Equivalent Hamiltonians for PT-Symmetric Versions of Dual 2-D Field Theories

H. F. Jones
Imperial College London

Outline

1. Systematics of Commutation Relations for Q, h

2. Q and h for PT-Symmetric Versions of Sine-Gordon and Massive Thirring Models†

3. How it works

4. Conclusions

†C. M. Bender, HFJ and R. J. Rivers, Phys. Lett. **B625** (2005) 333

1. Commutation Relations

Talking about Hamiltonians that are not Hermitian, but are:
- (i) PT-symmetric
- (ii) pseudo-Hermitian

i.e.

$$H^\dagger = \eta H \eta^{-1} \qquad (1.1)$$

Here η is **Hermitian** and **positive definite**, and is usefully written as

$$\eta = e^{-Q}$$

where Q is Hermitian.

Specialize to case where $H = H_0 + \varepsilon H_1$, where

- H_0 is Hermitian: $H_0^\dagger = H_0$
- H_1 is anti-Hermitian: $H_1^\dagger = -H_1$

Looking for a perturbative solution: $Q = \sum_r Q_r \varepsilon^r$ (r odd)

(i) Commutation relations for Q

In first place Eq. (1.1) becomes

$$\begin{aligned} H^\dagger &= e^{-Q} H e^{Q} \\ &= H + [H,Q] + \frac{1}{2!}[[H,Q],Q] + \frac{1}{3!}[[[H,Q],Q],Q] \\ &\quad + \cdots + \frac{1}{n!}\underbrace{[\ldots[H,Q],\ldots,Q]}_{n\ commutators} + \ldots \end{aligned}$$

Now insert $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$ and collect terms:

$$
\begin{aligned}
-2H_1 &= [H_0, Q_1] \\
0 &= [H_0, Q_3] + \frac{1}{2!}[[H_1, Q_1], Q_1] + \frac{1}{3!}[[[H_0, Q_1], Q_1], Q_1] \\
0 &= [H_0, Q_5] + \frac{1}{2!}([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) \\
&\quad + \frac{1}{3!}([[[H_0, Q_1], Q_1], Q_3] + perms) \\
&\quad + \frac{1}{4!}[[[[H_1, Q_1], Q_1], Q_1], Q_1] \\
&\quad + \frac{1}{5!}[[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1] \\
&\cdots\cdots
\end{aligned}
$$

Here coefficients are simple: just $1/n!$. But in each equation we can use previous equations to eliminate $[H_0, Q_r]$. They then become

$$
\begin{aligned}
[H_0, Q_1] &= -2H_1, \\
[H_0, Q_3] &= -\tfrac{1}{6}[[H_1, Q_1], Q_1], \\
[H_0, Q_5] &= -\tfrac{1}{6}\left([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]\right) \\
&\quad + \tfrac{1}{360}[[[[H_1, Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

$$
\begin{aligned}
[H_0, Q_7] \quad = \quad & -\tfrac{1}{6}[[H_1, Q_3], Q_3] \\
& -\tfrac{1}{6}([[H_1, Q_1], Q_5] + [[H_1, Q_5], Q_1]) \\
& +\tfrac{1}{360}([[[[H_1, Q_1], Q_1], Q_1], Q_3] + perms) \\
& -\tfrac{1}{15120}[[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

$\ldots\ldots$

Q1: What are these coefficients, and what is general coefficient?

Well, c_n, coefficient of 2n-fold commutator, is

$$\begin{aligned} c_n &= -\frac{2B_{2n}}{(2n)!}, \\ &= \text{coefficient of } z^{2n} \text{ in } -z\coth\tfrac{1}{2}z \end{aligned}$$

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Q2: Why?

Well, $\exists$ a recursion relation for c_n:

$$c_n = -\frac{1}{(2n)!} - \underbrace{\sum_{r=1}^{n} \frac{c_{n-r}}{(2r+1)!}}_{\textit{from } H_0 \textit{ comm}^s} \qquad (1.2)$$

- same as one of recursion relations for c_n in

$$-z \coth \tfrac{1}{2} z = \sum_{n=0}^{\infty} c_n z^{2n}.$$

Multiply both sides by $e^{z/2}$:

$$-z(e^z + 1) = \sum_{n=0}^{\infty} c_n z^{2n} (e^z - 1),$$

$i.e.$

$$-z\left(1 + \sum_{n=0}^{\infty} \frac{z^n}{n!}\right) = \sum_{m=0}^{\infty} c_m z^{2m} \sum_{s=1}^{\infty} \frac{z^s}{s!}$$

Equating coefficients of z^{2n+1} gives precisely Eq. (1.2).

