Dynamically Equivalent & Kinematically Distinct Pseudo-Hermitian Quantum Systems

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Outline

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Classical Mechanics

Phase space: $\mathcal{M}$

Observables: $o_c : \mathcal{M} \to \mathbb{R}$

Dynamics: $$\frac{d}{dt} o_c(t) = \{o_c(t), H_c\}_{\mathrm{PB}}$$

H_c: **An observable called "Hamiltonian"**

H_c is real-valued.

Quantum Mechanics

Hilbert space: $\mathcal{H}$

Observables: Hermitian operators

$$o : \mathcal{H} \to \mathcal{H}$$

Dynamics: $$i\,\hbar \frac{d}{dt}\psi(t) = H\,\psi(t)$$

H: **An observable called "Hamiltonian"**

H is Hermitian.

Inner product of $\mathcal{H}$: $\langle\cdot|\cdot\rangle$

Definition: $o:\mathcal{H}\to\mathcal{H}$ is **Hermitian** if $\langle\psi|o\,\phi\rangle=\langle o\,\psi|\phi\rangle$ for all $\psi,\phi\in\mathcal{H}$.

Theorem: Let $o:\mathcal{H}\to\mathcal{H}$ be a linear operator. Then $\langle\psi|o\,\psi\rangle\in\mathbb{R}$ for all $\psi\in\mathcal{H}$ **iff** o is **Hermitian**.

There is no escape from Hermiticity.

In Classical Mechanics the geometry of the phase space is fixed.

In Quantum Mechanics the geometry of the Hilbert Space is fixed.

Is this absolutism justified in QM?

Answer: Yes, because up to unitary-equivalence there is a single separable Hilbert space.

Defining inner product of $\mathcal{H}$: $\langle\cdot|\cdot\rangle$

Any other inner product:

$$\langle\cdot,\cdot\rangle_{\eta_+} = \langle\cdot|\eta_+\cdot\rangle$$

η_+ **is an everywhere-defined, positive-definite, invertible operator (metric operator).**

$$\mathcal{H}_{\eta_+} := (\mathcal{H}, \langle\cdot,\cdot\rangle_{\eta_+})$$

Unitary Operator $U : \mathcal{H}_{\eta_+} \to \mathcal{H}$:

$$\langle \Psi, \Phi \rangle_{\eta_+} = \langle U\Psi | U\Phi \rangle$$

$\rho = \sqrt{\eta_+}$ **viewed as mapping** $\mathcal{H}_{\eta_+}$ **onto** $\mathcal{H}$ **is a unitary operator.**

States vectors Ψ and $\psi := \rho\,\Psi$ are in 1-to-1 correspondence.

Observables $o : \mathcal{H} \to \mathcal{H}$ and $O := \rho^{-1} o\, \rho : \mathcal{H}_{\eta_+} \to \mathcal{H}_{\eta_+}$ are in 1-to-1 correspondence.

Expectation values are identical:

$$\frac{\langle \Psi, O\Psi \rangle_{\eta_+}}{\langle \Psi, \Psi \rangle_{\eta_+}} = \frac{\langle \psi | o\psi \rangle}{\langle \psi | \psi \rangle}$$

Let a quantum system be described by $(\mathcal{H}, o, h)$ where o stands for an arbitrary observable and h the Hamiltonian. Then the same system may be described by $(\mathcal{H}_{\eta_+}, O, H)$ where $H = \rho^{-1} h\, \rho$.

$(\mathcal{H}, o, h)$ and $(\mathcal{H}_{\eta_+}, O, H)$ provide two physically equivalent representations of the system.

Though these representations are equivalent, the very fact that they exist reveals a symmetry of the formulation of QM.

Different reprsentations may prove appropriate for different systems.

Definition: An everywhere defined, Hermitian, invertible operator $\eta:\mathcal{H}\to\mathcal{H}$ is called a **pseudo-metric operator**.

Definition: A **positive-definite** pseudo-metric operator $\eta_+:\mathcal{H}\to\mathcal{H}$ is called a **metric operator**.

