The Physics of Non-Hermitian Operators

Accepted Abstracts and Posters

Ahmed, Zafar	**$\mathcal{PT}$-symmetry in conventional quantum physics** Investigations during the last few years show that complex $\mathcal{PT}$-symmetric or pseudo-Hermitian Hamiltonians possess real discrete spectrum and several other features akin to a Hermitian Hamiltonian. These developments not only show us new directions but also demand a re-visit to the conventional physics wherein $\mathcal{PT}$-symmetry could have been invoked as it is being discussed nowadays. After an introduction of $\mathcal{PT}$-symmetric quantum mechanics some instances of conventional quantum physics possessing some essence of $\mathcal{PT}$-symmetry shall be presented.
Bender, Carl	**Some new results on $\mathcal{PT}$-symmetric classical and quantum mechanics** This talk will report on two calculations completed this fall. The first concerns the coordinate-space trajectories in some $\mathcal{PT}$-symmetric classical-mechanical theories. Depending on the initial conditions, it is possible to find arbitrarily long periodic $\mathcal{PT}$-symmetric classical trajectories. The longer trajectories originate from smaller regions of initial conditions. There is an interesting resemblance to the so-called period-doubling route to chaos. The second concerns the perturbative construction of the C operator for a $\mathcal{PT}$-symmetric square well. The result has some notable similarities and differences from that for the $\mathcal{PT}$-symmetric cubic oscillator.
Berkdemir, Ayşe; Berkdemir, Cüneyt and Sever, Ramazan	**Woods-Saxon square approximation to the solution of Dirac equation for the Woods-Saxon potential with spin symmetry** The Dirac equation for the Woods-Saxon potential with spin symmetry is solved analytically by the method of reducing of hypergeometric functions. A new effective potential which is the form of Woods-Saxon square is introduced by exchanging with the spin-orbit term of the Dirac equation to evaluate the spectra of energy eigenvalues for the special atomic mass numbers. The full solution of the problem is analyzed by the Nikiforov-Uvarov method and the spectrum for the first four levels is given for the several spin-orbit quantum numbers.
Berkdemir, Cüneyt	**Bound State Solutions of the Dirac Equation for the pseudospin-symmetry Morse potential** The energy spectra of the bound states of the Morse potential with pseudospin symmetry is obtained by applying the Nikiforov-Uvarov (NU) method as well as Pekeris approximation. Certain bound-states of the Dirac particle moving in the attractive scaler $S(\vec{r})$ potential and repulsive vector $V(\vec{r})$ potential are calculated numerically, under the conditions of exact pseudospin symmetry ($\Sigma = V + S =$constant). The spectrum for the first four levels is given for the several pseudo-orbital and pseudospin quantum numbers.
Berry, Michael	**The optical singularities of bianisotropic crystals** The electric and magnetic polarization states for plane waves in arbitrary linear crystals, where all four fields are coupled, can be characterized by theirtypical singularities in direction space:degeneracies, points where two refractive index eigenvalues coincide;C points, where the electric or magnetic field is circularly polarized;and L lines, where either field is linearly polarized. A 4×4 matrix formalism enables extensive numerical and visual exploration of the singularities in the general case (which involves 65 crystal parameters), incorporating bianisotropy, natural and Faraday optical activity, and absorption, as well as special cases where one or more effect is absent. For crystals whose anisotropy is weak but which are otherwise general, an unusual pertubation theory leads to a powerful 2×2 formalism capturing all the essential singularity phenomena, including the principal feature of the general case, namely the separation between theelectric and magnetic singularities.

Caliceti, Emanuela	**Perturbation theory for $\mathcal{PT}$-symmetric Hamiltonians** Perturbation theory for selfadjoint operators is well established and has been successfully employed in order to obtain interesting results on the spectrum of $\mathcal{PT}$-symmetric Hamiltonians. When the unperturbed operator is non selfadjoint $\mathcal{PT}$-symmetric, perturbation theory is by far more problematic and less known. This is mainly due to the fact that $\mathcal{PT}$-symmetric Hamiltonians admit Jordan structures, so that they are not diagonalizable in the standard sense. Such Jordan blocks play an important role as ingredients of perturbation theory in the sense of Kato, because they determine a difference between the geometric and the algebraic multiplicity of the eigenvalues. When the two multiplicities coincide and the spectrum is real the Hamiltonian is similar to a selfadjoint one; moreover perturbation theory becomes the standard one when the eigenvalues are simple, i.e. both multiplicities are equal to one. Generically for one-dimensional Schroedinger operators only the geometric multiplicity is guaranteed to be one. As a preliminary approach to the problem we study the perturbation theory for $\mathcal{PT}$-$\mathcal{PT}$-symmetric matrices.
