Malati

Mathematics learning and teaching initiative

ALGEBRA

Module 7

The quadratic function

Grade 9

TEACHER DOCUMENT

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Overview of Module 7

In this Module learners explore and analyse the characteristics of the quadratic function $y = ax^2 + bx + c$ and the effect of the parameters a, b and c on the behaviour of the function and form of the graph of the function.

Traditionally the quadratic function is not explored in Grade 9 in South African schools. However, computing technologies available to schools now provide unprecedented access to representations of variables as quantities with changing value. Computing technologies facilitate explorations of algebra by providing learners with continual access to numerical, graphical and symbolic representations of functions.

This Module has been designed to use the (TI-82) graphing calculator. We enclose a useful manual with key procedures for the TI-82 graphing calculator.
However, if learners have access to a computer teachers may use

- the *Graphmatica* computer graphing program as an alternative. You can install Graphmatica from our software page. Here are some guidelines and activities using Graphmatica as a teaching aid (it is a PDF file).
- the enclosed *polynomial graphing java applet* as an alternative. You can install the graphing applet from our software page.
- a graphing applet on the world-wide web as alternative, for example *Function Grapher* at http://147.4.150.5/~matsrc/Graf/Graf.html

We also include some Excel activities that teachers may use as optional extra activities.

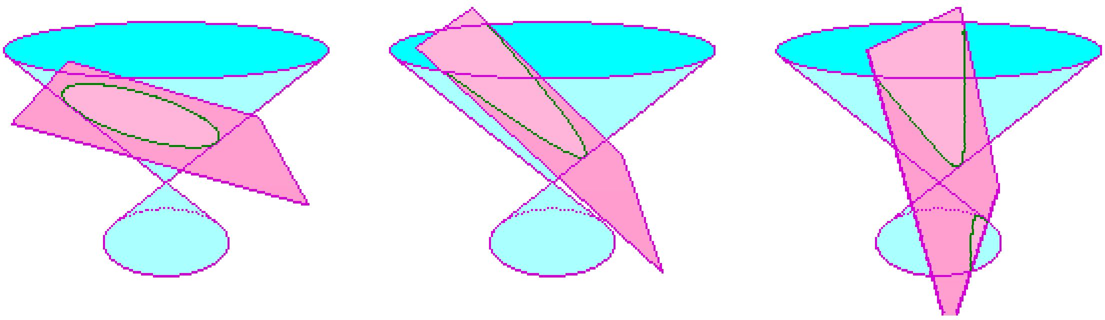
Teachers should also note that they can install Graph Paper Printer from our software page so that learners can draw some point-by-point graphs on graph paper.

What is important about the dynamic nature of the variable is not only the change in the values of variables but the effects of those changes on the value of other (dependent) variables. Families of functions like the linear, quadratic, exponential, etc. are studied because they depict different kinds of ways variables effect each other, and they are reasonable models for relationships among real world quantities.

Rather than allowing learners to roam freely in problem solving explorations, the suggested sequence of activities intends to enhance learners' development of the particular concepts.

Learners are not expected to master each concept and procedure when they *first* encounter it , but rather to *develop* their mathematical understandings continually.

Learners should be familiarised with the *shape* of a rectangular parabola. The parabola is one of the classic conic sections studied by the Greeks: If a parallel cross-section is cut through a cone, the shape of the edge of the cut is a **parabola**.



Elips *Parabola* *Hyperbola*

ACTIVITY 1: Classifying functions by their graphs

In Module 5 you classified the graphs of the following algebraic expressions as a "straight line" or "another curve". Now use the graphing calculator and classify them as "straight line", "parabola" or "another curve". (Remember that you are only looking at a part of the graph). Rewrite the algebraic expression in the correct column in the table.

(a) $y = 10 - x^2$	(b) $y = 5x - 15$	(c) $y = x$	(d) $y = x^3$
(e) $y = -4x^2$	(f) $y = \frac{x^2}{5}$	(g) $y = \frac{x}{5}$	(h) $y = -6x$
(i) $y = (x + 6)^2$	(j) $y = 15$	(k) $y = x^2 + 18$	(l) $y = 2x - x^2$
(m) $y = 50 - 5x$	(n) $y = 10x$	(o) $y = x^2$	(p) $y = x^3 + 9x$
(q) $y = \frac{5}{x}$	(r) $y = \frac{5}{x^2}$	(s) $y = x + 8$	(t) $y = 10 - x$
(u) $y = -3x^2 + 5$	(v) $y = -4x - 8$	(w) $y = x^4 - 5$	(x) $y = x^2 - 5$

Makes a straight line	**Makes a parabola**	**Makes another curve**

Look at the expressions that you have put in the column "makes a parabola". The functions that they represent are also called **quadratic functions**. Their graphs are called **parabolas**.

- What do the quadratic function expressions have in common?
- Write down three *other* expressions that make *parabolas*.
  Check your answers with the graphing calculator.
- Is the graph of $y = 3x^2 + 2x + 1$ a line, a parabola or some other shape? Explain your thinking!

The formula or algebraic rule for a **quadratic function** is often written as

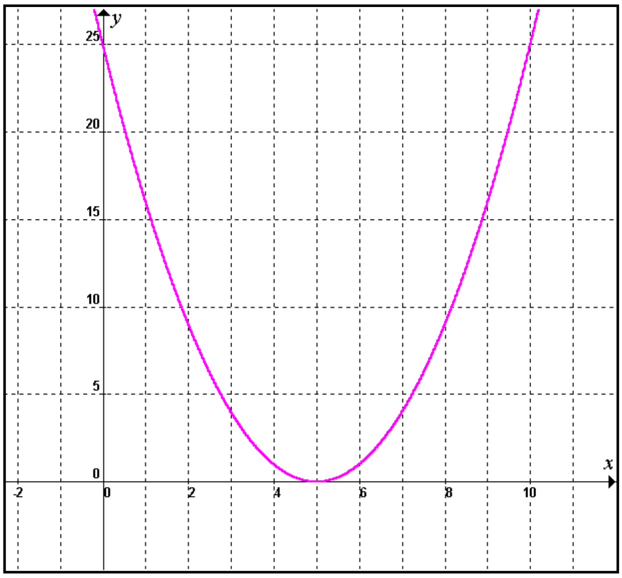
$$y = ax^2 + bx + c \qquad \text{or} \qquad y = a(x - p)^2 + q \qquad (a \neq 0)$$

