# Choice of physical operators and uniqueness of the metric

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# Outline

- Context
  non-Hermitian many-body Hamiltonians resulting from
  - generalized Dyson-Maleev boson mapping of bifermion operators
  - $\mathcal{PT}$-symmetry considerations
  - Phenomenology (eg coupling to environment)

- Simple non-hermitian quadratic boson Hamiltonian
  (Swanson, J Math Phys **45** (2004) 585)

- Quasi- or pseudo-Hermitian Hamiltonians and modified inner product
  - Choice or construction of the metric
  - Uniqueness of the metric
  - Physical observables

- Moyal bracket construction of the metric

- Conclusions

# Non-Hermitian Hamiltonians from Dyson-Maleev boson mappings

Schematic notation, suppressing single particle state indices and summation

$$
\begin{aligned}
c^\dagger c^\dagger &\longleftrightarrow f(B^\dagger, B) = B^\dagger - B^\dagger B^\dagger B \\
cc &\longleftrightarrow g(B^\dagger, B) = B \\
c^\dagger c &\longleftrightarrow h(B^\dagger, B) = Bt\dagger B
\end{aligned}
$$

- operators $c$ refer to fermions, $B$ to bosons
- characteristic feature: functions $f, g$ and $h$ all finite, *but* $g \neq f^\dagger$
- Result: 2-body fermion interaction terms typically map to structures

$$\alpha B_i^\dagger B_j^\dagger B_k B_l + \beta B_l^\dagger B_k^\dagger B_j B_i$$

  with $\alpha \neq \beta$ ie manifestly *non-Hermitian* wrt standard inner product in boson Fock space
- link to Hermitian fermion problem guarantees real spectrum
- complete spectrum also obtained from full boson Fock space with *a posteriori* selection of *physical subspace* (see PHHQP1, Czech J Phys **54** No 1, 2004)

Clearly similar boson Hamiltonians may be studied from a purely phenomenological point of view (absorption, coupling to environment)

or from the point of view of generalizing the framework of QM, eg *à la* $\mathcal{PT}$-symmetry

(variational principle?)

# Simple quadratic non-Hermitian boson Hamiltonian

(Swanson, J Math Phys **45** (2004) 585)

$$H = \omega(a^\dagger a + \frac{1}{2}) + \alpha a^2 + \beta {a^\dagger}^2$$
$$\omega, \alpha, \beta \text{ real; } \alpha \neq \beta$$

Diagonalize $H$ by generalized Bogoliubov transformation to a canonical pair of boson operators

$$c = g_1 a^\dagger - g_3 a$$
$$d = g_4 a - g_2 a^\dagger$$

$g_i$ generally complex; $[d, c] = 1 \Leftrightarrow g_1 g_4 - g_2 g_3 = 1$

$$H = \Omega c d + \tilde{\alpha} c^2 + \tilde{\beta} d^2 + \epsilon + \frac{1}{2}\omega$$

Setting $\tilde{\alpha} = \tilde{\beta} = 0$ and introducing $|n_R\rangle = \frac{1}{\sqrt{n!}} c^n |0_d\rangle$, with $|0_d\rangle$ the $d$-boson vacuum $d|0_d\rangle = 0$, gives

$$H|n_R\rangle = (n + \frac{1}{2})\Omega |n_R\rangle \equiv E_n |n_R\rangle$$

with

$$\Omega = \sqrt{\omega^2 - 4\alpha\beta}$$

Consider

$$H^\dagger n_L\rangle = (n+\frac{1}{2})\Omega|n_L\rangle = E_n|n_L\rangle$$

where $\langle m_L|n_R\rangle = \delta_{mn}$

Utilises $U|n_R\rangle = |n_L\rangle$

$$\Rightarrow U = \sum_n |n_L\rangle\langle n_L|$$

with properties

$$\begin{aligned} H^\dagger U &= UH \\ Uc &= d^\dagger U \\ Ud &= c^\dagger U \end{aligned}$$

Swanson offers explicit form of $U$ ito $a^\dagger$, $c$ and $d$ and $g_i$'s (based on SU(1,1) disentangling formula): not particularly illuminating

Considers $W_{\varphi n} \equiv \langle n_R|U\exp(-iHt)|\varphi_R\rangle$
and concludes from explicit calculation of $W_{02}$ and $W_{20}$ that transition probability defined "à *la* Bender" is unitary, while it is violated without the metric $U$ (no surprises here!).