Dorey, Patrick	**On the reality (and unreality) of $\mathcal{PT}$-symmetric quantum mechanics** In this talk I shall discuss how techniques borrowed from the theory of integrable quantum field theory can be used to prove spectral reality for some $\mathcal{PT}$-symmetric systems, and to analyse how this reality is lost for some other systems.
Feinberg, Joshua	**"Single Ring Theorem" and the Disk-Annulus Phase Transition** Some time ago, an analytic method was developed to study in the large N limit non-hermitean random matrices that are drawn from a large class of circularly symmetric non-Gaussian probability distributions, thus extending the existing Gaussian non-hermitean literature. One obtains an explicit algebraic equation for the integrated density of eigenvalues from which the Green's function and averaged density of eigenvalues could be calculated in a simple manner. Thus, that formalism may be thought of as the non-hermitean analog of the method due to Brézin, Itzykson, Parisi and Zuber for analyzing hermitean non-Gaussian random matrices. A somewhat surprising result is the so called "Single Ring" theorem, namely, that the domain of the eigenvalue distribution in the complex plane is either a disk or an annulus. In this paper we extend previous results and provide simple new explicit expressions for the radii of the eigenvalue distiobution and for the value of the eigenvalue density at the edges of the eigenvalue distribution of the non-hermitean matrix in terms of moments of the eigenvalue distribution of the associated hermitean matrix. We then present several numerical verifications of the previously obtained analytic results for the quartic ensemble and its phase transition from a disk shaped eigenvalue distribution to an annular distribution. Finally, we demonstrate numerically the "Single Ring" theorem for the sextic potential, namely, the potential of lowest degree for which the "Single Ring" theorem has non-trivial consequences.
Feinberg, Joshua	**Quantized Normal Matrices: Some Exact Results and Collective Field Formulation** We formulate and study a class of U(N)-invariant quantum mechanical models of large normal matrices with arbitrary rotation-invariant matrix potentials. We concentrate on the U(N) singlet sector of these models. In the particular case of quadratic matrix potential, the singlet sector can be mapped by a similarity transformation onto the two-dimensional Calogero-Marchioro-Sutherland model at specific couplings. For this quadratic case we were able to solve the $N-$body Schrödinger equation and obtain infinite sets of singlet eigenstates of the matrix model with given total angular momentum. Our main object in this paper is to study the singlet sector in the collective field formalism, in the large-N limit. We obtain in this framework the ground state eigenvalue distribution and ground state energy for an arbitrary potential, and outline briefly the way to compute bona-fide quantum phase transitions in this class of models. As explicit examples, we analyze the models with quadratic and quartic potentials. In the quartic case, we also touch upon the disk-annulus quantum phase transition. In order to make our presentation self-contained, we also discuss, in a manner which is somewhat complementary to standard expositions, the theory of point canonical transformations in quantum mechanics for systems whose configuration space is endowed with non-euclidean metric, which is the basis for constructing the collective field theory.

Geyer, Hendrik	**Choice of physical operators and uniqueness of the metric** The question of uniqueness of the metric relevant to a quasi-hermitian quantum mechanical Hamiltonian has recently been discussed from various points of view. Within the simple setting of a quasi-hermitian quadratic boson Hamiltonian we explore and compare the constructiuon of a metric dictated by different choices of physical operators to combine with the Hamiltonian to form an irreducible set.
Graefe, Eva-Maria	**Non-Hermitian two-level systems: Crossing scenarios and Landau-Zener-type transitions** The behavior of systems described by (small) symmetric non-Hermitian Hamiltonian matrices is analyzed. Special attention is payed to the unfamiliar non-Hermitian crossing scenarios. The possibility of coherent control of the system dynamics is discussed based on a generalized Landau-Zener scenario with a decaying level. Counterintuitively, the probability of a diabatic passage is not affected by the nonhermiticity, i.e. by the decay. In addition, a generalized nonlinear non-Hermitian two-state system is discussed which models the dynamics of a decaying Bose-Einstein condensate in a double-well potential in a mean-field approach.