Definition: A linear operator $H:\mathcal{H}\to\mathcal{H}$ is said to be **pseudo-Hermitian** if there is a pseudo-metric operator $\eta:\mathcal{H}\to\mathcal{H}$ satisfying $H^\dagger=\eta\,H\,\eta^{-1}$.

Definition: Given a pseudo-metric operator η, any linear operator H that satisfies $H^\dagger=\eta\,H\,\eta^{-1}$ is called **η-pseudo-Hermitian**.

$\mathcal{U}_H :=$ **Set of all pseudo-metric operators satisfying** $H^\dagger = \eta\, H\, \eta^{-1}$

Fact: $\mathcal{U}_H$ **is either empty or an infinite set. It may or may not include a metric operator. If it does there is an infinity of metric operators in** $\mathcal{U}_H$**.**

Qxn: Can a $\mathcal{PT}$-symmetric differential operator H serve as the Hamiltonian for a unitary quantum system?

Unitarity

$\Downarrow$

H must be **Hermitian**

$\Downarrow$

H is **diagonalizable** &

has a **real spectrum**.

Theorem: Let $H : \mathcal{H} \to \mathcal{H}$ be a diagonalizable linear operator with a discrete spectrum. Then the following conditions are equivalent.

- **The spectrum of H is real.**

- **$\exists$ a metric operator η_+ such that H is η_+-pseudo-Hermitian.**

- **$\exists$ an inner product $\langle\cdot,\cdot\rangle_{\eta_+}$ satisfying** $\langle\cdot,H\cdot\rangle_{\eta_+}=\langle H\cdot,\cdot\rangle_{\eta_+}$.

- **$\exists$ a Hermitian operator $h:\mathcal{H}\to\mathcal{H}$ such that $H=\rho^{-1}h\,\rho$ for an invertible operator ρ.**

H **may serve as the Hamiltonian for a unitary quantum system if one defines the physical Hilbert space using the inner product** $\langle\cdot,\cdot\rangle_{\eta_+}$ **for some metric operator** η_+ **belonging to** $\mathcal{U}_H$.

H **is diagonalizable:** $\exists$ **bases** $\{\psi_n\}$ **and** $\{\phi_n\}$ **such that**

$$H\psi_n = E_n\psi_n$$

$$H^\dagger\phi_n = E_n\phi_n$$

$$\langle\psi_n|\phi_m\rangle = \delta_{mn}$$

$$\sum_n |\psi_n\rangle\langle\phi_n| = 1$$

$\{\psi_n, \phi_n\}$ **is called a biorthonormal system.**

Characterization of $\eta_+ \in \mathcal{U}_H$:

Let $\{\psi_n, \phi_n\}$ be as above.

Then **every** $\eta_+ \in \mathcal{U}_H$ is given by $\eta_+ = A^\dagger \sum_n |\phi_n\rangle\langle\phi_n| A$ where A is invertible and $[A, H] = 0$.

A characterizes the **arbitrariness** in η_+.

Pseudo-Hermitian QM:

- $\mathcal{H}$: a reference Hilbert space [e.g.,$L^2(\Gamma)$]
- $H:\mathcal{H}\to\mathcal{H}$: a non-Hermitian Hamiltonian.
- $\mathcal{H}_{\rm Phys}:=(\mathcal{H},\langle\cdot,\cdot\rangle_{\eta_+})$ for some $\eta_+\in\mathcal{U}_H$.
- Observables: $O:=\rho^{-1}o\,\rho$ where $\rho:=\sqrt{\eta}_+$ and $o:\mathcal{H}\to\mathcal{H}$ is a Hermitian operator.