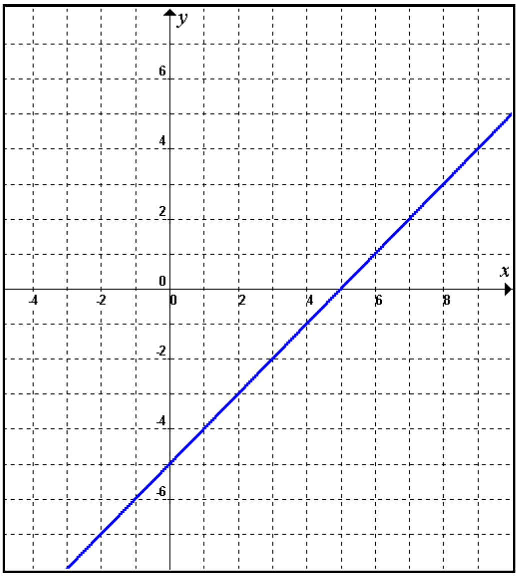
$$f(x) = ax^2 + bx + c \qquad\qquad f(x) = a(x - p)^2 + q$$

ACTIVITY 2: f and g

1. Here are two functions: $f(x) = (x-5)^2$ and $g(x) = x-5$
   How are the algebraic expressions of f and g different?
   Express f and g as flowdiagrams.

2. Here are pieces of the graphs of the two functions.
   Which one is f and which one is g? How do you know? How are the graphs different?





3. Below are pieces of the tables of the two functions. Complete it.
   How are the tables different?
   Don't just look at the numbers, but at how the values of f and g change as the values of x change.

$f(x)=(x-5)^2$

x	$f(x)$
– 5	
– 4	
– 3	
– 2	
– 1	
0	
5	
6	
7	
8	
9	
10	

$g(x)=x-5$

x	$g(x)$
– 5	
– 4	
– 3	
– 2	
– 1	
0	
5	
6	
7	
8	
9	
10	

ACTIVITY 3: *a*

How does different values of *a* influence the graph of $y = ax^2$?

1. Enter the following expressions into the graphing calculator and let the calculator draw the graphs:

$y_1 = x^2$

$y_2 = 4x^2$

$y_3 = \frac{1}{2}x^2$

$y_4 = 10x^2$

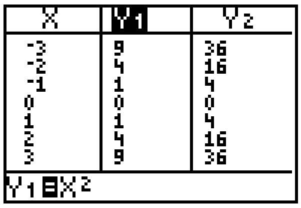
$y_5 = -4x^2$

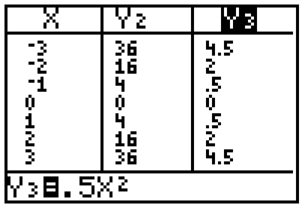
$y_6 = 25x^2$

$y_7 = -0{,}5x^2$

2. Describe the effect of a change in *a* on the graph of $y = ax^2$.

3. Make a table of y_1, y_2, y_3, y_4, . . . for values of x from -3 to 3, or use the TABLE facility of the graphing calculator.





Describe patterns in the values of y_1.

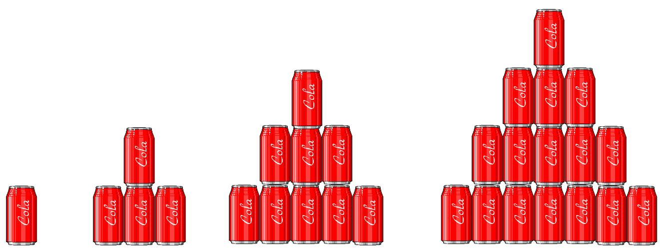
Describe patterns in the values of y_2.

⋮

Describe patterns *between* the values of y_1 and y_2 , y_1 and y_3 etc.

How is this different from the behaviour of a linear function?

4. Ann is a packer at the supermarket. She likes to pack the cans like this in the form of a pyramid if she can:



(a) Complete this table showing how many cans she needs for different pyramids:

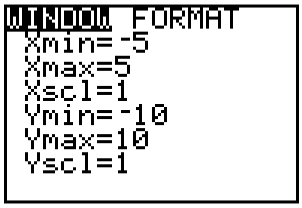
# rows	1	2	3	4	5	6	10	20	
# cans	1	4	9	16					625

(b) Ann has completed a pyramid with 27 rows. How many extra cans does she need to complete another row?

ACTIVITY 4: *c*

How does different values of c influence the graph of $g(x) = ax^2 + c$?

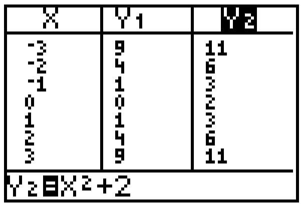
1. Enter the following expressions into the graphing calculator using the WINDOW as shown, and let the calculator graph the functions on the same screen.



$g(x) = x^2$
$g(x) = x^2 + 2$
$g(x) = x^2 + 4$
$g(x) = x^2 + 6$
$g(x) = x^2 – 2$
$g(x) = x^2 – 4$
$g(x) = x^2 – 6$

2. Describe the effect of a change in c on the graph of $g(x) = x^2 + c$.

3. Make a table of $g(x)$ for different values of c for values of x from –3 to 3, or use the TABLE facility of the graphing calculator.



Describe patterns!

