Malati

Mathematics learning and teaching initiative

ALGEBRA

Module 4

Towards Manipulation

Grade 8

Teacher document

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December 1999

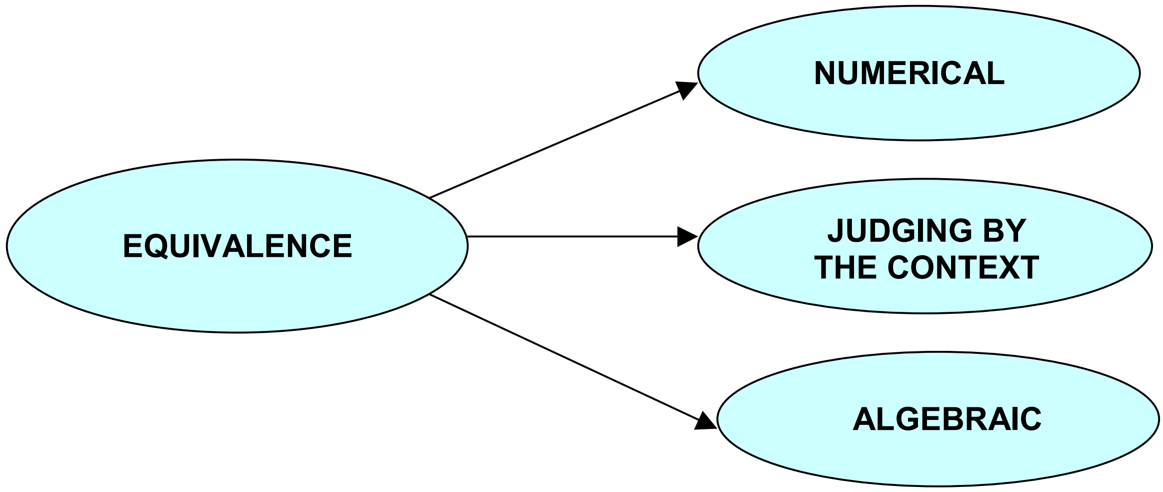
Overview of Module 4

The main purpose of this module is to develop the *meaning* of *algebraic equivalence* and to introduce the *usefulness* of algebraic equivalence. This idea of usefulness will then form the basis for developing manipulative skills. Here are therefore three dimensions of understanding:

1. the *meaning* of equivalence
2. the *usefulness* of equivalence
3. *skill* in using equivalence.

In the traditional curriculum children were usually thrown directly into meaningless manipulations of algebraic expressions supported by a body of *rules,* but with very little reflection on the *structure* of the expressions, for example *why* is $2x+3x=5x$? We believe that the kind of experiences in this module will contribute to the development of learners' “sense-making” and an avoidance of the “rules without reason” approach that is so prevalent.

We want to develop three kinds of approaches to the understanding of equivalence:



When using a context, the hidden mathematics should not be the major factor for judging equivalence, but rather that children develop understanding by using their common-sense.

We believe that at all times when learning is taking place the learner has to cope with competing cognitive schema, since the old schemas are never "deleted", but specific scenarios might trigger the old schema which might help them to reconstruct new knowledge. This module thus builds on the underlying ideas of structure in Module I and Module 2 in the Primary School.

The module is not about getting numerical *answers* but about reflecting on the *structure* and ultimately reflecting on generalisation. Children should use their structural knowledge to judge equivalence.

It should be remembered that by working through this module children are in the process of *constructing* (developing) their conception of algebraic equivalence and we should therefore not push to formal teaching (e.g. about like terms, distributive property). This knowledge should be a *product* of their reflections on generalisations and could be formalised at a later stage. Likewise, we do not provide extensive drill or consolidation activities to develop learners' *skill* in algebraic manipulation – we believe it is not appropriate at this level. If teachers value manipulation skill at this level they can provide learners with additional activities from the normal textbook.

