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# Coupling of channels in PT-symmetric models

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# I. A coupling of channels?

It's always been there!

Just recollect:

**(a)** Bender's PT symmetric potentials

$$V(x) = V_{symm}(x) + i V_{antisymm}(x),$$

$$V_{symm}^{(example)}(x) = \omega^2 x^2 + \lambda^2 x^4,$$

$$V_{antisymm}^{(example)}(x) = g x^3$$

in HO basis  $|n, \pm\rangle$

$$H = \begin{pmatrix} S & B \\ -B^T & L \end{pmatrix}, \quad P = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

**channels**, decoupled iff  $g \rightarrow 0$ .

(b) relativistic Sakata - Taketani:

$$H = \begin{pmatrix} 0 & K \\ I & 0 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

in fixed-frame evolution:

$$H \begin{pmatrix} x \\ y \end{pmatrix} = E \begin{pmatrix} x \\ y \end{pmatrix}$$

put  $x = E y$  and reduce to  $K y = E^2 y$

require  $E^2 > 0$ , **define**  $E_{\pm} = \pm\sqrt{E^2}$

and rotate to the **FV channels**.

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## II. Toy model with two coupled channels

**(a)** Hamiltonian:

$$H_{(kinetic)} = \begin{pmatrix} -\frac{d^2}{dx^2} & 0 \\ 0 & -\frac{d^2}{dx^2} \end{pmatrix},$$

$$H_{(interaction)} = \begin{pmatrix} V_a(x) & W_b(x) \\ W_a(x) & V_b(x) \end{pmatrix}.$$

**(b)** its  $\theta$ -pseudo-Hermiticity:

$$\theta = \theta^\dagger = \begin{pmatrix} 0 & \mathcal{P} \\ \mathcal{P} & 0 \end{pmatrix} = \theta^{-1}$$

**(c)** potentials [ $x \in (-1, 0)$ ]:

$$\text{Im } W_a(x) = X > 0,$$

$$\text{Im } W_b(x) = Y > 0,$$

$$\text{Im } V_a(x) = \text{Im } V_b(x) = Z,$$

**(d)** spin-like ( $\sigma = \pm 1$ ) symmetry:

$$\Omega = \begin{pmatrix} 0 & \omega^{-1} \\ \omega & 0 \end{pmatrix}, \quad \omega = \sqrt{\frac{X}{Y}} > 0.$$

**(e) solvable** (details see below)

**(f) physical** (details see below)

(g) **simple** in a modified Dirac's notation

$$H|E, \sigma\rangle = E|E, \sigma\rangle, \quad \Omega|E, \sigma\rangle = \sigma|E, \sigma\rangle$$

$$\langle\langle E, \sigma|H = E\langle\langle E, \sigma|, \quad \langle\langle E, \sigma|\Omega = \sigma\langle\langle E, \sigma|$$

$$\text{biog. : } 0 = \langle\langle E', \sigma'|E, \sigma\rangle \times \begin{cases} (E' - E) \\ (\sigma' - \sigma) \end{cases}$$

$$\text{cpl. : } I = \sum_{E, \sigma} |E, \sigma\rangle \frac{1}{\langle\langle E, \sigma|E, \sigma\rangle} \langle\langle E, \sigma|$$

$$\text{sp. : } H = \sum_{E, \sigma} |E, \sigma\rangle \frac{E}{\langle\langle E, \sigma|E, \sigma\rangle} \langle\langle E, \sigma|$$

$$\Omega = \sum_{E, \sigma} |E, \sigma\rangle \frac{\sigma}{\langle\langle E, \sigma|E, \sigma\rangle} \langle\langle E, \sigma|$$



III. Key problem:

Shall we be able to introduce the  
SGH metric?

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Subsummary:

Introducing a “physical”’ metric?

Perhaps only too easily!

The sleepers during the next two screens

will be also given the message

by the next speaker.

(a) **cc** metric  $\Theta$  for students:  $2 \times 2$ :

$$H \rightarrow \begin{pmatrix} -T & B \\ -B & T \end{pmatrix}, \quad \Theta = \begin{pmatrix} a & b \\ b & d \end{pmatrix}$$

$$\Theta H = H^T \Theta \implies 2bT = -B(a + d)$$

$$E \in \mathbb{R} \iff |T| \geq |B|, \quad B = T \sin \alpha$$

$$\theta_{1,2} > 0 \iff b \neq 0 \neq a + d = 2Z$$

and for  $a = Z(1 + \xi)$ ,  $d = Z(1 - \xi)$ ,

$$1 > \sqrt{\xi^2 + \sin^2 \alpha}.$$

We have an **interval** of  $\xi < \cos \alpha$  (!)

(b) biorthogonal “brabacket” basis:

$$\langle\langle n| H = \langle\langle n| E_n, \quad H |n\rangle = E_n |n\rangle$$

+ the Mostafazadeh’s universal formula:

$$\Theta = \Sigma |n\rangle\rangle s_n \langle\langle n|, \quad s_k > 0.$$

KG **cc** = **direct sum** of  $2 \times 2$  matrices:

$$H = \begin{pmatrix} 0 & B \\ I & 0 \end{pmatrix}, \quad \Theta = \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

gives  $\theta_{1,2} > 0$  for all  $d = aE^2 > 0$ ,

i.e., **full intervals** of  $|b_n| < |a_n E_n|$

(recommended choice:  $a = 1/E$ ).

