + 23 + 7	(72 + 50) – 23 + 7	(72 – 50) – 23 + 7
72 + (50 + 23) + 7	72 + (50 –23) + 7	72 – (50 – 23) + 7
72 + 50 + (23 + 7)	72 + 50 – (23 + 7)	72 – 50 – (23 + 7)

Is your rule still correct?

Teacher Notes: Activities 1 and 2

The aim of these activities is to get the pupils to reflect on the structures in the tables and to see whether they can formulate a rule about the equivalent expressions with brackets and those without brackets.

In reflecting on the nature of the structure of the expressions the pupils will for example realise that since the numbers in the expressions are the same, it has to the nature and position of the *operations* that make the difference.

In order to focus the discussion on the role of the brackets, for example, pupils can be asked why the expressions in the first two rows always have the same answer.

The following response from a pupil shows that the pupil understands that the position of the brackets in the second row makes it a redundant bracket:

> "*We work from left to right in those expressions, so if the **bracket is there**, we still work from left to right."*

Ask the pupils what causes the other positions of the brackets to make a difference in some of the answers.

<u>**ACTIVITY 3**</u>

MARBLES

> We know that 44 – 24 + 10 and 44 – (24 + 10)
> are **not equivalent** numerical expressions.

1. Are you *sure* 44 – 24 + 10 and 44 – (24 +10) are not equivalent?
   Why do you think they are not equivalent?

2. Let us consider the following problem:

   I had 44 marbles. I played two games. In the first game I lost 24 marbles and in the second I gained 10. How many marbles do I have now?

   (a) Write down a computing instruction using the numbers 44, 24 and 10 to solve this problem.

   (b) Which of the expressions 44 – 24 + 10 and 44 – (24 +10) did you get?
   Did you use any other expressions?

3. Let us consider the following problem:

   I had 44 marbles. I played two games and lost in both. In the first game I lost 24 marbles and in the second 10. How many marbles do I have now?

   (a) Write down a computing instruction using the numbers 44, 24 and 10 to solve this problem.

   (b) Which of the expressions 44 – 24 + 10 and 44 – (24 + 10) did you get?
   Did you use any other expressions?

Teacher Notes: Activity 3

The aim of this kind of activity is to provide a context in which the pupils can discuss the structures that have dealt with in the first two activities.

In the discussion encourage reflection on the following aspects:

- The meaning of the numbers in the expression. For example, ask the pupils:
  What is the number that we must subtract in 44 – 24 + 10?
  What is the number that we must subtract in 44 – (24 + 10)?
- How the computational instructions, for example 44 – 24 + 10 and 44 – (24 + 10) are *different* and yet the *same.* (equivalent expressions)

Note that it we do not deal with the distributive law formally at this stage.

ACTIVITY 4

Addition and Subtraction

A	B	C	D
35 + (12 + 3)	35 − (12 +3)	35 – (12 – 3)	35 + (12 − 3)
35 + 12 + 3	35 − 12 + 3	35 + 12 – 3	35 + 12 − 3

E	F
(75 + 38) − 23 + 5	(75 + 38) − (23 + 5)
75 + 38 − 23 + 5	75 + 38 − 23 + 5

1. In each column, evaluate the expression with the brackets and the one without the brackets.

2. In which cases are the expressions with the brackets and without brackets equivalent?

3. Use the data in the table to write down a rule about expressions with brackets that are equivalent to expressions without brackets.

4. Write two expressions using addition and subtraction in which the expressions with the brackets and without brackets are equivalent?

5. Write two expressions using addition and subtraction in which the removal of the brackets does not make a difference.

6. Is your rule correct?

Teacher Note: Activity 4

The aim of this activity is to focus on the rule established in Activity 1. In Activity 1 we looked at expressions and what happens if we insert brackets in different positions.
In this activity, however, the focus is on the expression with brackets and what happens when the brackets are removed.

ACTIVITY 5

In which of the cases below is the sentence *true*? Explain!

1. (12 – 6) + 2 = 12 – 6 + 2
2. 18 – (7 + 5) = 18 – 7 + 5
3. 82 – 17 + 15 + 3 = 82 – 17 + (15 + 3)
4. (82 – 17) – (15 +3) = (82 – 17) – 15 – 3
5. 69 – 18 –(27 – 2) = (69 – 18) – (27 – 2)
6. 69 – 18 –27 – 2 = 69 – 18 – (27 – 2)
7. (82 – 17) + (15 – 3) = 82 – 17 + 15 – 3

Teacher Notes: Activity 5

Use this activity in a group-setting.

Make it clear to the pupils that a *true sentence* is one in which the two expressions on either side of the equality symbol are equivalent (have the same *answer*).

An assessment can be given after this activity to establish whether pupils can apply the rule of brackets in creating or establishing whether two expressions were equivalent. (See next page)

Note:

- We have observed that pupils who are able to *explain* the equivalence of two expressions based on the syntactical rule for brackets, are not always able to *create* an expression that is equivalent to a given expression using this rule.
- Consolidation activities in which pupils reflect on their rule while creating expressions, can be given to these pupils, for example:

  Use the given number expressions on the left to build equivalent expressions containing brackets around the last two numbers in the expressions.
  Note: Keep the order in which the numbers are written the same.

	Number expression	Equivalent number expression
1	96 – 56 + 18	
2	79 + 58 + 9	
3	83 – 25 – 39	
4	75 + 35 – 14	
5	83 + 25 – 39 – 14	
6	96 – 65 + 8 – 17	

ASSESSMENT 1

Emma is given the following number expression:

$$\mathbf{150 + 85 - 54 - 16 + 38 - 3}$$

Emma wants to create many equivalent expressions to
150 + 85 – 54 – 16 + 38 – 3 by inserting brackets into the expression.