In this module the reflection process should also introduce children to the different meanings of algebraic statements:

- *Equivalent expressions:*
  e.g. $7a-(3a+2)$ and $4a-2$ are equivalent since they have the same values for *all values* of the variable. The algebraic statement $7a-(3a+2)=4a-2$ thus means that the statement is true for *all values* of the variable. It is called an *algebraic identity*.

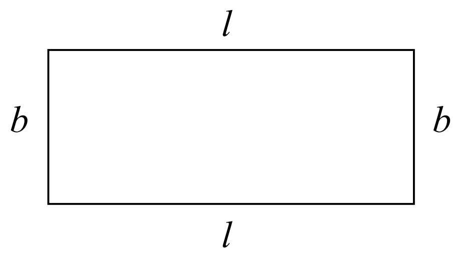
- *Non-equivalent expressions:*
  e.g. $4x+12$ and $7x+3$ are not equivalent since they have different values for all values of x except for $x=3$. An algebraic statement like $4x+12=7x+3$ thus means that the statement is true only for *some values* of the variable. It is called an *algebraic equation*.

  e.g. $5k+3$ and $5k+7$ are not equivalent since they have different values for *all* values of k. The algebraic statement $5k+3=5k+7$ thus means that the statement is true for *no value* of the variable. It is called an *algebraic impossibility*.

Making life easier

1. The perimeter of any rectangle with length l cm and breadth b cm can be calculated with different formulas, for example.

   - Perimeter = $l+b+l+b$
   - Perimeter = $2l+2b$
   - Perimeter = $2(l+b)$



   (a) Determine the perimeter of a rectangle with breadth 12 cm and length 23 cm by using each of the above formulas.

   (b) Determine the perimeter of a rectangle if the breadth is 27 cm and the length is 31 cm. Show all your calculations. Which formula did you use? Explain why.

   (c) Determine the breadth of a rectangle if the perimeter is 164 cm and the length is 31 cm. Show all your calculations. Which formula did you use? Explain why.

2. It costs R13,30 per square metre to paint a cube. Moses calculates the cost, in rands, of painting all the sides of a cube with length x metres, like this:

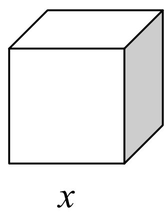
   $$13{,}3\times x\times x+13{,}3\times x\times x+13.3\times x\times x+13{,}3\times x\times x+13{,}3\times x\times x+13{,}3\times x\times x$$

   Bingo uses this formula:

   Cost = $13{,}3(x^2+x^2+x^2+x^2+x^2+x^2)$

   Rolene uses this formula:

   Cost = $79{,}8x^2$



   (a) Determine the cost to paint a cube with sides 1,7 m by using all three of the formulas.

   (b) Determine the cost to paint a cube with sides 4,6 m.
   Which formula did you use? Explain *why*.

   (c) Determine the length of the sides of the cube if the cost is R977,55.
   Which formula did you use? Explain *why*.

In mathematics it is possible to have formulas and expressions which, although they look *different*, are the *same*. We say they are *equivalent.*
You are going to learn some very useful techniques to transform formulas or expressions to equivalent ones to make mathematical problem solving easier.

***Teachers notes:* Making life easier**

This activity revisits the idea of equivalence and introduces the usefulness of it. This idea of usefulness will form the bases for developing manipulative skills.

Looking different 1

1. Complete the following table

x	**1**	**2**	**5**	**12**	**19**	**37**	**45**
$2x+5x$							
$3x+4x$							
$12x-5$							
$7x$							
$6x+x$							
$9x-2x$							

(a) What do you notice in the table?

(b) Determine the value of $2x+5x$ if x = 19. Discuss your method.

(c) Determine the value of x if $9x-2x$ = 35. Discuss your method.

2. Complete the following table.

x	**1**	**2**	**5**	**12**	**19**	**37**	**45**
$5x+x$							
$24x-18x$							
$3x+2x$							
$6x$							
$4x+2x$							
$16x-10x$							

(a) What do you notice in the table?