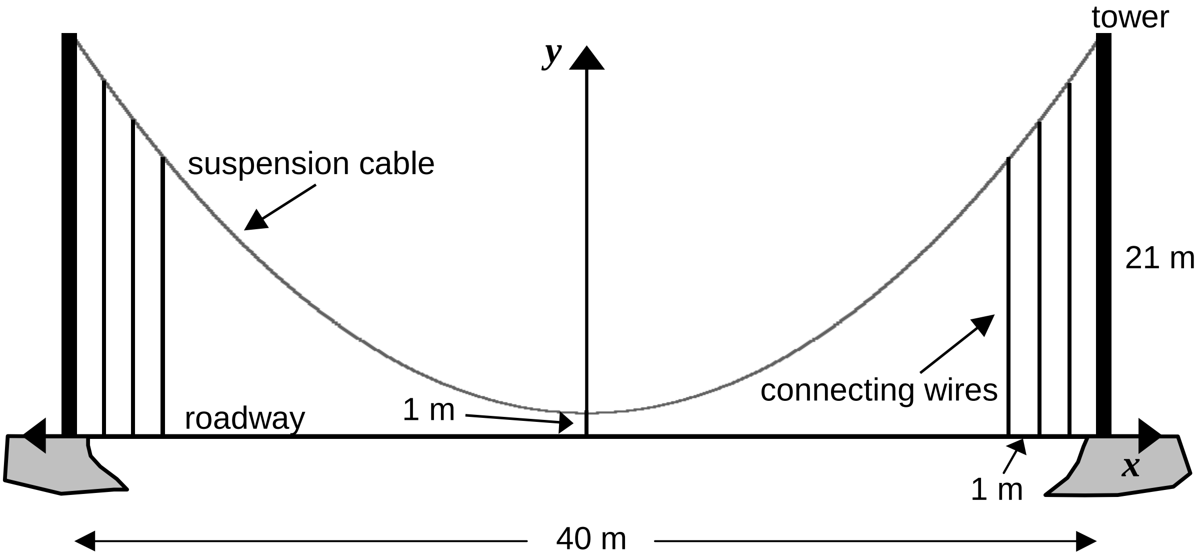
4. Now enter the following expressions into the graphing calculator, and let the calculator draw the graphs on the same screen. Describe all patterns.

$h(x) = 2x^2$	$p(x) = 3x^2$
$h(x) = 2x^2 + 2$	$p(x) = 3x^2 + 2$
$h(x) = 2x^2 + 4$	$p(x) = 3x^2 + 4$
$h(x) = 2x^2 + 6$	$p(x) = 3x^2 + 6$
$h(x) = 2x^2 – 2$	$p(x) = 3x^2 – 2$
$h(x) = 2x^2 – 4$	$p(x) = 3x^2 – 4$
$h(x) = 2x^2 – 6$	$p(x) = 3x^2 – 6$

ACTIVITY 5: Building bridges

The famous Golden Gate hanging bridge in San Francisco, USA

Engineers must build a hanging bridge over a river. The roadway of the bridge will be hung from a suspension cable with connecting wires, 1 m apart. At its lowest point, the connecting wire is 1 m long. The suspension cable is supported by two towers at the ends, which are each 21 m high and are 40 m apart.



It is very important that the lengths of the connecting wires should be *exact*, otherwise the bridge is unsafe!
This is how engineers do it: They imagine a system of axis as shown above, then the form of the suspension cable can be described by the formula $y = 0{,}05x^2 + 1$, then they use this formula as a *model* to *calculate* the lengths of all the connecting wires *beforehand*.

Make a table giving the length of each of the connecting wires.
Describe patterns in the table.

Can you, after calculating a few values, use the *table* as a *model* to easily calculate the other lengths?

ACTIVITY 6: *a* and *c*

1. In each case write in the table below

   (a) where the graph of the functions cuts the y-axis (i.e. the ***y-intercept***)
   (check your answers with the graphing calculator)

   (b) the direction of the graph (up or down)

Function	**y-intercept**	**Direction (up or down)**
$y = x^2 - 9$		
$y = -2x^2 + 8$		
$y = 5 + x^2$		
$y = -2 - 3x^2$		
$y = \frac{x^2}{4}$		
$y = x^2 - 2x - 3$		
$y = 2 + 5x - 4x^2$		

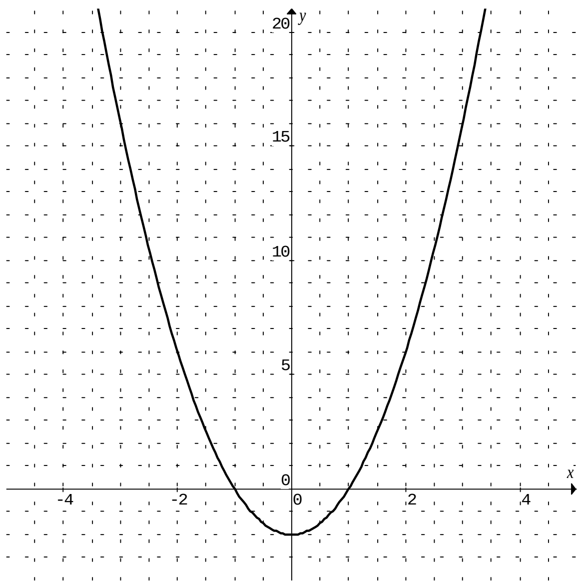
2. Write a rule about the about the y-intercept of a quadratic function.

3. Write a rule about the direction of the graph of a quadratic function.

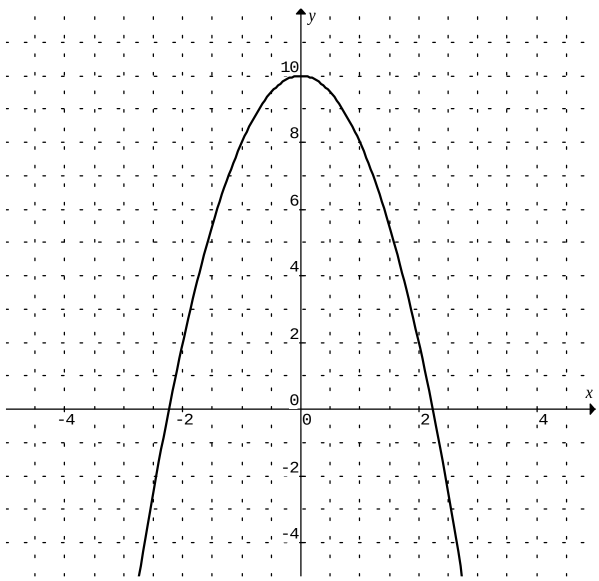
ACTIVITY 7:   Matching graphs and expressions

For each of the four functions below, find the corresponding graph which represents a section of the function. Explain your thinking in each case.