Observation: coefficients $g_i$ appearing in Bogoliubov transformation are not uniquely determined;
only products $g_l g_4, g_2 g_3, g_l g_2$ and $g_3 g_4$ fixed. This leaves an undetermined scale factor

$$g_l \to \lambda g_l \quad , \quad g_3 \to \lambda g_3$$
$$g_2 \to 1/\lambda g_2 \quad , \quad g_4 \to 1/\lambda g_4$$

$\Rightarrow U$ so constructed definitely not unique

## Quasi-Hermitian Hamiltonian, modified inner product and consideration of a *set* of observables

(Scholtz, Geyer & Hahne Ann Phys **213** ('92) 74)

Linear operator (metric) $T$ on Hilbert space $\mathcal{H}$

$T : \mathcal{H} \to \mathcal{H} \ni$

(i) $\mathcal{D}(T) = \mathcal{H}$

(ii) $T^\dagger = T$ (Hermiticity)

(iii) $(\varphi, T\varphi) > 0 \; \forall \; \varphi \; \in \; \mathcal{H}$ and $\varphi \neq 0$ (positive definiteness)

(iv) $\|T\varphi\| \leq \|T\| \; \|\varphi\| \; \forall \; \varphi \in \mathcal{H}$ (boundedness)

(v) $TA_i = A_i^\dagger T \;\; \forall \; A_i$

Notes:

- Hellinger-Toeplitz theorem: $(i) \Rightarrow (iv)$ (Kretschmer & Szymanowski, PLA 2004)
- Relaxing (iii) leads to pseudo-Hermiticity (studied by Mostafazadeh in a series of papers, by Znojil, Solombrino, Ahmed and others)

Define *new scalar product* $(\varphi, \psi)_T \equiv (\varphi, T\psi) \Rightarrow (\varphi, A_i\psi)_T = (\varphi, TA_i\psi) = (\varphi, A_i^\dagger T\psi) = (A_i\varphi, \psi)_T$

| |
|---|
| Uniqueness of $T$: iff $T$ is defined wrt a *set* of operators $A_i$ which is *irreducible* on $\mathcal{H}$<br>(Scholtz, Geyer & Hahne) |

Set of operators $A_i$ is irreducible if no proper subset of states in $\mathcal{H}$ is left invariant by *all* operators $A_i$

Since $H$ is trivially reducible (each eigenstate is invariant under $H$), a unique metric is not necessarily obtained from $TH = H^\dagger T$ only (viz Swanson's construction $UH = H^\dagger U$)

Accordingly, not all transition matrix elements calculated wrt an inner product defined by $U$ are physical (no *unique* measurement prediction) $\Longleftrightarrow$ different quantum mechanical systems

## Construction of metric $T$ for the quadratic non-Hermitian boson Hamiltonian

$$H = \omega(a^\dagger a + \tfrac{1}{2}) + \alpha a^2 + \beta {a^\dagger}^2 \ .$$

with $\omega$, $\alpha$, $\beta$ real; $\alpha \neq \beta$

$a$ and $a^\dagger$ standard boson annihilation and creation operators Rewriting to position $x$ and momentum $p$ makes it simple to check that $H$ is $\mathcal{PT}$-symmetric

Rescaling $a \to \lambda a$ and $a^\dagger \to \lambda^{-1} a^\dagger$ immediately shows that $H$ can equivalently be written in the Hermitian form

$$\tilde{H} = \omega(a^\dagger a + \tfrac{1}{2}) + \sqrt{\alpha\beta}(a^2 + {a^\dagger}^2) \ .$$

which can be diagonalized with a standard Bogoliubov transformation

$\Longrightarrow$ SHO with frequency $\Omega = \sqrt{\omega^2 - 4\alpha\beta}$

In general: if $S$ hermitizes $H$, then $H$ is quasi-Hermitian wrt $T = S^\dagger S$. For

$$H = \omega(a^\dagger a + \frac{1}{2}) + \alpha a^2 + \beta {a^\dagger}^2$$

$S = (\frac{\alpha}{\beta})^{\hat{n}/4}$ (with $\hat{n} = a^\dagger a$) transforms $H$ into $\tilde{H}$ above, where $\tilde{H} = SHS^{-1} = \tilde{H}^\dagger$

As noted, via standard *unitary* Bogoliubov transformation this yields the same (correct) spectrum as obtained by Swanson. Note that *both* $H$ and $\hat{n}$ are quasi-Hermitian wrt $T = (\frac{\alpha}{\beta})^{\hat{n}/2}$.