Here are two of the Emma's equivalent expressions:

$$(150 + 85 - 54 - 16 + 38) - 3$$

$$(150 + 85 - 54) - 16 + 38 - 3$$

1. Explain what you understand by *equivalent expressions*.

2. Explain why the position of the brackets in Emma's expressions makes her expressions equivalent to the expression 150 + 85 – 54 – 16 + 38 – 3 which has no brackets.

3. Write down several other equivalent expressions that Emma could create.

ASSESSMENT 2:

1. Explain whether the following sentences are true or not.

   (a) 89 – (16 + 20) = 89 – 16 + 20

   (b) 97 + 27 + (32 – 14) = 97 + 27 + 32 – 14

2. Fill in the missing operation signs in order to make the sentences true.

   (a) 27 + 15 ___ 12 = 27 + (15 – 12)

   (b) 27 – (15 ___ 12) = 27 – 15 – 12

   (c) 27 ___ (15 ___ 12) = 27 – 15 + 12

   (d) 27 ___ 15 + 12 = 27 ___ (15 – 12)

   (e) 27 – (15 + 12) = 27 ___ 15 ___ 12

Teacher Notes: Assessment

After the assessment a brief feedback session with the whole-class is recommended in which the following ideas can be emphasised:

- Sometimes when we insert brackets or remove brackets the result of the expression changes.
- Every time when there is a subtraction before the brackets and we remove the brackets the answer is changed.
- When we insert brackets before a subtraction the answer changes.

ACTIVITY 6

HAVE FUN

Here are three numbers: f; u; n

Fatima writes down the following algebraic expressions:

$$f - u + n \quad \text{and} \quad f - (u + n)$$

Is it possible to create numerical expressions for $f - u + n$ and $f - (u + n)$ so that these numerical expressions are equivalent?

Are the algebraic expressions $f - u + n$ and $f - (u + n)$ equivalent?
Explain your answer.

Teacher Notes: Activity 6

Before doing this activity a whole-class discussion can be conducted in which the pupils can be told that we are going to work with numbers that will be represented by letters *(the letter as a number)*.

We can start the discussion by saying:

Let us suppose we have three numbers a , b and c and we create the expressions

$a - b - c$ and $a - (b - c)$.

Are these expressions equivalent?

We observed that the pupils now used the rule of brackets to explain why the two expressions were not equivalent.

The aim of Activity 6 is to make it explicit to the pupils that two algebraic expressions are equivalent if they are *numerically* equivalent for *all numbers.*

In looking for numbers that will make the expressions $f - u + n$ and $f - (u + n)$ numerically equivalent, the pupils can be encouraged to find as many cases as possible. However, it must in general be stressed that even if there are infinitely many possible solutions for making two expressions numerically equivalent, if we find just one set of numbers for which the expressions are not numerically equivalent, then the two expressions are *not* equivalent. For example, in this specific case, if f is any number, and u is any number and $n = 0$, the expressions $f - u + n$ and $f - (u + n)$ will have the same answer (i.e. they have the same value for an infinite number of numbers), yet for $f = 2$, $u = 3$ and $n = 4$, they do *not* have the same answer, and therefore they are *not* equivalent.

The pupils who found numbers that make the expressions numerically equivalent, for example, if $f = 0$ and $u = 8$ and $n= 0$, the two expressions $0 - 8 + 0$ and $0 - (8 + 0)$ are equivalent, leads to discussions about whether f, u and n could be the same number.

ACTIVITY 7

1. Use the numbers a, b and c and the operations $+$ and $-$ to create as many number expressions as you can.

2. Now create as many different expressions by inserting brackets for each of the expressions you created. Which of the expressions with brackets are equivalent to the expressions without brackets? Explain.

3. $a \square b \square c = a \square (b \square c)$

   $a \square b \square c = a \square (b \square c)$

   $a \square b \square c = a \square (b \square c)$

   $a \square b \square c = a \square (b \square c)$

   Put subtraction and addition symbols into the expressions above so that the algebraic sentence will be true for all numbers.

Teacher Notes: Activity 7

The pupils should create four expressions:

a + b + c   a + b – c   a – b + c   a – b – c

The aim of this activity is to re-enforce the idea that algebraic expressions are equivalent only if the expressions are equivalent *for all numbers.*

Note:
For pupils who still have difficulty in understanding the rule of brackets, this activity provides operational support for the structural view of these expressions in that the pupils can be allowed to substitute numbers into the expressions to check for equivalence.

ACTIVITY 8

You are given the numbers $a = 56$, $b = 20$ and $c = 30$ and these structures:

$$a - b - c$$

$$a - b + c$$

$$a + b - c$$

1. In which of these structures will you get the largest result if the numbers $a = 56$, $b = 20$ and $c = 30$ are inserted into the structures? Explain your answer.

2. If brackets are inserted, which of the structures below will give the largest result if the numbers $a = 56$, $b = 30$ and $c = 20$ are inserted into the structures?

$$a - (b - c)$$

$$a - (b + c)$$

$$a + (b - c)$$

   Explain your answer!