- **Hermitian Hamiltonian**: $h := \rho\, H \rho^{-1}$
- **Pseudo-Hemritian rep.**: $(\mathcal{H}_{\rm Phys}, O, H)$
- **Hermitian rep.**: $(\mathcal{H}, o, h)$

$$
\begin{array}{ccc}
\mathcal{H}_{\rm Phys} & \stackrel{\rho}{\longrightarrow} & \mathcal{H} \\
\updownarrow & & \updownarrow \\
O = \rho^{-1} o \rho & \longleftarrow & o
\end{array}
$$

Pseudo-Hermitian Canonical Quantization

Definition: Let $\mathcal{H} = L^2(\mathbb{R})$. Then

$$X := \rho^{-1} x\, \rho, \qquad P := \rho^{-1} p\, \rho.$$

are called the **pseudo-Hermitian position** and **momentum** operators.

$$[X, P] = i\hbar 1$$

η_+-Pseudo-Hermitian Quantization:

$$x_c \longrightarrow X, \quad p_c \longrightarrow P, \qquad \{\cdot,\cdot\}_c \longrightarrow \frac{1}{i\hbar}[\cdot,\cdot]$$

Classical Hamiltonian: H_c

Either express H **as** $H = f(X, P)$

or express $h = f(x, p)$ **and let**

$H_c = \lim_{\hbar\to 0} f(x_c, p_c)$.

For $H = p^2 + V(x)$ **with** V **complex**, h, X, P **are generally nonlocal operators.**

$$A = \sum_{\ell=0}^{\infty} a_\ell(x) p^\ell$$

Both representations $(\mathcal{H}_{\rm Phys}, H, O)$ **&** $(\mathcal{H}, h, o)$ **involve nonlocal operators.**

Indefinite-Metric QM

Definition: A pseudo-metric operator η such that $\pm\eta$ is not positive-definite is called an **indefinite metric**.

$\exists\Psi \in \mathcal{H}_\eta - \{0\}$ such that $\langle\Psi,\Psi\rangle_\eta \leq 0$.

$\langle\cdot,\cdot\rangle_\eta$ is an **indefinite quadratic form** ("inner product").

Fix an indefinite-metric operator η.

Consider Hamiltonian operators H that are η-pseudo-Hermitian.

Physical Hilbert space:= invariant subspace $\mathcal{H}_+ \subset \mathcal{H}_\eta$ of η where η is positive-definite.

If $\mathcal{H}_+$ is invariant under H there is a consistent theory.

Suppose that

1. $\exists \mathcal{H}_\pm \subset \mathcal{H}_\eta$ invariant under η and H.
2. $\pm\eta$ is positive-definite on $\mathcal{H}_\pm$.
3. $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$.
4. $\mathcal{H}_+ \perp \mathcal{H}_-$ with respect to $\langle\cdot,\cdot\rangle_\eta$.

Then $\Psi_i = \Psi_i^+ + \Psi_i^-$ with $\Psi_i^\pm \in \mathcal{H}_\pm$ and

$$\langle\Psi_1, \Psi_2\rangle_+ := \langle\Psi_1^+, \Psi_2^+\rangle_\eta - \langle\Psi_1^-, \Psi_2^-\rangle_\eta$$

is a (positive-definite) inner product.

Nevanlinna (1952,1954)

For $\mathcal{PT}$-symmetric systems, $\eta = \mathcal{P}$ defines an indefinite inner product $\langle\cdot,\cdot\rangle_{\mathcal{P}}$ and the corresponding Nevanlinna inner product $\langle\cdot,\cdot\rangle_+$ is the $\mathcal{CPT}$-inner product.

In terms of the L^2-inner product,

$$\langle\cdot,\cdot\rangle_+ = \langle\cdot,\cdot\rangle_{\eta_+}$$

for some metric operator $\eta_+ \in \mathcal{U}_H$.

$$\mathcal{C} = \eta_+^{-1}\mathcal{P}$$

$\mathcal{C}$ is just a grading operator:

$$\mathcal{C}\,\Psi^{\pm} = \pm\Psi^{\pm} \quad \text{for} \quad \Psi^{\pm} \in \mathcal{H}_{\pm}$$

$$\mathcal{C}^2 = 1$$

$\mathcal{H}_{\pm}$ are invariant under H:

$$[\mathcal{C}, H] = 0$$

$$\mathcal{C} = \eta_+^{-1}\mathcal{P}$$

$\mathcal{C}^2 = 1$ **restricts** η_+.