Commutation relations for h

Equivalent Hermitian Hamiltonian is

$$\begin{aligned} h &= e^{-Q/2} H e^{Q/2} \\ &= H + \frac{1}{2}[H,Q] + \frac{1}{4\times 2!}[[H,Q],Q] + \frac{1}{8\times 3!}[[[H,Q],Q],Q] \\ &\quad + \cdots + \frac{1}{2^n \times n!}\underbrace{[\ldots[H,Q],\ldots,Q]}_{n\ commutators} + \ldots \end{aligned} \tag{1.3}$$

Now insert $H = H_0 + \varepsilon H_1$, $Q = \sum_r Q_r \varepsilon^r$, $h = \sum_{r\ even} h_r \varepsilon^r$:

First few equations are

$$
\begin{aligned}
h_0 &= H_0 \\
h_2 &= \frac{1}{2}[H_1, Q_1] + \frac{1}{4 \times 2!}[[H_0, Q_1], Q_1] \\
h_4 &= \frac{1}{2}[H_1, Q_3] + \frac{1}{4 \times 2!}([[H_0, Q_1], Q_3] + [[H_0, Q_3], Q_1]) \\
&\quad + \frac{1}{8 \times 3!}[[[H_1, Q_1], Q_1], Q_1] \\
&\quad + \frac{1}{16 \times 4!}[[[[H_0, Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

$$
\begin{aligned}
h_6 &= \frac{1}{2}[H_1, Q_5] + \frac{1}{4 \times 2!}([[H_0, Q_1], Q_5] + [[H_0, Q_5], Q_1]) \\
&+ \frac{1}{4 \times 2!}[[H_0, Q_3], Q_3] \\
&+ \frac{1}{8 \times 3!}([[[H_1, Q_1], Q_1], Q_3] + perms) \\
&+ \frac{1}{16 \times 4!}([[[[H_0, Q_1], Q_1], Q_3], Q_1] + perms) \\
&+ \frac{1}{32 \times 5!}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1] \\
&+ \frac{1}{64 \times 6!}[[[[[[H_0, Q_1], Q_1], Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

Again, coefficients are simple: just $1/(2^n n!)$. But eliminating $[H_0, Q_r]$ gives

$$
\begin{aligned}
h_0 &= H_0 \\
h_2 &= \frac{1}{4}[H_1, Q_1] \\
h_4 &= \frac{1}{4}[H_1, Q_3] - \frac{1}{192}[[[H_1, Q_1], Q_1], Q_1] \\
h_6 &= \frac{1}{4}[H_1, Q_5] - \frac{1}{192}([[[H_1, Q_1], Q_1], Q_3] + perms) \\
&\quad + \frac{1}{7680}[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

$\cdots\cdots$

Q3: What are these coefficients, and what is general coefficient?

Well, a_n, coefficient of $(2n-1)$-fold commutator, is

$$a_n = -\frac{E_{2n-1}(0)}{2^{2n-1} \times (2n-1)!},$$
$$= \text{coefficient of } z^{2n-1} \text{ in } \tanh(z/4)$$

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Q4: Why?

Well, recursion relation for a_n is:

$$a_n = \frac{1}{2^{2n-1} \times (2n-1)!} + \underbrace{\sum_{r=1}^{n} \frac{c_{n-r}}{(2^{2r} \times (2r)!}}_{from\ H_0\ comm^s} \qquad (1.4)$$

Can obtain same recursion relation for coefficients in

$$\tanh(z/4) = \sum_{n=1}^{\infty} a_n z^{2n-1}$$

Thus

$$z \sinh(z/2) = \sum_{n=1}^{\infty} \frac{z^{2n}}{2^{2n-1} \times (2n-1)!}$$

and

$$\begin{aligned}(z \coth(z/2))(\cosh(z/2) - 1) &= -\Big(\sum_{m=0}^{\infty} c_m z^{2m}\Big)\Big(\sum_{r=1}^{\infty} \frac{z^{2r}}{(2^{2r} \times (2r)!}\Big) \\ &= -\sum_{n=1}^{\infty} z^{2n} \sum_{r=1}^{n} \frac{c_{n-r}}{(2^{2r} \times (2r)!}\end{aligned}$$

Subtracting gives

$$\begin{aligned}\sum_{n=1}^{\infty} a_n z^{2n} &= z[\sinh(z/2) - \coth(z/2)(\cosh(z/2) - 1)] \\ &= z \tanh(z/4) \text{ !}\end{aligned}$$