Teacher Notes: Activity 8

In question one the *numbers cannot be ignored* in establishing which of the three structures will give the largest result. One can of course decide which structure will give the largest result *without knowing what this result is*. For example, it can be argued that $a - b + c$ will give the largest result because in this expression we will only be subtracting 20 from *a*. In the other two expressions we are subtracting 30 in the one and 20 and 30 in the other. If none of the pupils provide this kind of explanation encourage them to reflect on the structures and justify which expression will give the largest result without doing a calculation.

The aim of question 2 is to see whether the pupils will use their *syntactical knowledge*, that $a - b + c$ is equivalent to $a - (b - c)$ in finding the largest result for the numbers $a = 56$, $b = 30$ and $c = 20$.

Enrichment Activity

CYRIL'S CONJECTURE

$$a + (b + c)$$

$$a - (b - c)$$

$$a - (b + c)$$

$$a + (b - c)$$

Given the structures above, Cyril says: "*a + (b + c) gives the largest result*".

Is Cyril's statement true?

How will you convince your group of your answer?

Teacher Notes: Cyril's Conjecture

This activity has the following aims:

1. Pupils need to realise that the kind of statement made by Cyril is called a *conjecture* or an informed guess based on the information he is presented with. In the discussion one could for example ask the pupils why they think Cyril made this conjecture.
2. Pupils also need to realise that if one makes a conjecture it is equally important to provide a *justification* or *explanation* in an effort to *convince* others of its truth.
3. Pupils need to realise that each expression firstly represent *A NUMBER*, in other words $a - (b - c)$ for example needs to be viewed as *A NUMBER.* This number will depend on the numbers a, b, and c, in other words the value of $a - (b - c)$ will change, so there is no way of telling which expression will be the largest.

ACTIVITY 9

Equivalent Expressions

Numerical expressions that have the same numerical values (same answers) are called ***equivalent numerical expressions.***

1. $35 - 12 + 7$ and $25 + 5$
2. $(35 - 12) + 7$ and $35 - 12 + 7$

Note: we can use the same numbers to create equivalent expressions and we can sometimes use different numbers.

Algebraic expressions that have the same numerical value for ***all*** values of the numbers represented by letters, are called ***equivalent algebraic expressions***.

1. $a - b + c$ and $a - (b - c)$
2. $a + (b - c)$ and $a + b - c$

Look at the following pairs of expressions.
In each case explain whether the two expressions are equivalent or not.

1. $a - (50 + 25)$ and $a - 50 - 25$
2. $120 - (50 + a)$ and $120 - 50 - a$
3. $78 - (a + b)$ and $78 - a + b$
4. $b + 49 - a$ and $b + (49 - a)$

ACTIVITY 10

PERIMETER OF SQUARES

Jason determines the perimeter of a square by adding the four sides of the square.

Jamie determines the perimeter of a square by multiplying one of the lengths of the side of the square by four.

Will they *always* get the same answers for the perimeter, no matter what the length of the side of the square?

Write an algebraic expression for Jason's method of finding the perimeter of *any square.*

Write an algebraic expression for Jamie's method of finding the perimeter of *any square.*

Mark say's that he has another method of finding the perimeter of *any square.*

Mark writes down the following algebraic expression for the perimeter of any square:

$$2 \times x + 2 \times x$$

Do you accept Mark's expression for finding the perimeter? Explain.

Mark says that his expression and Jamie's and Jason's expressions are equivalent algebraic expressions. Do you agree with this statement? Explain.

Teacher Notes: Activity 10

In this activity the pupils will build the following algebraic expressions to represent different ways of finding the perimeter of a square:

$x + x + x + x$ and $2 \times x + 2 \times x$ and $4 \times x$ which are equivalent algebraic expressions.

Note:

1. We do not at this stage expect the pupils to manipulate the algebraic expressions to justify their equivalence.
2. We look at numerical structures explored in the first Module.

What kinds of justifications are the pupils able to provide at this stage for the equivalence of these algebraic expressions?

- The pupils can use the structure of the problem to justify the equivalence of the expressions.
- The pupils can substitute different values for the lengths of the sides and check to see whether they get the same answer. How can one be sure that it will give the same answer for *all values of the lengths of the sides*?

In this activity, the idea of an algebraic expression as a representation of the final result, is an important shift in how pupils are expected to view an algebraic expression. In the previous activities the algebraic expressions were viewed as computational processes in which the operations could not be performed, for example, in the expression a – b – c, the focus is on the letter being viewed as representing "a number". In the previous activities we do not explicitly address the duality of the expression a – b – c as being both a computational procedure as well as "a number".

We are now more explicitly confronting the pupils with the double meaning of algebraic expressions, namely that of computational procedures and that of the object produced, namely $4 \times x$ is both a process (four times the length of the side of the square) and a result (the perimeter of the square).

ACTIVITY 11

SIPHO'S PROBLEM

Here are the numbers x and y.

Sipho creates the following algebraic expressions: $x + x \times y$ and $2 \times x \times y$.

Sipho says that the algebraic expressions $x + x \times y$ and $2 \times x \times y$ are equivalent, because if $x = 1$ and $y = 1$ then

$x + x \times y$	and	$2 \times x \times y$
$= 1 + 1 \times 1$		$= 2 \times 1 \times 1$
$= 1 + 1$		$= 2$
$= 2$		

Do you agree? Explain your answer.

Teacher Notes: Activity 11

In this activity the pupils are presented with a situation in which the specific numbers x = 1 and y = 1 will make the algebraic expressions numerically equivalent for those specific numbers. The aim of the activity is see whether the pupils will merely accept a single case or check for other cases. In the discussions we can encourage reflection on why these specific numbers make the numerical expressions equivalent.