The $\mathcal{CPT}$**-inner product** $\langle\cdot,\cdot\rangle_{\eta_+}$ **that is defined by** η_+ **is not the most general permissible inner product supporting a unitary evolution.**

Calculation of the Most General η_+

1. **Spectral Method:**

$$\eta_+ = A^\dagger \sum_n |\phi_n\rangle\langle\phi_n| A$$

$$[A, H] = 0.$$

2. **Perturbation Theory**

Perturbative Calculation:

Let $H = H_0 + \epsilon H_1$ where H_0 is Hermitian and H_1 is anti-Hermitian.

- Let $Q := -\ln\eta_+$, i.e., set $\eta_+ = e^{-Q}$
- Expand: $Q = \sum_{k=1}^{\infty} Q_k\,\epsilon^k$.
- Insert in $H^\dagger = \eta_+ H \eta_+^{-1}$ and use Baker-Campbell-Hausdorff identity to solve for Q_k perturbatively.

$$[H_0, Q_1] = -2H_1,$$

$$[H_0, Q_2] = 0$$

$$[H_0, Q_3] = \frac{1}{12}[[[H_0, Q_1], Q_1], Q_1]$$

$$\cdots = \cdots$$

$$[H_0, Q_k] = f_k(H_0, H_1, Q_1, \cdots, Q_{k-1})$$

η_+ **that makes** $\mathcal{C} = \eta_+^{-1}\mathcal{P}$ **an involution** ($C^2 = 1$) **corresponds to a particular class of solutions.**

If $H_0 = p^2$, in the x-representation

$$[H_0, Q_k] = f_k(H_0, H_1, Q_1, \cdots, Q_{k-1})$$

is just the wave equation:

$$(-\partial_x^2 + \partial_y^2)\langle x|Q_k|y\rangle = f_k(x, y)$$

which can be solved exactly.

Solution is not unique.

Imaginary Cubic Potential

$$H = \frac{p^2}{2m} + i\epsilon\, x^3$$

Calculate Q_ℓ, X, P, h, H_c, **etc.**

$$Q_1 = \frac{1}{4}\{x^4, \frac{1}{p}\} + \frac{3}{4}\{x^2, \frac{1}{p^3}\} + \frac{a_1}{p^5} + \frac{ib_1}{p^5}\mathcal{P}$$

$$Q_2 = \frac{a_2}{p^{10}} + \frac{b_2}{p^{10}}\mathcal{P}$$

$$Q_3 = \sum_{\ell=1}^{5} j_\ell \{x^{2\ell}, \frac{1}{p^{15-2\ell}}\} + \frac{a_3}{p^{15}} + i\left[\sum_{\ell=1}^{4} k_\ell \{x^{2\ell}, \frac{1}{p^{15-2\ell}}\} + \frac{b_3}{p^{15}}\right]\mathcal{P}$$

a_i **and** b_i **are arbitrary real numbers**

j_ℓ **and** k_ℓ **are functions of** a_i **and** b_i**.**

Terms involving $\mathcal{P}$ have been missed in an earlier calculation that was based on making an ansatz for the solution.

For $\mathcal{CPT}$-metrics $a_2 = b_2 = 0$.

$$H = \frac{p^2}{2m} + i\epsilon\, x^3$$

$$m = \hbar = 1$$

$$X = x + \frac{i}{8}\left(\{x^4, \frac{1}{p^2}\} + 9\,\{x^2, \frac{1}{p^4}\} + \frac{20a_1}{p^6} - 4b_1\,\{x, \frac{1}{p^5}\}\,\mathcal{P}\right)\epsilon + \mathcal{O}(\epsilon^2)$$

$$P = p + \frac{i}{4}\left(2\{x^3, \frac{1}{p}\} + 3\{x, \frac{1}{p^3}\} - \frac{4b_1}{p^4}\mathcal{P}\right)\epsilon + \mathcal{O}(\epsilon^2)$$

$$h = \frac{p^2}{2} + \frac{3}{16}\left(\{x^6, \frac{1}{p^2}\} + 22\{x^4, \frac{1}{p^4}\}\right.$$

$$+(510 + 10a_1)\{x^2, \frac{1}{p^6}\} +$$

$$\left.\frac{8820 + 140a_1}{p^8} - \frac{4}{3} b_1\{x^3, \frac{1}{p^5}\} \mathcal{P}\right) \epsilon^2$$

$$+\frac{1}{4}\left(15a_2(\{x^2, \frac{1}{p^{11}}\} + \frac{44}{p^{13}})\right.$$

$$\left.+ib_2\{x^3, \frac{1}{p^{10}}\} \mathcal{P}\right) \epsilon^3 + \mathcal{O}(\epsilon^4).$$