(b) Each of the group members must write a different algebraic expression that will be equivalent to $24x-18x$. Discuss your answers.

(c) What is the shortest expression that will be equivalent to $24x-18x$?

(d) Determine the value of $16x-10x$ if x = 33,2.

We say that expressions in the tables are *equivalent* if they produce the same output numbers for the same input numbers.
For example, $4x+2x$ and $6x$ are equivalent expressions, because they produce the same output values for the same input values.
We can explain the equivalence in the following way:
$4x+2x$ means $4\times x+2\times x$ = $(x+x+x+x)+(x+x)$ = $6\times x=6x$

3. Choose different algebraic expressions that will be equivalent to $13x+6x$. Complete the table and check if the different expressions are really equivalent.

x	**1**	**2**	**5**	**12**	**19**	**37**	**45**
$13x+6x$	19	38	95	228	361	703	855

(a) Determine the value of $13x+6x$ if x = 13,7.

(b) Determine the value of x if $13x + 6x$ = 703

4. Complete the table and see if the different expressions are equivalent.

x	**1**	**2**	**5**	**12**	**19**	**37**	**45**
$4x+13x$							
$13x+4x$							
$13x+4$							
$17x$							
$13+4x$							
$17+x$							

(a) Which of the expressions are equivalent?

(b) Which of the expressions are not equivalent?

(c) Try to find the shortest expression for $13x + 4x$. Check and discuss your answer.

(d) Try to find the shortest expression for $13x + 4$. Check and discuss your answer.

5. Complete the following table.

x	1	2	5	12	19	37	45
$2x + 6$							
$4x + 2$							
$(2x + 6) + (4x + 2)$							
$6x + 8$							

What do you notice? Try to explain this.

6. Complete the following table.

x	1	2	5	12	19	37	45
$(3x + 4) + (5x + 2)$							
$(2 + 8x) + 4$							
$6x + 5 + 1 + 2x$							
$14 + x$							
$14x$							
$8x + 6$							

(a) Which of the expressions are equivalent?

(b) Which of the expressions are not equivalent?

(c) Try to find the shortest expression for $6x + 5 + 1 + 2x$. Check and discuss your answer.

(d) Try to find the shortest expression for $8x + 6$. Check and discuss your answer.

We can say expressions that produce the same output numbers for the same input numbers are *equivalent* and can replace one another.
Long expressions can be *simplified* to a more convenient, equivalent expression by:

- *Combining like* terms: $13x+4x$ can be replaced by the simpler expression $17x$, because they will give the same result for any value of x. We say $13x$ and $4x$ are combinable or like terms.
- *Non-combining* or *unlike* terms: $8x+6$ cannot be replaced by $14x$. We say $8x$ and 6 are non-combinable or unlike terms.
- *Rearranging* the terms of the expression, because the order in which numbers are *added* does not affect the answer, for example:
  $(3x+4)+(5x+2)$ = $3x+4+5x+2$ = $3x+5x+4+2$ = $8x+6$

7. Determine the value of the following expressions if x = 7,3.

   (a) $4x+3+6x+2$

   (b) $12x+3$

   (c) $(2+4x)+(7x+5)$

   (d) $1{,}3x+3{,}7x+2{,}6x+2{,}4x$

Teacher notes: **Looking different 1**

These activities introduce the meaning of equivalence. Learners must take time to explore the tables and attach meaning to it. The problem types of finding function values and finding input values are also explored where possible.

In number 3 Learners get the opportunity to construct their own equivalent expressions.

Number 5 is about the addition of two expressions.

Numbers 4 and 6 give learners the opportunity to explore typical mistakes.

In number 7 learners must experience the usefulness of first changing the expression to an easier, equivalent expression before they find the function values. This idea can be made explicit in a classroom discussion.