ACTIVITY 12

Alan's Algebraic Expressions

Here are the numbers 24, 52, m and n.

1. Alan uses these numbers to create the following pairs of algebraic expressions. Look carefully at each of the pairs of algebraic expressions and decide whether they are equivalent or not. Explain your answer in each case.

   (a) $24 - m - n$ and $52 - (m + n)$

   (b) $n - m - n$ and $52 - (m + n)$

   (c) $24 - m - n$ and $m - (m + n)$

   (d) $24 - m - 52$ and $52 -(m + 24)$

   (e) $m - 52 - 24$ and $m - (52 + 24)$

   (f) $m - 24 - 52$ and $m - (52 + 24)$

2. Can you use all four numbers 24, 52, m and n to create two equivalent expressions?

Teacher Notes: Activity 12

In this activity we used some of the earlier structures but now we give:

1. Two expressions in which the one expression has constants and unknowns and the other only unknown numbers, for example, n – m – n and 52 – (m + n).
   The aim is to see what kind of justification the pupils' will give:
   - Will they proceed to substitute different numbers for m and n?
   - Will they view the second structure as 52 – m – n explain the non equivalence on the basis that since the two expressions have the same operational structure, the only difference being that the first number in the one expression is an unknown?
   - Will they realise that since one of the numbers is a constant and in the other expression all the numbers are unknowns the two expression can obviously not be equivalent?

   In the whole-class discussion we can see whether pupils can establish what condition is necessary for the two expressions to be equivalent, for example the two expressions n – m – n and 52 – (m – n) will only be equivalent if n = 52.

2. Two expressions in which the order of the numbers are different, for example, m – 24 – 52 and m – (52 +24).
   What justification strategies for the equivalence of the expressions are possible?
   - Will the pupils first use the commutative property of addition and change m – (52 +24) to m – (24 + 52) and then use the rule of brackets?
   - Will the pupils use the rule of brackets and then decide whether m – 24 – 52 and m – 52 – 24 are equivalent?
   - Will the pupils justification be solely semantic in that they view the expression as one in which the same number (52 and 24) is being subtracted from the number m.

   Are the pupils who view the two expressions in this way moving towards a "pseudo-structural" stage in their cognitive development of algebraic expressions? These pupils have not given up the procedural view but they both processes as "m subtract a number"

ACTIVITY 13

HELPING GARY

Are the following algebraic expressions equivalent?

$(b + 2) - (b - 5)$ and $(b + 2) - b + 5$

To decide, Gary replaces the number b with 8 in the expressions:

$(b + 2) - (b - 5)$	and	$(b + 2) - b + 5$
$= (8 + 2) - (8 - 5)$		$= (8 + 2) - 8 + 5$
$= 10 - 3$		$= 10 - 8 + 5$
$= 7$		$= 7$

Gail says to Gary that she can show that the two expressions are equivalent without replacing b with numbers. How do you think Gail shows Gary that the two expressions are equivalent?

Teacher Notes: Activity 13

The pupils have the syntactical know-how of justifying the equivalence of the two expressions. For example, both expressions can be written as b + 2 − b + 5 if the brackets are removed.

Those pupils who need to work procedurally with the definition of algebraic equivalence, namely by substituting different values of b into the two expressions, will soon realise that the answer is always 7.

The pupil can now reflect on the structures and try to explain why the value of the expression is 7 *for all values of b*.

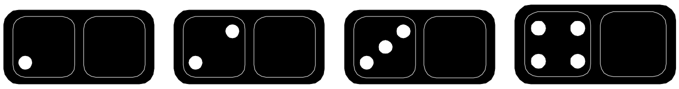
While dealing with the language of algebra in the context of the structure of algebraic expressions, the power of this language needs to be explored. The pupils need to appreciate the power of the algebraic language in dealing with generality.

The following activities have been designed to focus on the algebraic language as a means to express generality. We have chosen general structures with which the pupils are familiar in the numerical context. These structures can now be revisited in a context in which they have to look for pattern.

Dumi's Dynamic Dominoes 1

Dumi has a box filled with magical dominoes. He decides to arrange the dominoes in order of the number of dots on the domino. He starts with the domino with the smallest number of dots as shown below:

1^{st} domino 2^{nd} domino 3^{rd} domino 4^{th} domino 5^{th} domino 6^{th} domino

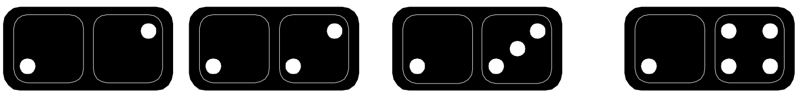


1. How many dots are there on Dumi's 5^{th} domino?
2. How many dots are there on Dumi's 6^{th} domino?
3. How many dots are there on Dum's 11^{th} domino?
4. Which domino has 21 dots?

Dumi's Dynamic Dominoes 2

Dumi has another box also filled with magical dominoes. He decides to arrange the dominoes in order of the number of dots on the domino. He starts with the domino with the smallest number of dots as shown below:

1^{st} domino 2^{nd} domino 3^{rd} domino 4^{th} domino 5^{th} domino 6^{th} domino

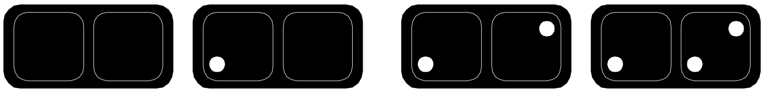


1. How many dots are there on Dumi's 5^{th} domino?
2. How many dots are there on Dumi's 6^{th} domino?
3. How many dots are there on Dumi's 11^{th} domino?
4. How many dots are there on Dumi's 29^{th} domino?
5. Which domino has 11 dots?
6. Which domino has 22 dots?
7. Which domino has 29 dots?