Classical Hamiltonian

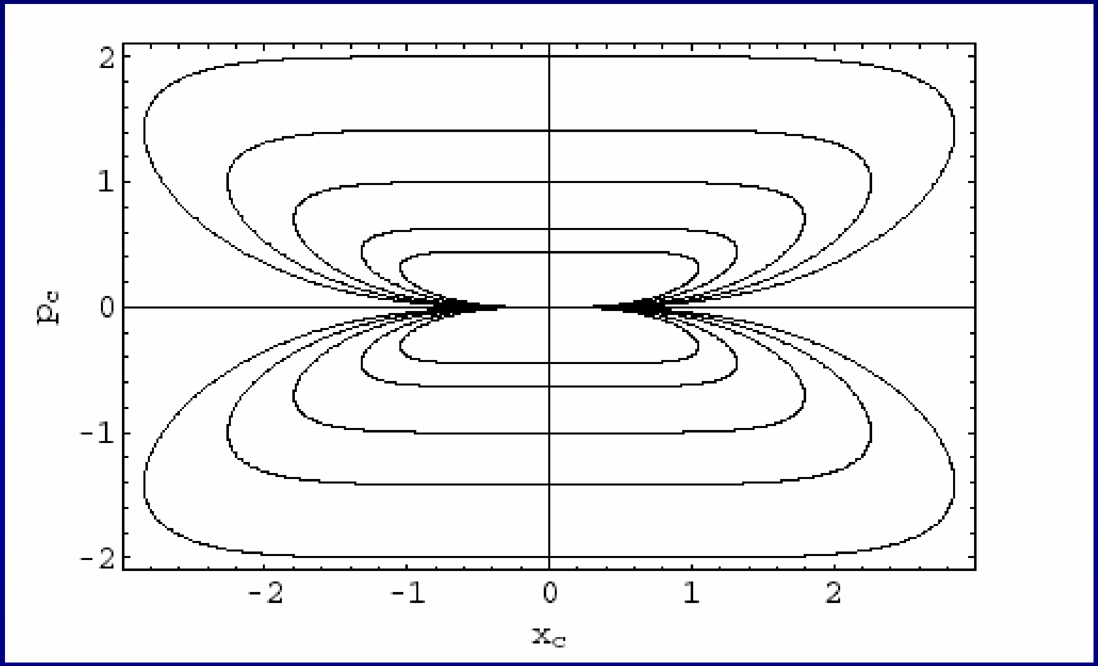
$$H_c = \frac{{p_c}^2}{2m} + \frac{3}{8} m \, \epsilon^2 \, \frac{x_c^6}{{p_c}^2} + \mathcal{O}(\epsilon^4)$$

H_c is **independent** of the choice of **the metric operator**.

This is true in all orders of the perturbation theory.

Classical Phase Space Orbits of

Imaginary Cubic Potential



$$H_c = \frac{p_c^2}{2m} + \frac{3}{8}\, m\, \epsilon^2\, \frac{x_c^6}{p_c^2} + \mathcal{O}(\epsilon^4)$$

$$m = 1, \quad \epsilon = 0.1$$

QM of Klein-Gordon Fields

$$\mathcal{H} = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

$$\xi \in \mathcal{H} \quad \Leftrightarrow \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \xi_i \in L^2(\mathbb{R}^3)$$

$$\langle \xi | \zeta \rangle := \int d^3x \, [\xi_1(\vec{x})^* \zeta_1(\vec{x}) + \xi_2(\vec{x})^* \zeta_2(\vec{x})]$$