Looking different 2

1. Complete the table.

x	**1**	**3**	**5**	**8**	**11**	**17**	**38**
$2x+3x$							
$5x$							

What do you notice in the table?

We can say the two algebraic expressions $2x+3x$ and $5x$ are ***equivalent***, because they have the same values for any value of the input variable x.
We can say: $2x+3x = 5x$ for all values of x.
An algebraic statement like $2x+3x = 5x$ which is true for all values of the input variables, is called an ***algebraic identity***.

2. Complete the following table.

x	1	3	5	8	11	17	38
$4x+12$							
$7x+3$							

(a) Is $4x+12 = 7x+3$ an algebraic identity? Discuss by using the table.

(b) Are there any values of x where the two expressions have the same numerical value?

3. Complete the following table.

x	1	3	5	8	11	17	38
$(5+12x)+(3x+20)$							
$10x+50$							

(a) Is $(5+12x)+(3x+20) = 10x+50$ an algebraic identity? Discuss by using the table.

(b) Are there any values of x where the two expressions have the same numerical value?

Sometimes the values of the two expressions are not equal for <u>all</u> values of the input variable.
For example $4x+12$ and $7x+3$ are therefore not equivalent expressions, because they have different values for all values of x, except for x = 3.
In this case, the "="-sign means that the values of the two expressions will be equal only for *some* values of x.

4. Complete the table.

x	1	3	5	8	11	17	38
$10x+40$							
$10x+50$							

(a) Is $10x+40$ = $10x+50$ an algebraic identity?

(b) Is it possible to find any value for x where $10x+40$ = $10x+50$?

$10x+40$ = $10x+50$ for *no values* of x. We call this an *algebraic impossibility.*

(c) Write down other algebraic impossibilities.

***Teacher notes:* Looking different 2**

In this activity, the learners are introduced to the three different meanings of algebraic statements.

Looking different 3

1. (a) Say in words the meaning of 12 + (3 + 5) and 12 + 3 + 5.

   (b) Are 12 + (3 + 5) and 12 + 3 + 5 equivalent?

   (c) Say in words the meaning of $12x+(7x+5)$ and $12x + 7x + 5$.

   (d) Are $12x+(7x+5)$ and $12x + 7x + 5$ equivalent? Use a table to check.

2. (a) Say in words the meaning of 12 + (3 – 5) and 12 + 3 – 5.

   (b) Are 12 + (3 – 5) and 12 + 3 – 5 equivalent?

   (c) Say in words the meaning of $12x+(7x-5)$ and $12x + 7x - 5$.

   (d) Are $12x+(7x-5)$ and $12x + 7x - 5$ equivalent? Use a table to check.

3. (a) Say in words the meaning of 12 – (3 + 5) and 12 – 3 – 5.

   (b) Are 12 – (3 + 5) and 12 – 3 – 5 equivalent?

   (c) Say in words the meaning of $12x-(7x+5)$ and $12x-7x-5$.

   (d) Are $12x-(7x+5)$ and $12x-7x-5$ equivalent? Use a table to check.

4. (a) Say in words the meaning of 12 – (3 – 5) and 12 – 3 + 5.

   (b) Are 12 – (3 – 5) and 12 – 3 + 5 equivalent?

   (c) Say in words the meaning of $12x-(7x-5)$ and $12x-7x+5$.

   (d) Are $12x-(7x-5)$ and $12x-7x+5$ equivalent? Use a table to check.

   (Tables on the next page)

For any numbers x, y, and z:

- $x+(y+z)=x+y+z$
- $x+(y-z)=x+y-z$
- $x-(y+z)=x-y-z$
- $x-(y-z)=x-y+z$

We can therefore *replace* expressions of the left hand types with those on the right hand and vica versa.