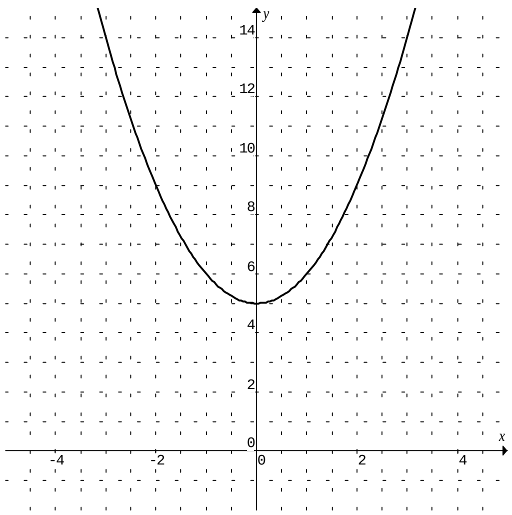
Function	Graph	Explanation
$f(x) = -2x^2 + 10$		
$f(x) = x^2 + 5$		
$f(x) = 5x^2$		
$f(x) = 2x^2 - 2$		



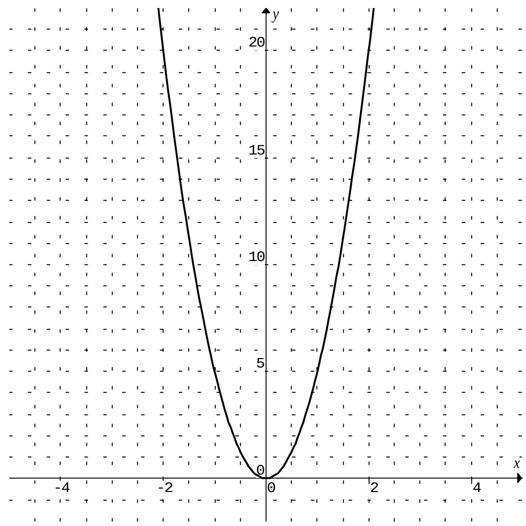
graph 1



graph 2



graph 3



graph 4

ACTIVITY 8:

1. Three functions are given by their algebraic formula:

   $f(x) = x^2 - 9$ $\qquad g(x) = 9x^2$ $\qquad h(x) = -x^2 + 9$

   (a) For each function, where will the graph cross the y-axis?

   (b) In which direction will each parabola point? (up or down)

2. Think about the parabolas represented by $y = 3x^2$ and $y = 5x^2$.

   How do they look different?

3. Think about the parabolas represented by $y = x^2 + 10$ and $y = x^2 - 10$.

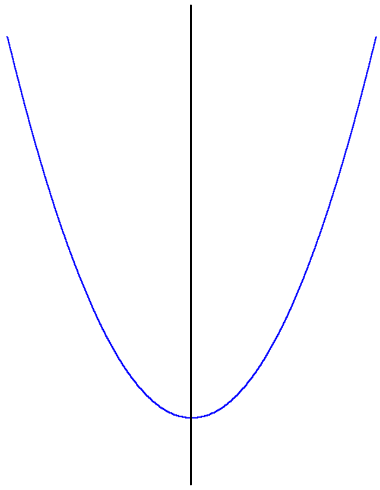
   How do they look different?

4. Think about the parabolas represented by $y = 9x^2$ and $y = -9x^2$.

   How do they look different?

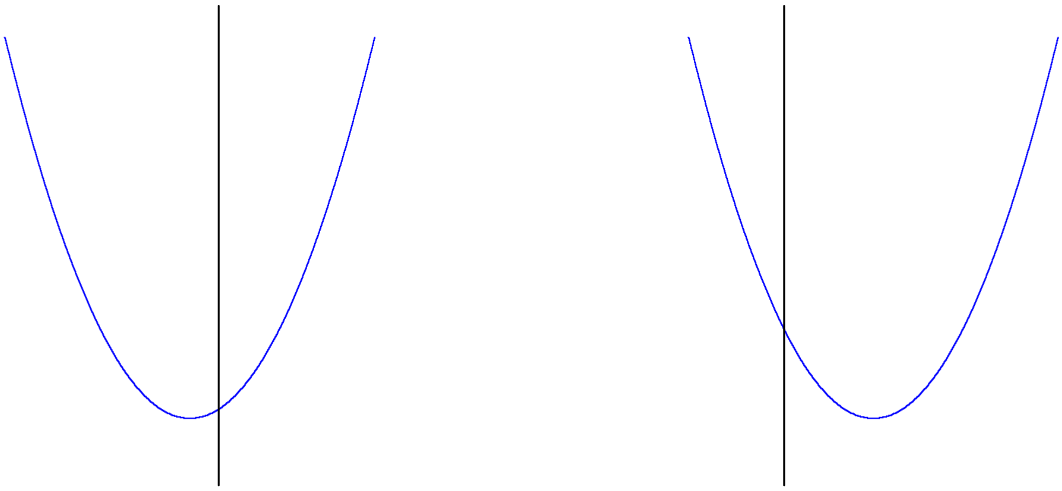
ACTIVITY 9:

1.



What do you think will happen if you fold the parabola on the vertical line?