Notes:

- $H$ and $\hat{n}$ are clearly irreducible on $\mathcal{H}$ (when we restrict to either subspace of $n$ even or $n$ odd)
- $\hat{n}$ is not quasi-Hermitian wrt $U$ (previous construction) *without* further conditions on $g_i$ (use $Uc = d^\dagger U$ and $Ud = c^\dagger U$)
  enforcing $U\hat{n} = \hat{n}U$ requires $g_l/g_2 = g_4^*/g_3^*$, which fixes the $g_i$ *uniquely* (the other condition on the $g_i$'s is consistent with $[d, c] = 1$
- boundedness of $T = (\frac{\alpha}{\beta})^{\hat{n}/2}$ ? Consider $T^{-1}$ and $H^\dagger$ (more general cure?)
- $\tilde{H}$ is Hermitian for $\alpha\beta > 0$;
  the condition $\omega^2 - 4\alpha\beta \geq 0$ therefore does *not* define the full parameter space for real eigenvalues – only validity of Bogoliubov transformation

- even for $\alpha\beta < 0$ and $|\alpha\beta|$ small enough, eigenvalues are still real – transform to *anti*-Hermitian interaction and use 2nd-order perturbation
- however, real eigenvalues *do not* guarantee that $T$ exists - see SGH Annals paper

# Construction of metric $T$ for different choices of observables

Choose respectively $n$, $x$ and $p$ as the physical observable which, together with $H$, forms the set of irreducible operators which defines the metric uniquely.

The startegy here will be to use Swanson's results for $U$ (equivalent to $T$) and demand in turn that $U\theta = \theta U$ for each of $\theta = n\ , x\ , p$. Tansform between $a\ , a^\dagger \leftrightarrow c\ , d$ and use $Uc = d^\dagger U$ and $Ud = c^\dagger U$

This produces the following results:

Number operator $n$: $Un = nU$ produces a difference equation $U(n+1) = \frac{g_3}{g_2} U(n)$ with solution

$$U(n) = \left(\frac{g_3}{g_2}\right)^n = \left(\frac{\alpha}{\beta}\right)^{n/2}$$

For $x$ and $p$ one finds simple differential equations with solutions

$$U(x) = \exp\left(\frac{\alpha - \beta}{2(\omega - \alpha - \beta)} x^2\right)$$

and

$$U(p) = \exp\left(-\frac{\alpha - \beta}{2(\omega + \alpha + \beta)} p^2\right)$$

These results have also been obtained and discussed by Jones, JPA **38** (2005) 1741

One can now continue to calculate matrix elements of various physical quantities, by recalling that $S = \sqrt{T}$ hermitizes $H$ and that the inverse tansformation can be used to obtain the operators $X$ and $P$ which should be viewed as equivalent to $x$ and $p$, viz $X = S^{-1} x S$, etc

# Moyal bracket calculation of the metric

Using the standard relations $a^\dagger = (\hat{x} - i\hat{p})/\sqrt{2}$ and $a = (\hat{x} + i\hat{p})/\sqrt{2}$ the quadratic boson Hamiltonian

$$H = \omega a^\dagger a + \alpha a a + \beta a^\dagger a^\dagger$$

is easily expressed (up to a constant) as

$$H = A\hat{p}^2 + B\hat{x}^2 + iC\hat{p}\hat{x}\,,$$

with

$$A = (\omega - \alpha - \beta)/2,\; B = (\omega + \alpha + \beta)/2,\; C = (\alpha - \beta)\,.$$

On the level of the Moyal bracket formulation this becomes

$$\begin{aligned} H(x,p) &= Ap^2 + Bx^2 + iCpx; \\ H^\dagger(x,p) &= Ap^2 + Bx^2 - iCpx + C\,. \end{aligned}$$

The equation for the metric (with $\hbar$ explicitly shown) is

$$
\begin{aligned}
& C\,(\hbar - 2\,i\,p\,x)\;T \\
+\;& \hbar\,\Big((C\,p - 2\,i\,B\,x)\,T^{(0,1)} + (C\,x + 2\,i\,A\,p)\,T^{(1,0)}\Big) \\
+\;& \hbar^2\,\Big(B\,T^{(0,2)} - A\,T^{(2,0)}\Big) = 0\,,
\end{aligned}
$$

where $T = T(x,p)$ and $T^{(m,n)} = \frac{\partial^{m+n}T}{\partial x^m \partial p^n}$.