Günther, Uwe; Kirillov, Oleg and Stefani, Frank	**Bundle stratification of $\mathcal{PT}-$symmetric 4×4 matrix systems and fourth-order spectral branch points of MHD α^2-dynamos** The spectra of $\mathcal{PT}-$symmetric quantum systems are, in general, highly complicated complex varieties over the parameter spaces of these systems. Parameter space regions of exact $\mathcal{PT}-$symmetry with purely real eigenvalues adjoin other regions with broken $\mathcal{PT}-$symmetry and pairwise complex conjugate eigenvalues. The boundaries between different regions correspond to (exceptional) hypersurfaces in the parameter space and have certain system defined codimensions. Hypersurfaces (strata) of lower codimension may intersect at sub-hypersurfaces (sub-strata) of higher codimension forming in this way a hierarchical structure of dimensionally nested hypersurfaces of increasing codimension (and increasing “exceptionality”), i.e. they form a so called stratified manifold. In case of matrix systems, the most general parameter space can be identified with the matrix space itself. Endowing this space with a natural fiber bundle structure, regarding matrix orbits (equivalence classes) over a given Jordan structure as fibers, the hierarchical hypersurface structure of the spectral variety can be studied with the help of nested Jordan structures. As result, one arrives at a so called bundle stratification. We perform such a bundle stratification explicitly for a $\mathcal{PT}-$symmetric 4×4 matrix systems and use the obtained results to comparatively analyze the spectral structure of the spherically symmetric α^2-dynamo of magnetohydodynamics in the vicinity of a fourth-order branch point.
Heiss, Dieter	**Reflections upon the thermodynamic limit of the Lipkin model** The Lipkin model is a toy model describing an interacting many Fermion system giving rise to a phase transition implying symmetry breaking. While it is well understood for finite particle numbers, the limit of infinite particle number appears rather elusive. Owing to its popularity and to its general aspects, such limit is of interest. There are indirect indications (such as the study of exceptional points) that the Hamilton operator, while hermitian for finite particle number, may not be selfadjoint in the thermodynamic limit.
Jones, Hugh	**Equivalent Hamiltonians for PT-symmetric versions of dual 2-dimensional field theories** First we discuss the systematics of the commutation relations that appear in the perturbative calculation of the operator $Q(\equiv -\ln \eta)$ when the Hamiltonian $H \equiv H_0 + gH_1$ is such that H_0 is Hermitian and H_1 anti-Hermitian. We then apply these results to find exact expressions for the Q operator for PT-symmetric, non-Hermitian versions of the dual Sine-Gordon and massive Thirring models. The equivalent Hermitian Hamiltonians, constructed via Q, turn out to be just the conventional models with appropriately defined masses.

Kirillov, Oleg and Günther, Uwe	**Perturbation theory for non-self-adjoint operator matrices and its application to the MHD α^2-dynamo** We consider singular boundary value problems for linear non-self-adjoint m-th order $N \times N$ matrix differential operators on the interval $(0,1] \ni x$. The coefficient matrices in the differential expressions and the matrix boundary conditions are assumed to depend on the spectral parameter λ as well as on a vector-valued parameter distribution $\mathbf{p}(x)$. For simplicity, we restrict our attention to setups with a (boundary) singular point ($x = 0$) of regular (Fuchsian) type. Based on explicitly derived bi-orthogonal solution bases, we study perturbations of simple and multiple eigenvalues under small variations of the parameter distribution $\mathbf{p}(x)$. Special attention is paid to perturbations of semi-simple multiple eigenvalues (diabolical points; characterized by coinciding geometric and algebraic multiplicity and corresponding diagonal spectral decomposition of the operator) as well as to perturbations of non-derogatory eigenvalues (branch points, exceptional points; with non-trivial Jordan structures in the spectral decomposition) with one eigenvector (geometric multiplicity one) and several associated vectors forming a Keldysh chain of length equal to the algebraic multiplicity of the eigenvalue. Explicit formulae describing the bifurcation of the eigenvalues are derived. As application, the general technique is utilized for the investigation of the spectral properties of spherically symmetric MHD α^2-dynamos. Specifically, we provide an analytical description for the occurrence of spectral branch points under transitions from idealized boundary conditions to physically realistic boundary conditions. Furthermore, we develop a gradient technique with respect to the $\alpha-$profile $\alpha(x)$ of the dynamo which allows us to search for most efficient $\alpha-$perturbations to trigger magnetic field reversals.