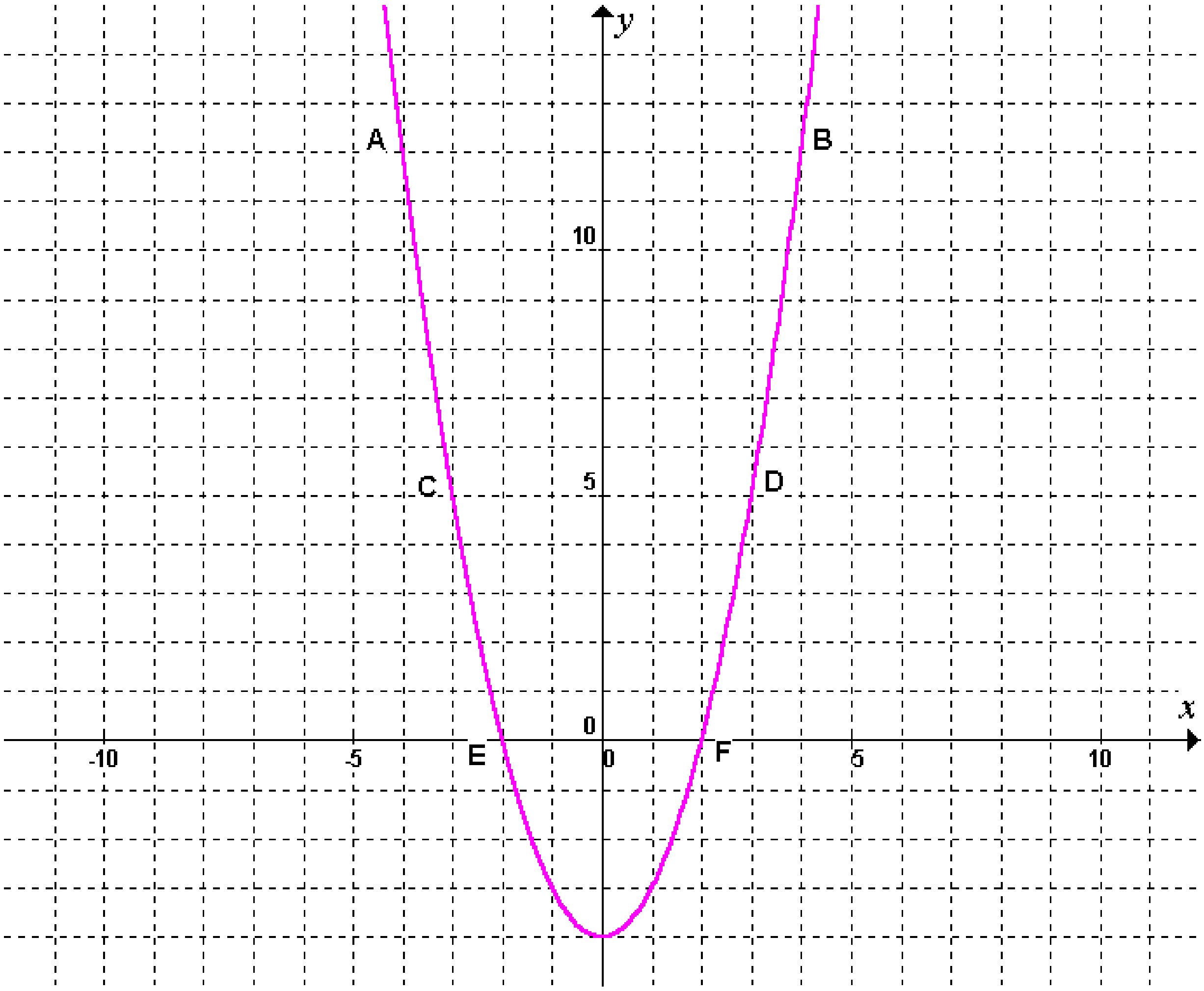
2.



(a) What will happen if you fold on the above vertical lines?

(b) What is the difference between the vertical lines we drew through the graphs in questions 1 and 2?

ACTIVITY 10:



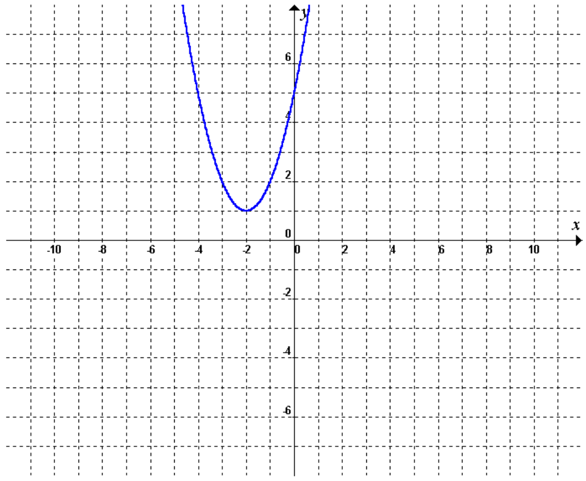
1. What is the distance from A to the y-axis?
2. What is the distance from B to the y-axis?
3. What is the distance from C to the y-axis?
4. What is the distance from D to the y-axis?
5. What is the distance from E to the y-axis?
6. What is the distance from F to the y-axis?

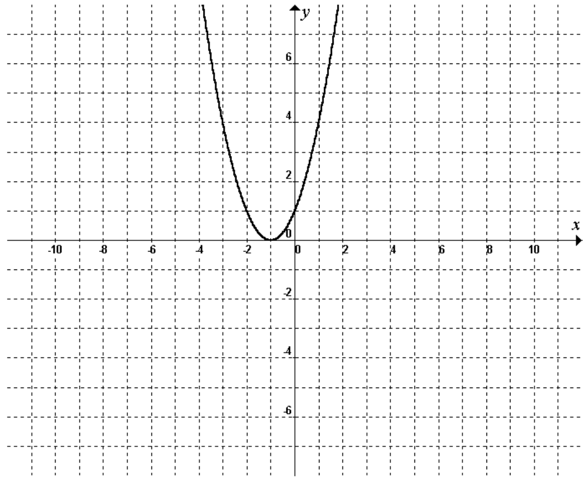
What do you observe?
Check your observation with other points on the parabola.

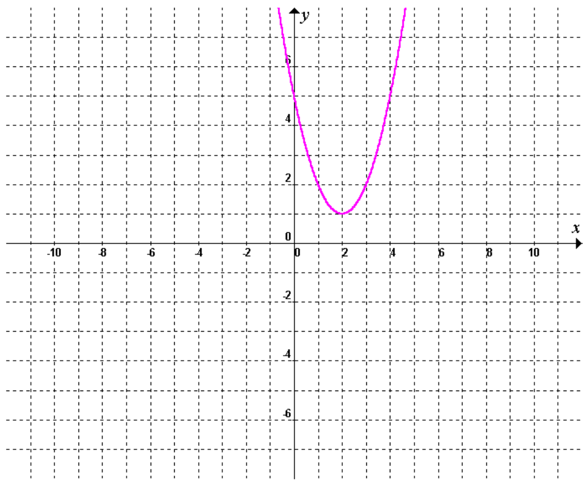
A line that divides a shape into two mirror images is called a ***line of symmetry*** of the figure.