5. What is the value of each expression if x = 23,45 and y = 13,92?

   (a) $25x-(7x+5)$

   (b) $13x+5-(10x-4)$

   (c) $12x+(4y-5x)+7x-2y$

   (d) $(3{,}3x+2{,}1y)-(1{,}4y-1{,}7x)-(4x+0{,}7y)$

***Teacher notes:* Looking different 3**

No. 1-4 links with the exercises from mathematical structure.

No. 5 helps learners to experience the meaning and usefulness of equivalence.

Discuss as a whole class the benefits if they first simplify the expressions before the find the function values.

Distribution 1

1. Write the following computational procedures in words. Which of them are the same?

   (a) $4 \times (8 + 5)$

   (b) 4×13

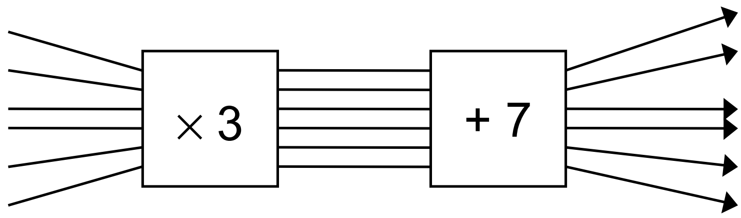
   (c) $4 \times 8 + 4 \times 5$

   (d) $4 \times 8 + 5$

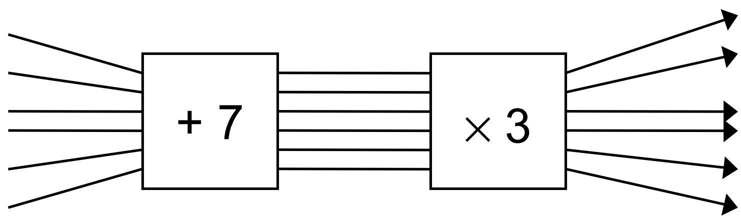
   (e) $(8 + 5) \times 4$

2. Complete the flow diagrams. Use the same input numbers every time.

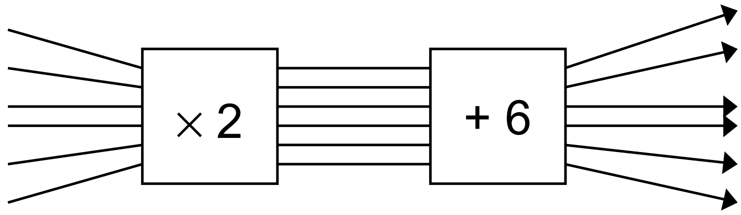
   (a)



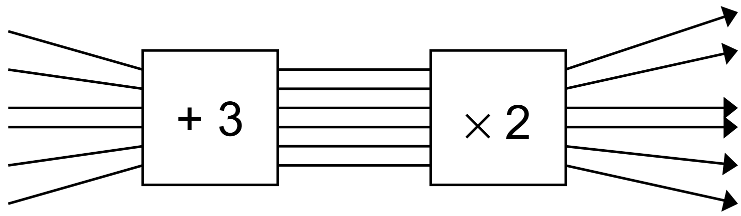
   (b)



(c)



(d)



(e) Discuss your answers.

(f) Which of the procedures produce the same output numbers?

(g) Write the flow diagrams as algebraic expressions.

3. Complete the following tables:

(a)

x	-12,3	-3,8	7	94	112	308
$12x+9$						
$3(4x+3)$						
$(3)(4x)+(3)(3)$						
$12x+3$						

(b)

x	-12,3	-3,8	7	94	112	308
$4x(2x+5)$						
$8x^2+20x$						
$(4x)(2x)+(4x)(5)$						
$(4x)(2x)+5$						

(c) What do you notice in the tables? Discuss!

An expression of the form $x(y+z)$ can be replaced with $xy+xz$.
And $xy+xz$ can be replaced with $x(y+z)$.
$x(y+z)$ and $xy+xz$ are equivalent expressions for all numbers x, y and z.
We call this the *distributive property of multiplication over addition*.