2. PT-Symmetric Versions of Sine-Gordon and Massive Thirring Models

(i) Modified Sine-Gordon model

Conventional S-G model, in [1+1], has

$$\mathcal{L}_{\mathsf{SG}} = \frac{1}{2}(\partial_\mu \varphi)^2 + \frac{M^2}{\lambda^2}(\cos \lambda\varphi - 1),$$

or correspondingly

$$\mathcal{H}_{\mathsf{SG}} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1 - \cos \lambda\varphi).$$

Modified version (PT-symmetric, but not Hermitian):

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\varphi)^2 + \frac{M^2}{\lambda^2}(1 - \cos \lambda\varphi - i\varepsilon \sin \lambda\varphi),$$

So $\mathcal{H}_0 = \mathcal{H}_{\mathsf{SG}}$, and $\mathcal{H}_1 = -i(M^2/\lambda^2)\sin\lambda\varphi$.

First few equations for Q_n are:

$$
\begin{aligned}
[H_0, Q_1] &= -2H_1, \\
[H_0, Q_3] &= -\tfrac{1}{6}[[H_1, Q_1], Q_1], \qquad (2.1) \\
[H_0, Q_5] &= -\tfrac{1}{6}\left([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]\right) \\
&\quad + \tfrac{1}{360}[[[[H_1, Q_1], Q_1], Q_1], Q_1]
\end{aligned}
$$

Ansatz for Q_1:

$$Q_1 = \xi_1 \int_x \Pi_x.$$

Then

$$
\begin{aligned}
[H_0, Q_1] &= \xi_1 \int_{xy} \left[\frac{1}{2} \partial_1 \varphi_x \partial_1 \varphi_x - \frac{M^2}{\lambda^2} \cos \lambda\varphi, \Pi_y \right] \\
&= i\xi_1 \int_x \Big(\underbrace{\partial_1^2 \varphi_x}_{\int \to 0} + \underbrace{\frac{M^2}{\lambda} \sin \lambda\varphi}_{\propto \mathcal{H}_1} \Big),
\end{aligned}
$$

$\because [\varphi_x, \Pi_y] = i\delta_{xy}$

So $[H_0, Q_1] = -2H_1$, if $\boxed{\xi_1 = 2/\lambda}$

Similarly set $Q_3 = \xi_3 \int_x \Pi_x$

Then (2.1) $\Rightarrow$ $\boxed{\xi_3 = \xi_1/3}$

Similarly $\boxed{\xi_5 = \xi_1/5}$

Seem to be generating series for $\tanh^{-1}\varepsilon$, giving all-orders result

$$Q = \frac{2\delta}{\lambda}\int_x \Pi_x$$

where $\delta = \tanh^{-1}\varepsilon$

N.B. Only makes sense for $|\varepsilon| < 1$.

Can verify result *a posteriori* by constructing

$$h = e^{-(\delta/\lambda)\int_x \Pi_x} H e^{(\delta/\lambda)\int_x \Pi_x}$$

This just shifts φ: $\varphi \to \varphi + i\delta/\lambda$.

Thus

$$\begin{aligned} \cos\lambda\varphi + i\varepsilon\sin\lambda\varphi &\equiv \text{sech}\delta\cos(\lambda\varphi - i\delta) \\ &\longrightarrow \text{sech}\delta\cos\lambda\varphi \end{aligned}$$

$\therefore$ h is again S-G model, but with bare mass M' given by

$$(M')^2 = M^2\text{sech}\delta = M^2(1-\varepsilon^2)^{\frac{1}{2}} \tag{2.2}$$

Now can understand restriction $|\varepsilon| < 1$: PT spontaneously broken for $|\varepsilon| > 1$.

(ii) Modified massive Thirring model

Conventional MT model, in [1+1], has

$$\mathcal{L}_{\mathsf{MT}} = \bar{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}g(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi),$$

or correspondingly

$$\mathcal{H}_{\mathsf{MT}} = \bar{\psi}(-i\nabla\!\!\!\!/ + m)\psi - \frac{1}{2}g(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi).$$

This is equivalent to S-G model, with correspondence

$$\frac{\lambda^2}{4\pi} = \frac{1}{1 - g/\pi}, \quad M^2 = m\Lambda \quad (\Lambda = \text{ren. scale})$$

In particular $\lambda = \sqrt{4\pi} \leftrightarrow g = 0$, the free fermion theory.