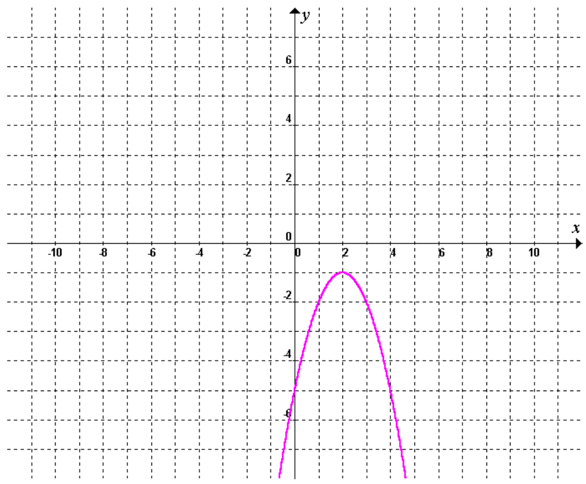
ACTIVITY 11:

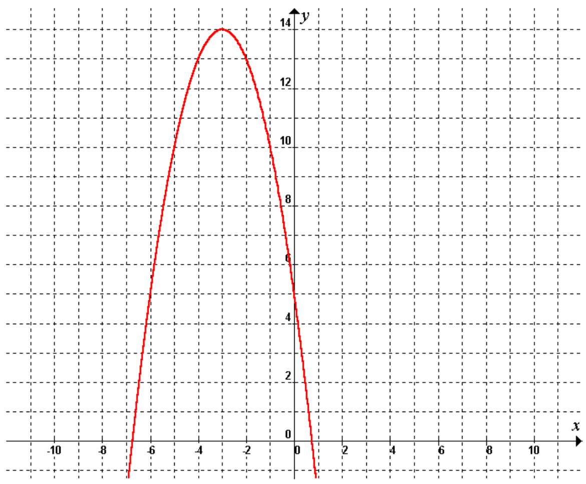
Write down the line of symmetry of each of these functions:

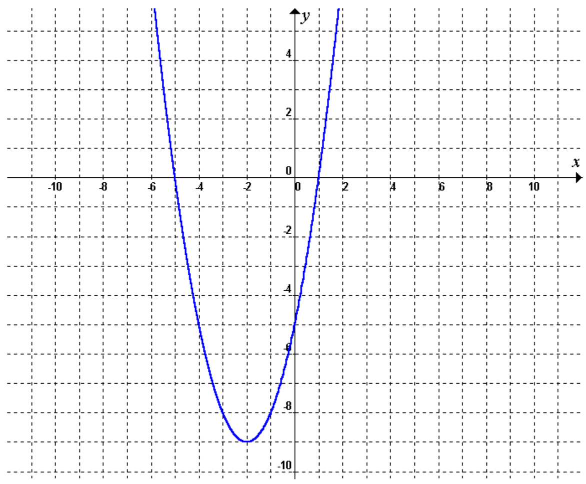












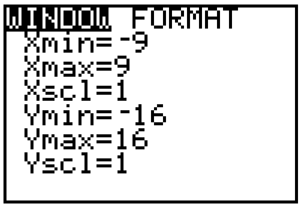
Challenge: Draw these graphs on your graphing calculator!

ACTIVITY 12: *b*

How does different values of *b* influence the graph of $y = ax^2 + bx + c$?

1. Enter the following functions into the graphing calculator using the WINDOW as shown and let the calculator graph the functions on the same screen.

   $y = x^2 + 2x + 1$
   $y = x^2 + 4x + 1$
   $y = x^2 + 6x + 1$
   $y = x^2 + 8x + 1$+++



   Describe what you observe.

2. Enter the following functions into the graphing calculator and let the calculator graph the functions on the same screen.

   $y = x^2 - 2x + 1$
   $y = x^2 - 4x + 1$
   $y = x^2 - 6x + 1$
   $y = x^2 - 8x + 1$

   Describe what you observe.

3. Complete the following table:

Function	Line of symmetry
$y = x^2 + 2x + 1$	
$y = x^2 + 4x + 1$	
$y = x^2 + 6x + 1$	
$y = x^2 + 8x + 1$	
$y = x^2 - 2x + 1$	
$y = x^2 - 4x + 1$	
$y = x^2 - 6x + 1$	
$y = x^2 - 8x + 1$	

   (a) What do you observe?

   (b) Make a conjecture about how you can deduce the line of symmetry of a quadratic function from its formula.

   (c) Test your conjecture by *predicting* the line of symmetry of each of the following functions, and then *checking* your prediction on your graphing calculator:

   $y = x^2 + 3x + 1$
   $y = x^2 + 5x + 2$
   $y = 2x^2 - 4x + 3$
   $y = 3x^2 + 6x - 4$

   (d) Refine your conjecture if necessary.

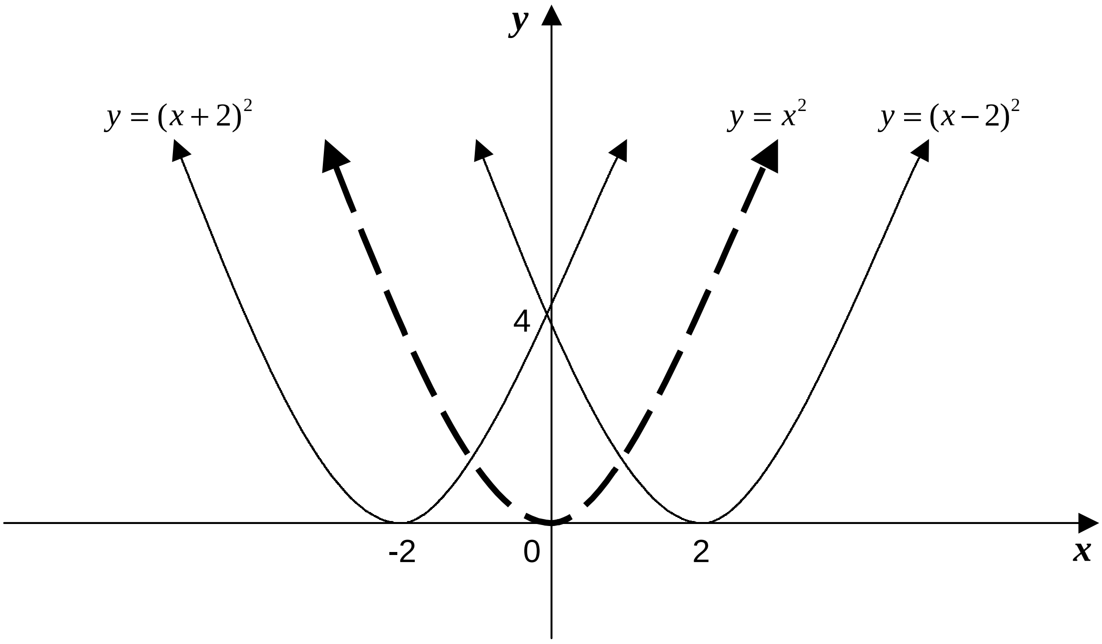
ACTIVITY 13:

Complete the table without the use of the graphing calculator.
You can of course *afterwards* check with the calculator!