Dumi's Dynamic Dominoes 3

Dumi has another box also filled with x magical dominoes. He decides to arrange the dominoes in order of the number of dots on the domino. He starts with a blank domino as shown below:

1^{st} domino | 2^{nd} domino | 3^{rd} domino | 4^{th} domino | 5^{th} domino | 6^{th} domino

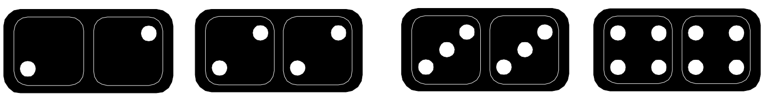


1. How many dots are there on Dumi's 5^{th} domino?
2. How many dots are there on Dumi's 6^{th} domino?
3. How many dots are there on Dumi's 14^{th} domino?
4. How many dots are there on Dumi's x^{th} domino?
5. Which domino has 14 dots?
6. Which domino has 22 dots?
7. Which domino has 29 dots?
8. Which domino has x dots?

Delshe's Dynamic Dominoes 1

Delshe has a box filled with x magical dominoes. In this box the dominoes only have an even number of dots on them. She also arranges the dominoes in order of the number of dots on the domino. She starts with the domino with the smallest number of dots as shown below:

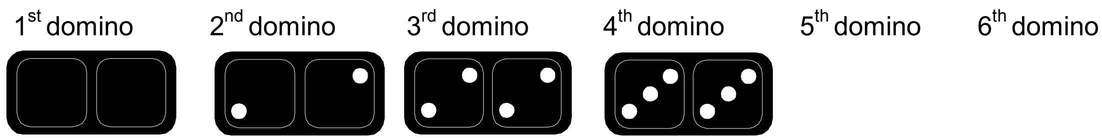
1^{st} domino | 2^{nd} domino | 3^{rd} domino | 4^{th} domino | 5^{th} domino | 6^{th} domino



1. How many dots are there on Delshe's 5^{th} domino?
2. How many dots are there on Delshe's 6^{th} domino?
3. How many dots are there on Delshe's 14^{th} domino?
4. How many dots are there on Delshe's x^{th} domino?
5. Which domino has 14 dots?
6. Which domino has 22 dots?
7. Which domino has 36 dots?
8. Which domino has x dots?

Delshe's Dynamic Dominoes 2

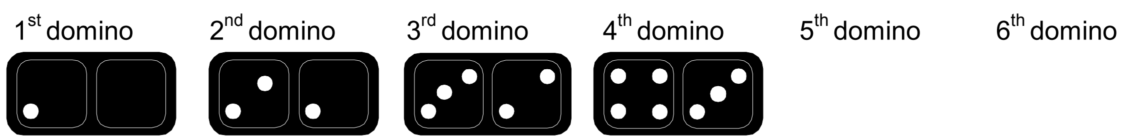
Delshe has another box filled with x dominoes. In this box the dominoes only have an even number of dots on them and there is also a blank domino. She also arranges the dominoes in order of the number of dots on the domino. She starts with the blank domino and then continues as shown below:



1. How many dots are there on Delshe's 5th domino?
2. How many dots are there on Delshe's 6th domino?
3. How many dots are there on Delshe's 14th domino?
4. How many dots are there on Delshe's xth domino?
5. Which domino has 14 dots?
6. Which domino has 22 dots?
7. Which domino has 68 dots?
8. Which domino has x dots?

Delshe's Dynamic Dominoes 3

Delshe has another box filled with x number of dominoes. In this box the dominoes only have an odd number of dots on them. She also arranges the dominoes in order of the number of dots on the domino. She starts with the domino with the smallest number of dots then continues as shown below:



1. How many dots are there on Delshe's 5th domino?
2. How many dots are there on Delshe's 6th domino?
3. How many dots are there on Delshe's 14th domino?
4. How many dots are there on Delshe's xth domino?
5. Which domino has 13dots?
6. Which domino has 29 dots?
7. Which domino has 55 dots?
8. Which domino has x dots?

<u>ACTIVITY 14</u> Beverly and Ben

Ben must decide whether the following algebraic expressions are equivalent.

$[(a + 3) + (a - 5)] + (a + 100) - (a - 5 + 10)$

and

$(a + 3 + a - 5) + a + 100 - a + 5 - 10$

Ben decides to substitute numbers for *a* in the expressions, for example, he substitutes $a = 5$:

$[(a + 3) + (a - 5)] + (a + 100) - (a - 5 + 10)$
$= [(5 + 3) + (5 - 5)] + (5 + 100) - (5 - 5 + 10)$
$= 8 + 0 + 105 - 10$
$= 103$

and

$(a + 3 + a - 5) + a + 100 - a + 5 - 10)$
$= (5 + 3 + 5 - 5) + 5 + 100 - 5 + 5 - 10$
$= 8 + 105 - 10$
$= 103$

Beverly says to Ben that she can show that the two expressions are equivalent without substituting numbers. How do you think Beverly shows Ben that the two expressions are equivalent?

<u>ACTIVITY 15</u> **Beverly and Ben again**

1. This is how Beverly tackles the problem 285 – 189

   She decides to subtract 100 from 200.
   Now she is left with 185 – 89.
   She then says that 89 = 85 + 4 and subtracts the 85 from the 185.