Kleefeld, Frieder	**How about non-Hermitian Quantum Theory and the Correspondence Principle** Some non-trivial consequences of a hypothetical correspondence principle between non-Hermitian classical and quantum physics are revealed.
Kriel, Johannes N	**Solving non-perturbative flow equations using Moyal products** We illustrate the use of the non-commutative Moyal product in the treatment of operator equations. In particular, we consider flow equations based on continuous unitary transformations. These equations, first proposed by Wegner, describe the continuous transformation of a Hamiltonian into a diagonal or block diagonal form. Using the Moyal formalism this operator equation can be written as a nonlinear partial differential equation. The implementation is non-perturbative in that the derivation of the PDE is based on an expansion controlled by the size of the system rather than the coupling constant. Applying this method to the Lipkin model leads to very accurate results for the spectrum and expectation values both at strong coupling and in the thermodynamic limit. The Moyal bracket construction can also be used to construct a differential equation to solve for the metric required in quasi-Hermitian quantum mechanics (see abstract by Scholtz).
Lavagno, A	**q-deformed quantum mechanics and q-Hermitian operators** Starting on a q-deformed canonical quantization rule, postulated on the basis of the non-commutative q-differential calculus, we study a generalized q-deformed Schrödinger equation. Such an equation of motion can be viewed as the quantum stochastic counterpart of a generalized classical kinetic equation, reproducing the q-deformed exponential stationary distribution. In this framework, q-deformed adjoint of an operator and q-hermitian operator properties occur in a natural way in order to satisfy the basic quantum mechanics assumptions.

Lévai, Géza	**On the pseudo-norm in some $\mathcal{PT}$-symmetric potentials** The indefinite sign of the pseudo-norm is one of the most characteristic features of $\mathcal{PT}$-symmetric quantum mechanical potentials. It represents a major difference with respect to ordinary (Hermitean) quantum mechanical systems not only from the conceptual point of view (i.e. the probabilistic interpretation of the wavefunction) but also in technical terms (i.e. the evaluation of unusual integrals). Although the knowledge of the pseudo-norm is essential in the analysis of $\mathcal{PT}$-symmetric problems (e.g. in the determination of the $\mathcal{C}$ charge operator), it has been calculated until now only for a handful of systems. It was found that the sign of the pseudo-norm typically alternates according to the n principal quantum number as $(-1)^n$, which is in relation with an i^n factor necessary to secure $\mathcal{PT}$ invariance of the wavefunction itself, at least in the case of unbroken $\mathcal{PT}$ symmetry. Recently the $(-1)^n$ type behaviour of the pseudo-norm was proven exactly for a class of potentials that are written in a polynomial form of $\mathrm{i}x$ (including the harmonic oscillator and the archetype of $\mathcal{PT}$-symmetric potentials, $V(x) = \mathrm{i}x^3$, for example). However, the situation can be more complicated as the nature of $\mathcal{PT}$-symmetric potentials varies according to a number of features. In particular, $\mathcal{PT}$ symmetry can be unbroken or spontaneously broken; the integration path can be defined along various contours of the complex x plane, including the real x axis too; potentials can have finite or infinite number of discrete levels; and they can be solvable analytically, numerically, or by some approximation techniques. Here we present the results of (mainly analytical) calculations on the pseudo-norm for some $\mathcal{PT}$-symmetric potentials with features mentioned above.
Mailybaev, Alexei A. and Kirillov, Oleg N.	**Berry phase around EP and DP degeneracies of non-Hermitian Hamiltonians** We study Berry phases when an open quantum system traverses adiabatically a small closed cycle in parameter space near either an exceptional point (EP) or a diabolic point (DP) of a non-Hermitian Hamiltonian. These are the points, where two energy levels coincide forming a Jordan block in case of EP and a semisimple eigenvalue in case of DP. We develop a general multidimensional theory, which gives Berry phases as expansions in powers of a small parameter describing the size of a cycle. Explicit formulae for this expansion up to second order terms are presented. It is shown that the main (zero-order) term is determined only by the levels involved into the degeneracy, while higher-order terms are influenced by the background (all the levels in the system). Both reversible and irreversible systems are considered, in which Berry phases are topological and geometrical, respectively
Mondragón, Alfonso	**Crossings and anti-crossings of energies and widths, changes of identity and geometric phases of unbound states** The local toological structure and the universal unfolding of the eigenenergy surfaces at a degeneracy of unbound states in parameter space are derived in the framework of the theory of the analytical properties of the Schrödinger radial wave functions. Then, the rich phenomenology observed in the coherent mixing of two unbound energy eigenstates when the control parameters of the quantum systems are varied, namely, crossings and anti-crossings of energies and widths, the change of identity of the poles of the S(E) matrix and the geometric phases acquired by the wave functions of the unbound stares, is fully explained in terms of the topological structure of the eigenenergy surfaces close to the degeneracy point.