Function	Line of Symmetry	y-intercept	Direction
$y = 2x^2 - 4x - 6$			
$y = -2x^2 + 4x + 6$			
$y = x^2 - 4x - 5$			
$y = x^2 + 3x - 4$			
$y = -5x^2 + 10$			
$y = 3x^2 - 6x$			
$y = 3x^2 - 6x + 1$			
$y = 3x^2 - 6x + 2$			
$y = 3x^2 - 6x + 5$			
$y = 3x^2 - 6x - 4$			

ACTIVITY 14: $y = ax^2 + q$ and $y = a(x - p)^2 + q$

Consider the following graphs:



1. How are these graphs different and how are they the same?

2. Write down the line of symmetry of each of these graphs.

3. Make a conjecture about the graph of the function $y = (x - p)^2$.

4. Use the graphing calculator to test your conjecture by drawing graphs of:
   $y = (x - 3)^2$
   $y = (x + 3)^2$
   $y = (x - 1)^2$
   $y = (x + 1)^2$

5. Refine your conjecture if necessary.

6. Use the graphing calculator to help you make and/or test a conjecture about
   - $y = 2x^2$ and $y = 2(x - 3)^2$
   - $y = -2x^2$ and $y = -2(x - 3)^2$
   - $y = ax^2$ and $y = a(x - p)^2$

7. Use the graphing calculator to help you make and/or test a conjecture about
   - $y = 2x^2 + 1$ and $y = 2(x - 3)^2 + 1$
   - $y = -2x^2 + 1$ and $y = -2(x - 3)^2 + 1$
   - $y = ax^2 + q$ and $y = a(x - p)^2 + q$

ACTIVITY 15:

Graph Art

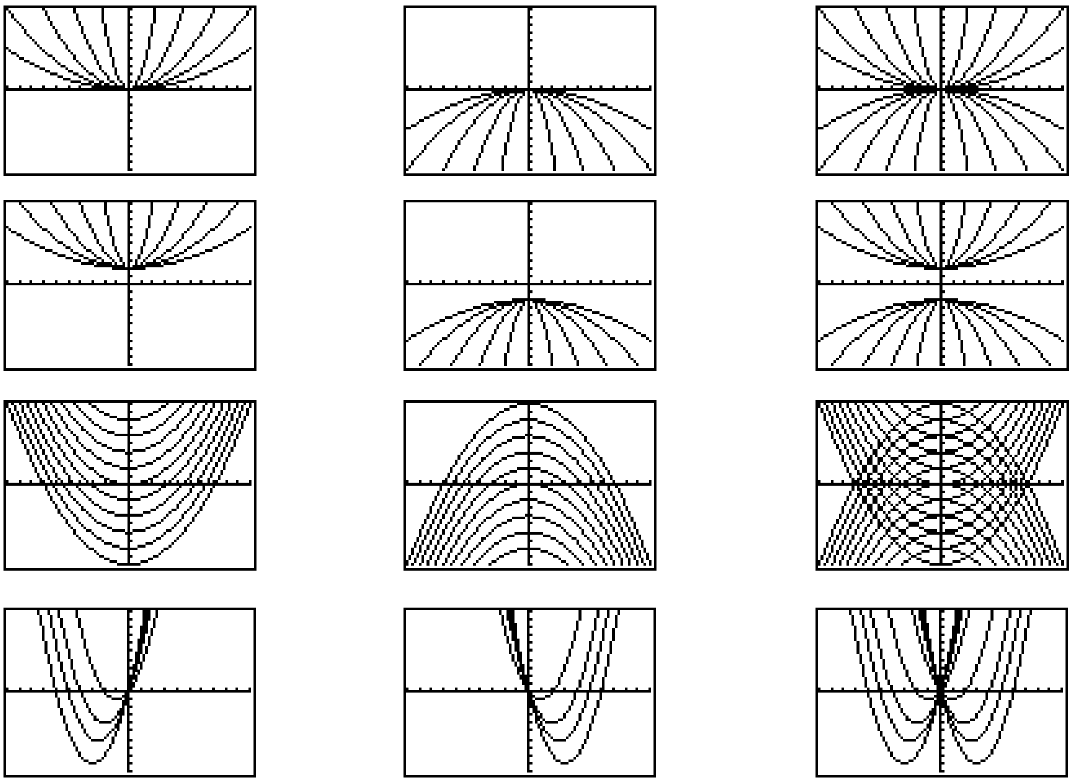
Make each of the designs below in your calculator window.

In each case *describe* the pattern in *words* and explain to your group exactly how you designed the pattern (write down the functions you used).

Note:
To begin, set the **WINDOW** on your TI-82 graphing calculator as shown.

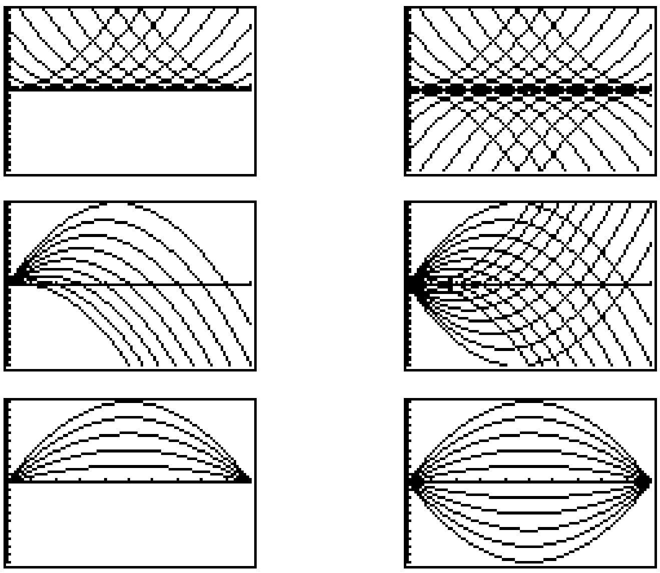
```
WINDOW FORMAT
 Xmin=-10
 Xmax=10
 Xscl=1
 Ymin=-10
 Ymax=10
 Yscl=1
```

Also clear all functions in **Y=**



Use this WINDOW for the last six:

```
WINDOW FORMAT
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=-20
 Ymax=20
 Yscl=1
```



Teacher Note: Activity 15

Teachers could use Activity 15 as a project or as further development and enrichment.