Let $H : \mathcal{H} \to \mathcal{H}$ be defined by

$$H = \frac{\hbar}{2}\begin{pmatrix} \lambda D + \lambda^{-1} & \lambda D - \lambda^{-1} \\ -\lambda D + \lambda^{-1} & -\lambda D - \lambda^{-1} \end{pmatrix}$$

$$D := \hbar^{-2}\left[\vec{p}^{\,2} + (mc)^2\right], \qquad \lambda, m \in \mathbb{R}^+$$

- H is σ_3-pseudo-Hermitian.
- H is diagonalizable.
- Spectrum$(H) = \mathbb{R}$

General Metric Operator

(Spectral Method $\sum \to \int$)

$$\eta_+ = \frac{1}{2} A^\dagger \begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix} A$$

$$S := \frac{1}{2} \ln[\lambda^2 D], \qquad [A, H] = 0$$

$H : \mathcal{H}_{\eta_+} \to \mathcal{H}_{\eta_+}$ **is Hermitian.**

$$\mathcal{H}_{\eta_+} := (\mathcal{H}, \langle \cdot, \cdot \rangle_{\eta_+})$$

$\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ **is a KG field:**

$$(\partial^\nu \partial_\nu + \mu^2)\psi(x^0, \vec{x}) = 0, \qquad \mu = mc/\hbar$$

$$\psi(x^0, \cdot)\,,\ \dot{\psi}(x^0, \cdot) \in L^2(\mathbb{R}^3)$$

$$\Psi(x^0) := \begin{pmatrix} \psi(x^0, \cdot) + i\lambda\,\dot{\psi}(x^0, \cdot) \\ \psi(x^0, \cdot) - i\lambda\,\dot{\psi}(x^0, \cdot) \end{pmatrix} \in \mathcal{H}$$

For all $\tau \in \mathbb{R}$, $\quad U_\tau \psi := \frac{1}{2\sqrt{\mu\lambda}}\,\Psi(\tau)$

$\mathcal{V}$:= **Vector Space of all KG fields**

$U_\tau : \mathcal{V} \to \mathcal{H}$ **is linear & invertible.**

General inner product $(\cdot,\cdot)$ on $\mathcal{V}$:

$$(\psi,\psi') := \langle U_\tau \psi, U_\tau \psi' \rangle_{\eta_+}$$

$$\eta_+ = \frac{1}{2} A^\dagger \begin{pmatrix} \cosh S & \sinh S \\ \sinh S & \cosh S \end{pmatrix} A$$

Demanding **Lorentz invariance** of (ψ, ψ') restricts the choice of A that fixes η_+ to a parameter $a \in (-1, 1)$ and a trivial scaling factor $\kappa \in \mathbb{R}^+$.

$(\cdot, \cdot)$ is **independent** of λ and τ.

Hilbert Space of KG Fields:

$$\mathcal{H}_a = (\mathcal{V}, (\cdot, \cdot)_a)$$

$$(\psi, \psi)_a = \int_\sigma J_a^\mu \, d\sigma_\mu, \qquad a \in (-1, 1)$$

$$J_a^\mu = \kappa\,(J^\mu + a\, J_{\mathrm{KG}}^\mu), \qquad \kappa \in \mathbb{R}^+,$$

$$J_{\mathrm{KG}}^\mu(x) := -\frac{i}{2\mu} \left\{ \psi(x)^* \; \overleftrightarrow{\partial^\mu} \; \psi(x) \right\}$$

$$J^\mu(x) := -\frac{i}{2\mu} \left\{ \psi(x)^* \; \overleftrightarrow{\partial^\mu} \, C \psi(x) \right\}$$

$$C := i D^{-1/2} \partial_0$$

$$\partial_\mu J_a^\mu = 0$$

$\psi_\pm :=$ $\pm$-**frequency part of** ψ

$$C\psi_\pm = \pm\psi_\pm$$

Nivanlinna Construction

($\mathcal{CPT}$-inner product): $a = 0$

$$(\psi,\psi')_0 = \kappa\ \left[(\psi_+,\psi'_+)_{\rm KG} - (\psi_-,\psi'_-)_{\rm KG}\right]$$