   Beverly has a problem now since she does not know what to do with the 4.
   Can you help her to finish the problem?
   Justify all the steps that were carried out on the basis of the rules that have been discussed.

2. This is Ben's method to solve this problem: 9527 – 586 – 386

   He says that he will subtract 386 from 586 and will be left with 200. He then subtracts the 200 from 9527 to give him the answer 9327.

   Ben was then asked to check his answer on a calculator and was surprised to get 8555 and not 9327.

   Can you explain to him by using the rules that have been discussed *why* his answer is incorrect.

ACTIVITY 16

Multiplication and Division

1. Are the following expressions equivalent?

$(8 \times 5) \times 2$
$8 \times (5 \times 2)$

2. Which of the following expressions are equivalent?

A	B	C	D
$(12 \times 3) \times 8 \times 6$	$12 \times (3 \times 8) \times 6$	$12 \times 3 \times (8 \times 6)$	$(12 \times 3) \times (8 \times 6)$

3. What rule can you write down about these expressions?

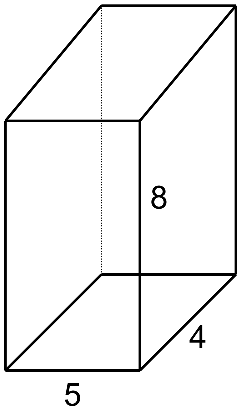
Teacher Notes: Activity 16

Ask the pupils to attempt to express the rule using letters (algebraic language). Start dripping the terminology "the algebraic language".

On the board draw two columns:

Spoken language	Algebraic language
	$(a \times b) \times c = a \times (b \times c)$

ACTIVITY 17



1. In how many different ways can you find the volume of the box?

2. Relate your observations to the rule that you formulated in Activity 16.

ACTIVITY 18

1. Are the following expressions equivalent?

   $(64 \div 8) \div 4$
   $64 \div (8 \div 4)$

2. Are the following expressions equivalent?

   $36 \div (18 \div 6) \div 3$
   $36 \div 18 \div (6 \div 3)$

3.

$18 \times 6 \times 3$	$18 \times 6 \div 3$	$18 \div 6 \times 3$	$18 \div 6 \div 3$
$(18 \times 6) \times 3$	$(18 \times 6) \div 3$	$(18 \div 6) \times 3$	$(18 \div 6) \div 3$
$18 \times (6 \times 3)$	$18 \times (6 \div 3)$	$18 \div (6 \times 3)$	$18 \div (6 \div 3)$

   (a) Evaluate the expressions in the table.

   (b) In each column mark the equivalent expressions.

   (c) Use the data in the table to write down a rule about expressions with brackets that are equivalent to expressions without brackets?

4. Use the numbers 250; 50 and 5 to form four columns as in the previous question. Check your rules.

Teacher Notes: Activity 18

Overview of $\times$ and $\div$ and the symmetry with $+$ and $-$

We saw that when we insert brackets in front of division operation the expression changes its value.

$54 \div 6 \div 3$ is not equivalent to $54 \div (6 \div 3)$
$54 \div 6 \times 3$ is not equivalent to $54 \div (6 \times 3)$

What was the situation when we inserted brackets in front of multiplication?

$54 \times 6 \times 3$	$54 \times (6 \times 3)$	Are they equivalent?
$54 \times 6 \div 3$	$54 \times (6 \div 3)$	Are they equivalent?

ACTIVITY 19

Let us try to remember what the situation was when we had addition and subtraction.

		Equivalent: yes or no?
$54 + 6 + 3$	$54 + (6 + 3)$	
$54 + 6 - 3$	$54 + (6 - 3)$	
$54 - 6 + 3$	$54 - (6 + 3)$	
$54 - 6 - 3$	$54 - (6 - 3)$	
$54 \times 6 \times 3$	$54 \times (6 \times 3)$	
$54 \times 6 \div 3$	$54 \times (6 \div 3)$	
$54 \div 6 \times 3$	$54 \div (6 \times 3)$	
$54 \div 6 \div 3$	$54 \div (6 \div 3)$	

Teacher Notes: Activity 19

Reflection on the rules and generalisation:

$$a+b+c=a+(b+c) \qquad a\times b\times c=a\times(b\times c)$$
$$a+b-c=a+(b-c) \qquad a\times b\div c=a\times(b\div c)$$
$$a-b+c\neq a-(b+c) \qquad a\div b\times c\neq a\div(b\times c)$$
$$a-b-c\neq a-(b-c) \qquad a\div b\div c\neq a\div(b\div c)$$

ACTIVITY 20

Keeping the expressions equivalent: Expressions with $\times$ and $\div$

$75 - (30 + 20)$ is not equivalent to $75 - 30 + 20$.
We can't just *delete* the brackets, we have to change the operations in order to keep the expressions equivalent:

$$75 - (30 + 20) = 75 - 30 - 20$$
$$75 - (30 - 20) = 75 - 30 + 20$$

- What was the rule that we have decided to use?
- What was our explanation for this rule?
- What is the case if we have multiplication and division?

When there is a multiplication operation in front of the brackets, we may remove the brackets without changing the answer. For example:

$25 \times (8 \times 7) = 25 \times 8 \times 7$; so we can just "remove" the brackets.

When we have division we have to be careful. If we want to "remove" the brackets without changing the answer we have to change some operation:

$54 \div (6 \div 3) = 54 \div 6 \square 3$

- What operation should we put in the box in order to have equivalent expressions?
- Use your calculator to check.
- What is your conclusion?