Modified version involves a "γ_5 - dependent mass":

$$\mathcal{H} = \bar{\psi}(-i\not{\nabla} + m(1+\varepsilon\gamma_5))\psi - \frac{1}{2}g(\bar{\psi}\gamma^{\mu}\psi)(\bar{\psi}\gamma_{\mu}\psi)$$

Here $\gamma_0 = \sigma_1$, $\gamma_1 = i\sigma_2$, $\gamma_5 \equiv \gamma_0\gamma_1 = -\sigma_3$.
Additional term is PT-symmetric but non-Hermitian.

First consider $g = 0$ ($\lambda = \sqrt{4\pi}$).

Write

$$\begin{aligned} Q_1 &= \int_{xy} \psi_x^{\dagger}(G_1)_{xy}\psi_y \\ H_0 &= \int_{xy} \psi_x^{\dagger} D_{xy}\psi_y, \end{aligned}$$

where $D = \gamma_0(-i\not\nabla + m) = -i\gamma_5\partial_1 + m\gamma_0$.

Using canonical equal-time anti-comm^n rel^n $\{\psi_x^\dagger, \psi_y\} = \delta_{xy}$, eq^n

$$[H_0, Q_1] = -2H_1,$$

reads

$$\begin{aligned} -2m\int \bar\psi\gamma_5\psi &= \int [\psi^\dagger D\psi, \psi^\dagger G_1\psi] \\ &= \int \psi^\dagger [D, G_1]\psi \\ &= \int \psi^\dagger[-i\gamma_5\partial_1 + \underline{m\gamma_0}, G_1]\psi \end{aligned}$$

Particular solution is $\boxed{G_1 = -\gamma_5}$

Similarly set $G_3 = -\xi_3 \gamma_5$.

Then (2.1) $\Rightarrow$ $\boxed{\xi_3 = 1/3}$

Similarly third eq^n gives $\boxed{\xi_5 = 1/5}$

Again, seem to be generating series for $\tanh^{-1} \varepsilon$, with all-orders result

$$Q = -\delta \int_x (\psi^\dagger \gamma_5 \psi)_x,$$

where again $\delta = \tanh^{-1} \varepsilon$

Check result by constructing

$$h = \exp\left(\frac{1}{2}\delta\int_x (\psi^\dagger\gamma_5\psi)_x\right) H \exp\left(-\frac{1}{2}\delta\int_x (\psi^\dagger\gamma_5\psi)_x\right)$$

By virtue of Lorentz-like comm$^{\mathrm{n}}$ relations

$$\begin{aligned} [\gamma_5, \gamma_0] &= 2\gamma_1, \\ [\gamma_5, \gamma_1] &= 2\gamma_0, \end{aligned}$$

this is just

$$\boxed{h = \bar{\psi}(-i\not{\nabla} + \mu)\psi}$$

where $\mu = m\,\mathrm{sech}\delta = m(1-\varepsilon^2)^{\frac{1}{2}}$, in agreement with (2.2).

N.B. This Q also works for $g \neq 0$, since

$$(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) = (\psi^\dagger\psi)^2 - (\psi^\dagger\gamma_5\psi)^2.$$

Each term on RHS commutes with $Q = -\delta\int_x (\psi^\dagger\gamma_5\psi)_x$.

Thus only effect on H is to change γ_5-dependent mass term $m\bar{\psi}(1+\varepsilon\gamma_5)\psi$ to a normal mass term $\mu\bar{\psi}\psi$.

Again, if $|\varepsilon| > 1$, PT symmetry is broken.

Another Solution

As usual, Q is not unique. Another possible (formal) solution is

$$G = i\gamma_5 \tanh^{-1}\left(\frac{\varepsilon m}{\not\nabla}\right)!$$

This mixes kinetic term and γ_5 mass term.
Corresponding h (for $g = 0$) is

$$h = \bar{\psi}\left[m - i\not\nabla\left(1 + \frac{\varepsilon^2 m^2}{\nabla^2}\right)^{\frac{1}{2}}\right]\psi$$

Strange way of writing free field theory with mass μ!
Equation of motion is

$$\Big[\partial_0^2 - \nabla^2\Big(1 + \frac{\varepsilon^2 m^2}{\nabla^2}\Big)\Big]\psi = -m^2\psi$$

i.e.

$$\partial^2\psi = -\underbrace{m^2(1-\varepsilon^2)}_{\mu^2}\psi \ !$$

This Q only defined in a limited range of p-space.

3. How does this work?

(i) Where does $\tanh^{-1}$ come from?

Important feature here: $Q_r = \alpha_r Q_1$ - all with same structure - so that $Q = (\sum_r^{\infty} \alpha_r \varepsilon^r) Q_1$.