Group-averaging method [Woddard (1993)]

Symplectic method [Kay & Wald (1991), Wald (1994)]

Greens's function method [Halliwel & Ortiz (1993)]

- **By construction** $U_\tau : \mathcal{H}_a \to \mathcal{H}_{\eta_+}$ **is unitary.**
- $\rho = \sqrt{\eta_+} : \mathcal{H}_{\eta_+} \to \mathcal{H}$ **is also unitary.**

$$\mathcal{H}_a \xrightarrow{U_\tau} \mathcal{H}_{\eta_+} \xrightarrow{\rho} \mathcal{H} = L^2(\mathbb{R}^3) \oplus L^2(\mathbb{R}^3)$$

Equivalent Representations:

Covariant Rep.: $(\mathcal{H}_a, H_a, O_a)$

Pseudo-Hermitian Rep.: $(\mathcal{H}_{\eta_+}, H, O)$

Hermitian Rep.: $(\mathcal{H}, h, o)$

$$h := \rho\, H\, \rho^{-1}, \qquad O = \rho^{-1} o\, \rho$$

$$H_a := U_\tau^{-1} H\, U_\tau\,, \qquad O_a = U_\tau^{-1} O\, U_\tau$$

H_a: **Generator of time-translations**

For $a = 0$:

$$h = \begin{pmatrix} \sqrt{p^2 + (mc)^2} & 0 \\ 0 & -\sqrt{p^2 + (mc)^2} \end{pmatrix}$$

So $\rho U_\tau : \mathcal{H}_0 \to \mathcal{H}$ is the Foldy-Whouthuysen transformation.

For $o = \vec{p} \otimes 1$, $O_0 = \vec{p}$.

For $o = \vec{x} \otimes 1$, $O_a = \vec{X}_a$ is a relativistic position operator.

$\vec{X}_0$ restricted to +-frequency fields is precisely the Newton-Wigner position operator.

Localized States:

$$\vec{X}_a \psi_a^{(\vec{x},\pm)} = \vec{x}\, \psi_a^{(\vec{x},\pm)}, \qquad C\, \psi_a^{(\vec{x},\pm)} = \pm \psi_a^{(\vec{x},\pm)}$$

One can define a probability density $\mathcal{J}_a^\mu$ associated with $\vec{X}_a$ but as is well-known [Peierls and Pauli] it cannot be covariant. It turns out not be even conserved ($\partial_\mu \mathcal{J}_a^\mu \neq 0$). But the total probability is conserved, because

$$\int d^3x \ \mathcal{J}_a^0 = \int d^3x \ J_a^0$$

$$\partial_\mu J_a^\mu = 0$$

Global Gauge Symmetry

$$\psi \longrightarrow e^{i\theta(C+a)}\psi$$

$$e^{i\theta(C+a)} \in \begin{cases} U(1) & \text{for } a \in \mathbb{Q} \\ \mathbb{R}^+ & \text{for } a \notin \mathbb{Q} \end{cases}$$

$\mathbb{R}^+ :=$ **multiplicative group of + reals**

Conclusion

- Not fixing the geometry of the Hilbert space reveals an infinite class of equivalent representations of QM.
- The classical limit of PT-symmetric non-Hermitian Hamiltonians may be obtained using the Hermitian rep. of the corresponding systems.
- *CPT*-inner product is not the most general permissible inner product.

- For imaginary cubic potential the classical Hamiltonian is insensitive to the choice of the metric.
- In the Hermitian rep. the Hamiltonian is generally nonlocal while in the pseudo-Hermitian rep. the basic observables are nonlocal. This seems to indicate a duality between non-Hermiticity and nonlocality.
- Pseudo-Hermiticity yields a consistent QM of free KG fields with a genuine probabilistic interpretation. The same holds for real fields and fields interacting with a time-independent magnetic field.

- A direct application of the method fails for interactions with other EM fields, for one can prove that in this case time-translations are non-unitary for every choice of an inner product on the space of KG fields (Klein Paradox).
- Application: Construction of Relativistic Coherent States [A.M. & F. Zamani].

Thank You for Your Attention