4. Investigate if you can replace an expression of the form $x(y-z)$ with $xy-xz$ and vica versa.

Teacher notes: **Distribution 1**

We expand the distributive property to algebraic expressions and try to link it to their existing knowledge in number 1. Number 2 highlights the idea that the order of operations is very important; you cannot simply interchange it around. The tables introduce the different ways of writing the expressions and a typical mistake is also included every time as the last expression in the table. Number 4 gives learners the opportunity to investigate the other possibility. This can be handled as a class discussion as well.

Factorisation and Expansion

1. We can write 12 as product of 3×4 or as 6×2 etc. Write the following numbers also as products.

   (a) 15 (b) 24

   (c) 64 (d) 8

2. We can write sum expressions of the form $ab + ac$ also as a product expression $a \times (b + c)$ or $a(b + c)$. Write the following sum expressions as product expressions:

   (a) $7x + 14$

   (b) $15 + 35x$

   (c) $4x^2 + 8x$

   (d) $2x^2 + 4x + 6$

3. We can write expressions of the form $ab - ac$ also as a product expression $a \times (b - c)$ or $a(b - c)$. Write the following expressions also as product expressions:

   (a) $9x - 9$

   (b) $12x - 27$

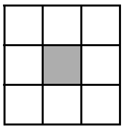
   (c) $6x^2 - 18x$

   (d) $3x^2 - 9x - 18$

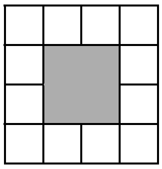
> Whenever we write a sum expression in the form of a product, we call it *factorisation*. If we therefore replace $ab + ac$ with the product $a(b + c)$ or $ab - ac$ with the product $a(b - c)$, we say that we *factorised* the initial expression $ab + ac$ or $ab - ac$.
>
> If we replace the product expression $a(b + c)$ with the sum expression $ab + ac$ or $a(b - c)$ with $ab - ac$, we call it *expansion.*

*New*town

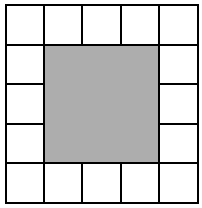
The town council of Newtown decides to beautify the town. They build square flower beds of different sizes and surround them with square tiles as shown in the sketches below.



1 by 1 bed



2 by 2 bed



3 by 3 bed

Different persons use different formulae to find the number of tiles in an n by n bed:

John uses the formula $2(n + 2) + 2n$

Jane uses the formula $4(n + 1)$

Joan uses the formula $4(n + 2) - 4$

Jim uses the formula $4n + 4$

Judy uses the formula $(n + 2)^2 - n^2$

1. Who is correct?

2. Can you *explain* each person's thinking to get the formula? Refer to the above sketches.

3. Show how each person would calculate the costs of a 12 by 12 bed if the tiles cost R1,34 each.

4. Which formula will *you* use to calculate the costs of a 12 by 12 bed? Explain your choice.

5. What is the largest bed that can be built for R150? Discuss your methods.

SCARCE METAL

Use your calculator in this activity where needed.
Share the work in your group.

1. The cost, in rands, of x g of a scarce metal is given by the formula

$$\text{Cost} = \frac{x^2 + 4x + 4}{x + 2}$$

Complete the table:

Mass (x g)	10	11	12	13	14	
Cost (R)						442

2. John says he uses this formula to calculate the cost of the same metal:

$$\text{Cost} = x + 2$$

Complete the table:

Mass (x g)	10	11	12	13	14	
Cost (R)						442

3. Calculate the cost of 115 g of the metal.

4. How much of the metal can you buy for R100?

LOOKING FOR PATTERNS IN ALGEBRAIC STRUCTURE

$$a(b+c)=ab+ac$$

Use the distributive structure above to find *equivalent forms* of the following algebraic expressions:

1. $x + x$

2. $x + y$

3. $xy + xy$

4. $xy + y$

5. $x(a + b) + x$

6. $x(a + b) + (a + b)$

7. $(c + d)\ (e + f)$

6. $(b + c)\ (b + c)$

7. $(b - c)\ (b + c)$

8. $ac + bc + ad + bd$

9. $a^2 + 2ab + b^2$

10 $a^2 + 7a + 10$

Teacher Notes: $a(b+c) = ab + ac$

1. The ability to transform expressions into equivalent forms is an important feature of algebraic work.

2. The distributive property is a powerful algebraic structure to develop an understanding of very simple equivalent forms , for example :

$b + b$
$= 1 \times b + 1 \times b$
$= b(1+1)$
$= 2b$

3. This structure can also be used to show that $x + y$ cannot be transformed

$x + y$
$= 1 \times x + 1 \times y$
$= 1(x + y)$

4 .To search for patterns in algebraic structure

Searching for the hidden pattern !
The expression $a^2 + 2ab + b^2$ does not match $ab + bc$ on first appearances, but on decomposing we find:

$a.a + ab + ab + b.b$
$= (a.a + a.b) + (ab + b.b)$
$= a(a + b) + b(a + b)$
$= (a + b)(a + b)$

$a^2 + 7a + 10$
$= a.a + 2a + 5a + 10$
$= (a.a + 2a) + (5a + 10)$
$= a(a + 2) + 5(a + 2)$
$= (a + 2)(a + 5)$

Distributive Property 1

Abeeda uses the distributive property to help her create an equivalent expression for

$$51 + 3x + 12$$

Abeeda first uses the rearrangement principle for addition and transforms

$51 + 3x + 12$ into $51 + 12 + 3x$

She now looks at the structure $x \times y + x \times z$ in which the two numbers, $x \times y$ and $x \times z$ are added.

Abeeda says that in the structure $51 + 12 + 3x$, there are also two numbers that are added, the one number is $51 + 12$ and the other number is $3x$.

If Abeeda chooses $x \times y$ to be the number $51 + 12$ and $x \times z$ to be the number $3x$,

What is Abeeda's number x?

What is Abeeda's number y?

What is Abeeda's number z?

Now write down her equivalent expression.

Vusi uses Abeeda's approach but does not rearrange the numbers. Vusi says that in the structure $51 + 3x + 12$, he sees two numbers, 51 and $3x + 12$ that are added.

Vusi now looks at the structure $x \times y + x \times z$ and chooses $x \times y$ to be the number 51 and $x \times z$ to be the number $3x + 12$.

What is Vusi's number x?

What is Vusi's number y?

What is Vusi's number z?

Now write down his equivalent expression.

Distributive Property 2

Trevor remembers the distributive property. He says that he can use this property to show that

$51 + 3x + 12$ is equivalent to $3(x + 21)$ for all the input numbers x.

He starts off by transforming $51 + 3x + 12$ into the following:

$$3 \times 17 \; + \; 3 \times x \; + \; 3 \times 4$$

Continue with this expression and use the distributive property to complete the transformation.

Teacher Notes on distribution

In these activities the learners focus explicitly on the additive structure of the distributive law $x \times y + x \times z$ *, which in this case represents two numbers that are added. The learners are encouraged to focus more complex expressions, which still has the same structure, namely two numbers that are added. When given the expression 51 +* $3x$ *+ 12 most learners will acknowledge that there are three numbers that are added. Few learners will view* $3x + 12$ *as a number, which has been decomposed into its partial sums. Thus in the case if Vusi, who sees* $3x + 12$ *as a number and using the distributive law the number can be represented as* $3 \times x + 3 \times 4 = 3(x + 4)$.

Vusi's other number is 51 and this can be decomposed into 3×21*. Note that the whilst the pupils are in fact doing "factorisation", the aim is not to focus on the process of factorisation but rather to focus on why two structures that look different in fact have the same structure. For example, in a + b and c + d + e, two numbers are added, in the structure c + d + e, if the one number is c the other number is d + e.*