Mostafazadeh, Ali	**Dynamically Equivalent but Kinematically Distinct Pseudo-Hermitian Quantum Systems** We give a brief review of pseudo-Hermitian quantum mechanics and discuss the consequences of leaving the metric operator on the Hilbert space as a freedom of the formulation of the quantum theory. We present a systematic method of constructing the most general metric operator for a quasi-Hermitian Hamiltonian with a complex potential and demonstrate its application in the study of the imaginary cubic potential. This reveals an infinite set of previously unnoticed CPT- and non-CPT-inner products. We determine the underlying classical system for this potential and show that unlike the associated quantum theory it is independent of the choice of the metric operator. As another perhaps more important application of pseudo-Hermitian quantum mechanics we survey a recent formulation of the quantum mechanics of first-quantized Klein-Gordon fields.

Nowak, Maciej A.	**Random Hermitian versus Random Nonhermitian Operators – unexpected links** During the lecture, I will explore some astonishing links between hermitian and non-hermitian random systems. First, I will show, how quaternion-valued Green's functions hide the information about the complex spectra of non-hermitian matrix models, in analogy to complex Green's function unraveling the properties of real eigenvalues in the case of hermitian random matrix models. Then I will show that for a large class of nonhermitian random matrices, several non-trivial properties of the spectrum can be understood via the method of conformal transformations applied to the pertinent hermitian models. Finally, I will discuss the analogies to Brownian motion in the space of nonhermitian random matrix models.
Scarfone, A.M. and Lavagno, A	**Stochastic quantization of many-body classical systems described by generalized entropies** We present a family of nonlinear Schroedinger equations containing a complex nonlinearity and describing, in the mean filed approximation, a system of collectively interacting particles. These evolution equations are obtained, in the framework of the stochastic quantization method originally introduced by Nelson, starting from a classical many body system whose kinetics is described by a nonlinear Fokker-Planck equation compatible with a very general entropic form. Due to the complex structure of the nonlinearity in the Schroedinger equation the time evolution of this class of systems is governed by non-Hermitian operators. As a consequence the whole family of evolution equations describe, in general, damped and dissipative systems. We detail the results to some relevant quantum systems obtained starting from generalized entropies already known in literature.
Scholtz, FG	**Metrics and more from Moyal products** The Moyal product standardly employed in non-commutative quantum mechanics is used to cast the equation for the metric in the form of a differential equation. For Hamiltonians of the form $p^2+iV(x)$ with $V(x)$ polynomial this is an exact equation. Solving this equation in perturbation theory recovers known results. Explicit criteria for the hermiticity and positive definiteness of the metric are formulated. Similar considerations can be applied to construct the Berry connection from which the Berry phase can be computed.
Uncu, Haydar and Demiral, Ersan	**Solutions of the Schrödinger equation for $\mathcal{PT}$ Symmetric Dirac Delta Potentials in 1D** We have studied bound states for the Schrödinger equation of a $\mathcal{PT}$-symmetric potential with 2N Dirac delta functions. The potential is given as $$V(x) = -\frac{\hbar^2}{2m\sum_{i=1}^{N}}\sigma_i(x-x_i)+\sigma_i^*\delta(x+x_i$$ where $0 < x_1 < x_2 < \ldots < x_N$, and σ_is are arbitrary complex numbers. We present a method to obtain an equation for the bound state eigenvalues of this $\mathcal{PT}$-symmetric Hamiltonian. We solve this eigenvalue equation numerically for two and four Dirac delta functions (N=1,2). We also utilize these equations to investigate where the $\mathcal{PT}$symmetry breaks down.