Let us investigate with other numbers:

$100 \div (20 \div 5)$ $100 \div 20 \square 5$

- What operation should we put in the box in order to have equivalent expressions?
- Check with your calculator.
- Is your assumption correct?

Can you try to explain why we had to change the operation in order to maintain the equivalence. Why could we not just remove the brackets?

$54 \div (6 \times 3)$ $54 \div 6 \square 3$

- What operation should we put in the box in order to have equivalent expressions?
- What is your assumption?

Let us investigate with other numbers:

$84 \div (7 \times 2)$ $84 \div 7 \square 2$

Put an operation in the box so that the two expressions will be equivalent:

$135 \div (15 \times 3)$ $135 \div 15 \square 3$

$105 \div (15 \div 3)$ $105 \div 15 \square 3$

$36 \div (6 \times 2) + 8$ $36 \div 6 \square 2 + 8$

Teacher notes: Activity 20

Emphasise that we are only dealing with a $\times$ inside the brackets, so that the pupils do not overgeneralise to + and $-$.

It is very difficult to find semantic support for the equivalence of the structures

$a \div (b \times c)$ and $a \div b \div c$.

If it fits the example of the box in Activity 12, it can be used to show continued division.

1. How does one find the height of a box if its volume is 48 and the area of its base is 6×2?

   height = $48 \div (6 \times 2) = 4$

2. The height of the same box is to be found.
   This time we are given the volume of the box to be 48 and one of its sides is 6.
   It is of course impossible to find the height with this information but we can find the area of one of the sides in which the height is.
   Hence the area will be $48 \div 6 = 8$.

   If we are now told that the other dimension is 2, the height can be found by
   $8 \div 2 = 4$

   The overall calculation of the height in this case is $48 \div 6 \div 2 = 4$

An Overview (Whole-class discussion)

We investigated two kinds of expressions:

(1) Expressions that have only addition and subtraction.
(2) expressions that have only multiplication and division.

We discovered that:

(1) When we insert brackets in front of addition or multiplication or when we remove them we still have and equivalent expression, the answer does not change.
(2) When we insert brackets in front of subtraction or division or when we remove them we have to change the operation which was inside the brackets if we want to get an equivalent expression.

Let pupils write the rules that they formulated in the algebraic language:

$$a+(b+c)=a+b+c \qquad a\times(b\times c)=a\times b\times c$$
$$a+(b-c)=a+b-c \qquad a\times(b\div c)=a\times b\div c$$
$$a-(b+c)=a-b-c \qquad a\div(b\times c)=a\div b\div c$$
$$a-(b-c)=a-b+c \qquad a\div(b\div c)=a\div b\times c$$

ACTIVITY 21

Beverly must solve the following problem: $360 \div 200$

Beverly knows that 200 is 2×100.
She decides to handle the division in two steps. She writes:

$$360 \div 2 \times 100$$

She then says "I must divide 360 by 2 which is 180 and then I must multiply the 180 by 100."

Ben says to her: "You will get 18 000! It does not make sense, because if 360 is divided by 200 the answer is obviously less than 2".

How can Ben help Beverly to correct her mistake by continuing with her method of splitting the 200 into 2×100 and using the rules he knows.

ACTIVITY 22

Use Beverly's idea of splitting the number that is divided in order to evaluate the following:

1. $525 \div 15$

2. $648 \div 24$

3. $368 \div 16$

Teacher Notes: Activities 21 and 22

Hint to the pupils after they have worked on the problems that there is more than one way of splitting the number and to find another way in which the split involves a different operation, for example, 15 as 5×3 or 15 as $30 \div 2$.

ACTIVITY 23

Beverly must solve the following problem: $1500 \div 25$

She writes $1500 \div (5 \times 5)$

Ben says that he has found another way of splitting 25 that makes it easier for him to divide.

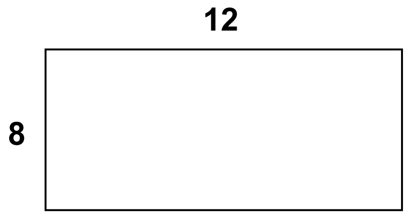
He writes $1500 \div (150 \div 6)$

How must Ben continue?

ACTIVITY 24

The Distributive Law

1.



Determine the perimeter of the above rectangle in two possible ways.

2. How many cool-drinks will be needed for a sports party for a rugby team (15 players) and a net-ball team (7 players) if there are 3 cool-drinks for each player?
   Do the calculation in two different ways.

3. 20 pupils in Grade 7 decide to have a party. R25 is collected from each pupil.
   In order to cover the expenses an additional R8 has to be collected from each pupil.
   What was the total cost of the party?
   Do the calculation in two different ways.

4. 35 pupils in Grade 8 decide to have a party. R30 is collected from each pupil.
   R7 was returned to each pupil after the total cost of the party was calculated.
   Determine the total cost of the party in two different ways.

ACTIVITY 25

Grade 6 pupils were solving the following problem:

Robea bought 7 blue pens for R2 each and 7 green pens for R 3 each
Write down a numerical expression that describes how much she paid for the pens.

Some of the pupils wrote the expression:

$7 \times 2 + 7 \times 3$ .... (a)

Other pupils wrote the expression:

$7(2 + 3)$ .... (b)

1. Are the expressions equivalent? Explain.

2. Which expression, in your opinion, describes the situation in the story better?

3. Expression (a) involves 3 calculations. What is the meaning of the results we get from each of these calculations?

4. Expression (b) involves 2 calculations. What is the meaning of the results we get from each of these calculations?

5. Do we get more information from expression (a) or does it just involve more calculations?

Teacher Note: Activity 25

Discuss with the whole class the conventional notation

$7 \times (2 + 3)$ and $7(2 + 3)$

The objective with Activity 25 is to:

- Reflect on the rules formulated in Activities 19 and 20.
- To generalise and to write the following rules in algebraic language:
  - multiplication distributes over addition
  - multiplication distributes over subtraction.