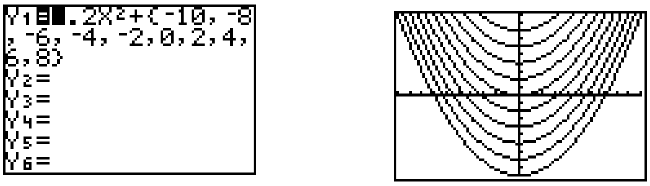
See the teacher notes for Activity 19: Graph Art in Module 5.

Learners will have to know much, or preferably *learn much* in order to reproduce the given artwork. They all involve *families* of functions where one of the parameters varies, as well as symmetry in the X- or Y-axis.

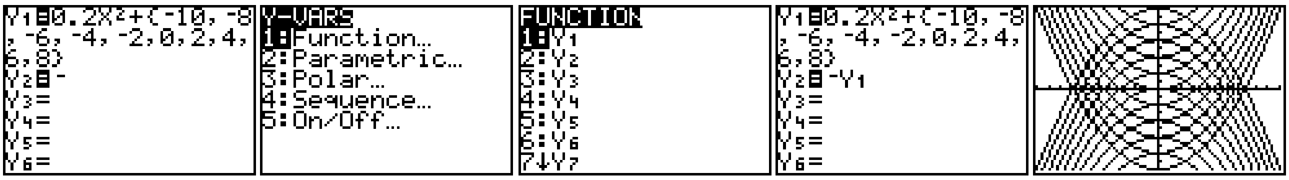
Entering all the functions can be hard work. However, the T--82 calculator enables us to work quite easily with such families of functions, as well as with symmetry.

A *family of functions* is a set of functions that are identical except for the value of a particular parameter. For example, f(x) = x^2 – 1, f(x) = x^2 + 2, and f(x) = x^2 + 3 are all member functions of a family, written as f(x) = x^2 + a. With a TI-82, we can easily work with a family of functions by following the procedures below. We will take as example the function family of the form $Y_1 = 0{,}2x^2 + a$, where a = {-10,-8,-6,-4,-2,0,2,4,6,8}.

1. Press the **Y=** key to define a function
2. Enter the rule for the function, by entering the following sequence of symbols: **0.2**X**^2 + {-10,-8,-6,-4,-2,0,2,4,6,8}**
3. Press GRAPH



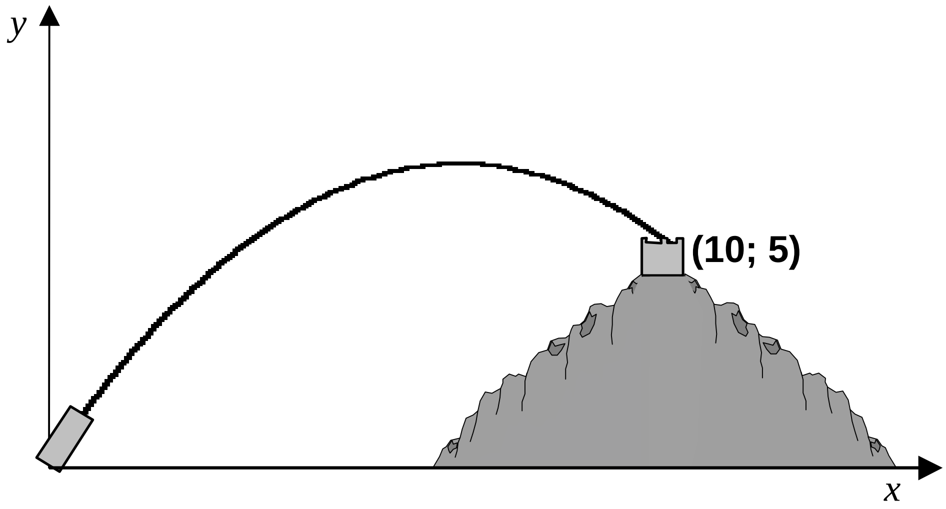
If functions Y_1 and Y_2 are symmetrical in the X-axis, then $Y_2 = – Y_1$. On the T--82 calculator we can easily define Y_2 in terms of Y_1: With the curser in Y_2, press 2nd Y-VARS , choose FUNCTION and then Y_1 . We can then graph Y_1 and Y_2 simultaneously:



ACTIVITY 16: Catapult

Imagine in the sketch below that you launch a stone from a catapult located at the origin (0; 0) and you want the stone to hit the enemy in their castle on the hill at the point (10; 5).

Assume that the trajectory (path) of the stone is described by a parabola.



1. If the path of the stone is described by the following formulae, will the stone hit or miss the target? If it will miss, is it too high or too low?

   (a) $y = -x^2 + 10x$

   (b) $y = -x^2 + 11x$

   Find the value of *b* for which $y = -x^2 + bx$ will hit the target.

   Describe your strategy.

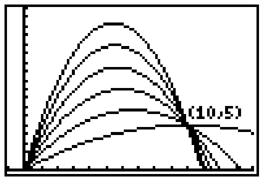
2. If the path of the stone is described by the following formulae, will the stone hit or miss the target? If it will miss, is it too high or too low?

   (a) $y = -0{,}05x^2 + 2x$

   (b) $y = -0{,}05x^2 + 6x$

   Find the value of *b* for which $y = -0{,}05x^2 + bx$ will hit the target.

   Describe your strategy.



3. Find the value of *a* for which $y = ax^2 + 5x$ will hit the target.

4. Find another formula that will hit the target. *Prove* that it will hit the target!

Teacher Note: Activity 16

Teachers could use Activity 16 as a project or as further development and enrichment.

Teachers may also be interested in Canon, the Excel equivalent of Catapult.