An exact solution for $T$ is found as a one parameter family of metrics, given by

$$T(x,p) = e^{r\,p^2 + s\,p\,x + t\,x^2}\,,$$

where

$$r \;=\; \frac{-C \pm \sqrt{C^2 - 4\,A\,B\,\hbar\,s\,(2i - \hbar s)}}{4\,B\,\hbar}\,,$$

$$t \;=\; \frac{C \pm \sqrt{C^2 - 4\,A\,B\,\hbar\,s\,(2i - \hbar s)}}{4\,A\,\hbar}\,,$$

$s$ being a free parameter.

Existence of above one parameter family of solutions of course reflects a one-parameter symmetry in the defining differential eq for $T(x,p)$.

(Earlier also noted by Geyer, Scholtz and Snyman (PHHQP2) and Jones, JPA **38** (2005) 1741)

$T(x,p) = e^{r\,p^2 + s\,p\,x + t\,x^2}$ is not the most general metric.

$$TH = H^\dagger T \Longrightarrow g(H)Tg(H^\dagger) \text{ also a solution .}$$

Moyal product formulation is

$$g(H(x,p)) * T(x,p) * e^{-i\hbar\partial_x\partial_p} g(H^\dagger(x,t))$$

For simplicity choice here restricted to $g = 1$; $\Longrightarrow$ particular choice of boundary conditions.

$T(x,p) = e^{r\,p^2 + s\,p\,x + t\,x^2}$ satisfies the hermiticity condition

$$A^*(x,p) = e^{-i\hbar\partial_x\partial_p} A(x,p)$$

provided that $s$ is chosen so that $r$ and $t$ are real;
ensured for $s = iq$ with $0 \le q \le 2/\hbar$ if $AB \ge 0$ and $q \le 0$ or $q \ge 2/\hbar$ if $AB \le 0$;
metric then Hermitian for all strength parameters $A$, $B$, $C$.

Construction does not guarantee that metric $T$ is positive definite

Check by verifying that the logarithm of the metric is hermitian $\Longrightarrow$ in Moyal product formulation one has to find the function corresponding to the logarithm of the metric operator and verify that it satisfies hermiticity relation above.

For arbitrary $s$ computation of the function corresponding to the logarithm of the metric operator is technically involved $\Longrightarrow$ find $\eta(x,p)$ such that $1+\eta+\frac{1}{2!}\eta*\eta+\frac{1}{3!}\eta*\eta*\eta*+\ldots=T$
Note that $\eta$ is not simply the logarithm of $T(x,p)$

For $s=0$ (and real $r$ and $t$) Moyal product reduces to an ordinary product
Now simply compute $\log T(x,p)$, giving
$-C\,p^2/2B\hbar$ or $C\,x^2/2A\hbar$ (depending on $r=0$ or $t=0$)
Recall $A=(\omega-\alpha-\beta)/2$, $B=(\omega+\alpha+\beta)/2$, $C=(\alpha-\beta)$

$\log T(x,p)$ now trivially satisfies hermiticity condition $\Longrightarrow$ metric positive definite

# Conclusions

- Non-Hermitian quadratic boson Hamiltonian offers a simple setting to explore various aspects of the role and construction of a metric in quasi-Hermitian quantum mechanics
- Explicit and exact (non-perturbative) forms of metric were constructed for different choices of an observable forming an irreducible set with the *given/chosen* Hamiltonian
- Moyal bracket construction offers an attractive way of constructing the metric (and explore eg semi-classical approximation)
- We still need a crisper focus on what we expect from, or say about observables/measurable quantities in quasi-Hermitian QM, including $\mathcal{PT}$-symmetric QM