<u>Enrichment Differential Activity</u> (Homogenous group setting)

Generalising the Distributive Law

Design an investigation for generalising the distributive law.

Teacher Notes:

A generalisation developed in an informal manner is referred to as generalisation of the first degree.

The enrichment activity is an example of the development of a generalisation at a formal level, referred to as generalisation of the second degree.
In this activity the pupil has to firstly ask "What is to be investigated?"
Once the pupil realises he has to investigate which of the operations + ; - ; × ; ÷ distributes over certain operations in the structure

$$a \nabla (b \Delta c) = (a \nabla b) \ \Delta \ (a \nabla c),$$

a systematic approach has to be followed to arrive at the generalisation.

Note that at this stage we have not dealt with the distributive law on the right, i.e.

$$(b \Delta c) \nabla a = (b \nabla a) \ \Delta \ (c \nabla a)$$

We know that $(20 + 30) \times 5 = 20\times5 + 30\times5$

ACTIVITY 26

Indicate with a ✓ where the distributive law can be used and with a ✗ where it cannot.

Then calculate the value of the expressions.

1. ___ $3(10 + 5)$
2. ___ $18(20 – 2)$
3. ___ $15 + (20 – 4)$
4. ___ $10(13 + 18 – 12)$
5. ___ $12(20 – 8 – 2)$
6. ___ $5(36 – 5 + 2)$
7. ___ $14 + (3 \times 7)$
8. ___ $4 \times (8 \times 2)$
9. ___ $2 \times (2 \times 2)$
10. ___ $7(5 + 100)$
11. ___ $252 \div (20 + 16)$
12. ___ $20 \div (5 \times 2)$
13. ___ $20 \div (5 – 4)$
14. ___ $2 \times (2 + 2)$
15. ___ $2 \times (2 \div 2)$
16. ___ $2 + (2 + 2)$
17. ___ $2 \div (2 + 2)$
18. ___ $2 \div (2 – 2)$

ACTIVITY 27

Use the distributive law to insert numbers into the boxes to make the following number sentences true.

1. $5\ (8 - \square\ ) = 5 \times 8 - 5 \times \square = 40 - 15 = 25$

2. $\square\ (20 + 18) = \square \times 20 + \square \times 18 = 60 + 54 = 114$

3. $11\ (\square + 6) = \square \times 11 + 6 \times 11 = 99 + 66 = 165$

4. $5\ (7 + \square\ ) = 35 + 65 = 100$

5. $\square\ (12 - 4) = \square \times 12 - \square \times 4 = 72 - 24 = 48$

6. $\square\ (\square + 17) = 34 + 4 = 38$

7. $3\ (19 - \square\ ) = 0$

ACTIVITY 28

DON'T CALCULATE !!!

1. Consider the following groups of expressions. Without calculating the value of the expressions, say which expressions in each group are equivalent and why.

   Discuss this in your groups.

   Write down your reasons for deciding which expressions are equivalent.

$312 - 117 - 59$
$312 - (117 - 59)$
$312 - (117 + 59)$
$300 - 100 - 50 + 12 - 17 - 9$
$(300 - 100 - 50) + (12 - 17 - 9)$
$(300 - 100 - 50) - (12 + 17 + 9)$

$7{,}93 \times 2{,}4 + 35{,}87$
$7{,}93 \times (2{,}4 + 35{,}87)$
$2{,}4 \times 7{,}93 + 35{,}87$
$35{,}87 \times 2{,}4 + 7{,}93$
$7{,}93 \times 2{,}4 + 7{,}932 \times 35{,}87$
$(35{,}87 + 2{,}4) \times 7{,}93$

2. Now consider the following algebraic expressions and decide which are equivalent:

(a)

$x + y$

$(x - m) + (y + m)$

$x - m + y - m$

$x - m + y + m$

$(x + k) + (y - k)$

(b)

$a + b - c$

$a - c + b$

$a - (c + b)$

$a - d + b - c + d$

$(a - d) + (b - d) - c$

3. Write equivalent expressions for:

$a - b$ and $x - (y + m)$

ACTIVITY 29

Losing Marbles

Carl has 298 marbles. He plays two games with his friend Simon. Carl loses marbles in both games. He goes home and tells his brother Craig about his games. Craig asks Carl how many marbles he lost in **each** game. Carl tells Craig that he lost 34 marbles in one of the games but was too embarrassed to tell him how many marbles he lost in the other game.

Carl writes down the following number expression to represent the number of marbles that he has left:

$$298 - a - 34$$

What does the *a* represent in the expression?

Carl asks Craig to tell him how he would calculate the number of marbles that he would have left.

In the table below are the three different ways that Craig explained to calculate the number of marbles that Carl has left. Write Craig's methods, using only symbolic language:

Craig's Verbal Explanation	Symbolic representation of Craig's verbal explanation.
"I will add the number *a* and 34. The number that I get, I will subtract from 298."	
"I will subtract the 34 from *a*. The number that I get I will subtract from 298."	
"I will subtract 34 from 298. The number that I get, I will subtract the number *a* from it."	

Do you agree with Craig's methods? Explain