$\therefore$ from

$$\begin{aligned} [H_0, Q_1] &= c_0 H_1, \\ [H_0, Q_3] &= c_1 [[H_1, Q_1], Q_1], \end{aligned}$$

get

$$[[H_1, Q_1], Q_1] = \alpha_3 \frac{c_0}{c_1} H_1 = \underline{4 H_1}$$

$(\alpha_3 = 1/3,\ c_0 = -2,\ c_1 = -1/6)$

Then eq$^{\text{n}}$

$$[H_0, Q_5] = c_1 ([[H_1, Q_1], Q_3] + [[H_1, Q_3], Q_1]) + c_2[[[[H_1, Q_1], Q_1], Q_1], Q_1]$$

becomes

$$-2\alpha_5 H_1 = \left[2(4c_1)\alpha_1\alpha_3 + (16c_2)\alpha_1^4\right] H_1$$

$\Rightarrow \alpha_5 = 1/5$

In fact all equations can be generated by expansion of

$$-2 \sum_{r\,odd}^{\infty} \alpha_r \varepsilon^r = \varepsilon \sum_{r=0}^{\infty} \underbrace{4^r c_r}_{\hat{c}_r} (\alpha_1 \varepsilon + \alpha_3 \varepsilon^3 + \alpha_5 \varepsilon^5 + \dots)^{2r} \qquad (3.1)$$

namely

$$
\begin{aligned}
-2\alpha_1 &= \hat{c}_0 \\
-2\alpha_3 &= \hat{c}_1\alpha_1^2 \\
-2\alpha_5 &= 2\hat{c}_1\alpha_1\alpha_3 + \hat{c}_2\alpha_1^4 \\
-2\alpha_7 &= \hat{c}_1(2\alpha_1\alpha_5 + \alpha_3^2) + 4\hat{c}_2\alpha_1^3\alpha_3 + \hat{c}_3\alpha_1^6
\end{aligned}
$$

$$\cdots\cdots$$

Multiplicities are no. of perm$^{\text{ns}}$ of a given commutator structure.

Then, assuming that α_r are indeed coefficients of arctanh, Eq. (3.1) reads

$$
\begin{aligned}
-2\delta &= \varepsilon \sum_r^{\infty} c_r (2\delta)^{2r} \\
&= \tanh\delta\,(-2\delta\coth\delta) \quad \checkmark\checkmark\checkmark
\end{aligned}
$$

(ii) Where does sech come from?

Well, with $[[H_1, Q_1], Q_1] = 4H_1$, $[H_1, Q_1] = -2H_0$, eq$^{\mathrm{s}}$ for h:

$$
\begin{aligned}
h_2 &= a_1[H_1, Q_1] \\
h_4 &= a_1[H_1, Q_3] + a_2[[[H_1, Q_1], Q_1], Q_1], \\
h_6 &= a_1[H_1, Q_5] + a_2([[[H_1, Q_1], Q_1], Q_3] + perms) \\
&\quad + a_3[[[[[H_1, Q_1], Q_1], Q_1], Q_1], Q_1],
\end{aligned}
$$

become

$$
\begin{aligned}
h_2 &= -2\hat{a}_1\alpha_1 H_0 \\
h_4 &= -2(\hat{a}_1\alpha_3 + \hat{a}_2\alpha_1^3)H_0 \\
h_6 &= -2(\hat{a}_1\alpha_5 + 3\hat{a}_2\alpha_1^2\alpha_3 + \hat{a}_3\alpha_1^5)H_0
\end{aligned}
$$

where $\hat{a}_r = 4^{r-1}a_r$.

Again, $h \equiv \sum h_r \varepsilon^r$ is generated by

$$\begin{aligned} & 1 - 2\varepsilon \sum_{r=1}^{\infty} \widehat{a}_r(\alpha_1\varepsilon + \alpha_3\varepsilon^3 + \alpha_5\varepsilon^5 + \ldots)^{2r-1} \\ = \; & 1 - \varepsilon \sum_{r=1}^{\infty} a_r(2\delta)^{2r-1} \\ = \; & 1 - \tanh\delta \times \tanh(\delta/2) \\ = \; & \mathrm{sech}\delta \text{ !!} \end{aligned}$$

4. Conclusions

(i) Have identified the coefficients of multiple commutators involved in perturbative expansions for Q and h:

$$
\begin{aligned}
H^{\dagger} &= e^{-Q} H e^{Q} \\
h &= e^{-\frac{1}{2}Q} H e^{\frac{1}{2}Q}
\end{aligned}
$$

(ii) Have found exact expressions for Q and h in PT-symmetric versions of Sine-Gordon and Massive Thirring models.

(iii) Have understood these solutions in terms of properties of coefficients of (i)