Verbaarschot, Jacobus	**Toda Lattices and Non-Hermitian Random Matrix Theories** Non-Hermitian Random Matrix Theories have applications to a wide range of physics problems. Among others we mention the distribution of the poles of S-matrix resonances, the Hatano-Nelson model and the Dirac spectrum of QCD at nonzero chemical potential. What these theories have in common is that in a parameter range of interest, their properties are determined by symmetries only and can be described in terms of a low-energy effective Lagrangian. Random Matrix Theories with the symmetries of the original problem are mapped onto the same effective Lagrangian and are, in many cases, the simplest mathematical representation of the universality class. They can be solved by a variety of techniques such as the complex orthogonal polynomial method, the super-symmetric method, the Riemann-Hilbert problem approach and other techniques discussed at this workshop. In fact, all these methods work because the problem has an underlying integrability structure. For systems with broken time reversal invariance generating functions for the spectral density are constrained by powerful integrability conditions, such as the Painlevé equation, introduced by Kanzieper, and the Toda lattice equation. The latter method will be discussed in detail and will be illustrated by applications to the above mentioned problems. We show that the spectral density can be understood in terms of a phase diagram of simple partition functions. Connections with the method of complex orthogonal polynomials will be made.

Weideman, JAC	**Spectral Differentiation Matrices for the Numerical Solution of Schrödinger's Equation** The spectral collocation method is a powerful numerical tool for solving all kinds of differential equations. By using weighted polynomial interpolants, one approximates continuous differential operators by their discrete (i.e., matrix) counterparts. In this manner a two-point boundary value problem can be approximated by a linear system, and a differential eigenvalue problem can be reduced to a matrix eigenvalue problem. The speaker and co-workers developed a MATLAB package, DMSUITE, for constructing such differentiation matrices. In this talk, we shall focus on the specific application of solving Schrödinger's equation using differentiation matrices based on Hermite and Laguerre interpolants. As a demonstration of the DMSUITE package we shall reproduce, from scratch and in real time, the numerical results of CM Bender et. al., "Complex WKB Analysis of energy-level degeneracies of non-Hermitian Hamiltonians", J. Phys. A: Math. Gen. Vol. 34 (2001) L31-L36.
Weigert, Stefan	**Detecting "Broken" $\mathcal{PT}$-Symmetry** A fundamental problem in the theory of $\mathcal{PT}$-invariant quantum systems is to determine whether a given system "respects" this symmetry or not. $\mathcal{PT}$-symmetry is said to be "unbroken" if the eigenstates of the Hamiltonian H are also eigenstates of the operator $\mathcal{PT}$. If the Hamiltonian possesses pairs of complex conjugate eigenvalues the symmetry is said to be "broken" since the commuting operators $\mathcal{PT}$ and $\mathcal{H}$ do not have a complete set of common eigenstates. It is shown in this contribution how to algorithmically detect the existence of complex eigenvalues for a given quasi-hermitean matrix. In other words, the procedure enables one to determine, for a given quantum system with finite-dimensional state space, whether $\mathcal{PT}$-symmetry is "broken." The procedure is based on theorems dating back to the second half of the 19th century which qualitatively locate the zeros of real polynomials in the complex plane. If a $\mathcal{PT}$-symmetric (or, more generally, a quasi-hermitean) perturbation depending on a real parameter is added to an initially hermitean Hamiltonian, the method outputs expressions, polynomial in the perturbation parameter, which can be used to study the emergence of complex conjugate pairs of eigenvalues. The interest and value of the present approach lies in the fact that it avoids diagonalization of the Hamiltonian at hand.
Znojil, Miloslav	**Coupling of channels in $\mathcal{PT}$-symmetric models** Standard Hermitian coupled-channel (CC) Hamiltonians proved relevant in atomic and nuclear physics as well as in some relativistic, cosmological, MHD, nonlinear (energy-dependent) or supersymmetric implementations of Quantum Theory. As long as the study of the latter five subjects also emerged within the quasi-unitary evolution context of the so called PT-symmetric Quantum Mechanics (PTSQM), we intend to review some of the most promising aspects of the proposed coupled-channel extension of the theory (CCPTSQM). Keeping in mind that even the textbook Feshbach-Villars approach to the Klein-Gordon equation represents a typical $\mathcal{PT}$-symmetric coupled-channel example we feel encouraged to introduce a certain natural generalization of the operator $\mathcal{PT}$ (in essence, we admit that the "pseudo-metric" P itself may be allowed non-Hermitian). The productivity of such an idea is then tested on a set of solvable examples.