**IV. Another subsummary:**

**Square wells with  $x \in (-1, 1)$ ?**

$$V(x) = V_{(Z)}(x) = -i Z \operatorname{sign}(x)?$$

**Why at all?**

**(a) = ODE** with constant coefficients:

$$-\frac{d^2}{dx^2} \varphi^{(m)}(x) + \sum_{j=1}^K V_{Z_{(m,j)}}(x) \varphi^{(j)}(x) = E \varphi^{(m)}(x), \quad m = 1, 2, \dots, K$$

= **solvable** by an ansatz for  $\varphi^{(m)}(x)$

$$= \begin{cases} C_L^{(m)} \sin \kappa_L(x + 1), & x < 0, \\ C_R^{(m)} \sin \kappa_R(-x + 1), & x > 0 \end{cases}$$

= **giving**  $Z_{(eff)}^{(m)}(K)$  as eigenvalues of

$$\begin{pmatrix} Z_{(1,1)} & Z_{(1,2)} & \cdots & Z_{(1,K)} \\ Z_{(2,1)} & Z_{(2,2)} & \cdots & Z_{(2,K)} \\ \vdots & \cdots & \cdots & \vdots \\ Z_{(K,1)} & Z_{(K,2)} & \cdots & Z_{(K,K)} \end{pmatrix}.$$

(b) **quantized** easily:

$$= \text{ansatz} \rightarrow \kappa_R = s + it = \kappa_L^*, \quad s > 0,$$

$$\rightarrow t = t_{\text{first curve}}(s) = Z_{(\text{eff})}^{(m)}(K)/(2s)$$

plus **matching** in the origin:

$$\rightarrow \kappa_L \cotan \kappa_L = -\kappa_R \cotan \kappa_R$$

gives the second, “universal” curve

$t = t_{\text{exact}}(s)$  with implicit definition

$$2s \sin 2s + 2t \sinh 2t = 0$$

$\rightarrow$  **energies** via **intersections**,

$$E_n = s_n^2 - t_n^2, \quad n = 0, 1, \dots$$

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**V. Technicalities are over at last.**

**Now there come THE IDEAS!**



**(a) the first idea: relax  $\mathbf{P} = \mathbf{P}^\dagger$**

pattern: if  $H^\dagger = \mathbf{R} H \mathbf{R}^{-1}$  and  $\mathbf{R} \neq \mathbf{R}^\dagger$ ,

we have the symmetry,

$$H \mathcal{S} = \mathcal{S} H, \quad \mathcal{S} = [\mathbf{R}^{-1}]^\dagger \mathbf{R}.$$

Let's choose  $\mathbf{R}^{-1} = \mathbf{R}^\dagger$  with  $\mathcal{S} = \mathbf{R}^2$  and

$$\mathbf{R} = \begin{pmatrix} 0 & \dots & 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & \dots & 0 & 0 \\ 0 & \mathcal{P} & 0 & \dots & 0 \\ \vdots & \dots & \dots & \dots & \vdots \\ 0 & \dots & 0 & \mathcal{P} & 0 \end{pmatrix}$$

at any  $K$ .

(b) the second idea: **take more:**

$\mathbf{R} = \mathbf{R}_{(K,L)} = \mathcal{P} \mathbf{r}_{(K,L)}$  such that

$$\mathbf{r}_{(K,L+1)} = \mathbf{r}_{(K,1)} \mathbf{r}_{(K,L)}, \quad L = 1, 2, \dots .$$

$$\left[ \mathbf{r}_{(K,L)} \right]^K = I, \quad \mathbf{r}_{(K,K-L)} = \left[ \mathbf{r}_{(K,L)} \right]^\dagger .$$

(c) necessity: **adapt  $H$  to  $\mathbf{R}$ :**

$$\mathbf{A} = \mathbf{r}_{(K,K-L)} \cdot \mathbf{A}^T \cdot \mathbf{r}_{(K,L)}$$

= solve a finite set of equations

(see the  $K = 2$  result above)

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**VI. Let us move now to  
THREE channels, with**

$$\mathbf{R}_{(3,1)} = \begin{pmatrix} 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \\ 0 & \mathcal{P} & 0 \end{pmatrix} = \mathbf{R}_{(3,2)}^\dagger = \mathbf{R}_{(3,2)}^{-1},$$

$$\mathbf{R}_{(3,2)} = \begin{pmatrix} 0 & \mathcal{P} & 0 \\ 0 & 0 & \mathcal{P} \\ \mathcal{P} & 0 & 0 \end{pmatrix} = \mathbf{R}_{(3,1)}^\dagger = \mathbf{R}_{(3,1)}^{-1}.$$

giving the **unique**

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & X \\ X & Z & X \\ X & X & Z \end{pmatrix}, \quad L = 1, 2$$

and **solutions** with the ‘first curve’

$$t = t^{(\sigma)}(s) = \frac{1}{2s} Z_{eff}(\sigma), \quad \sigma = 1, 2, 3$$

$$Z_{eff}(1) = Z + 2X \text{ and } Z_{eff}(2, 3) = Z - X$$

and with the **cc** coefficients

$$\left( C_{(1)}^{(a)}, C_{(1)}^{(b)}, C_{(1)}^{(c)} \right) \sim (1, 1, 1)$$

$$\left( C_{(2)}^{(a)}, C_{(2)}^{(b)}, C_{(2)}^{(c)} \right) \sim (1, -1, 0)$$

$$\left( C_{(3)}^{(a)}, C_{(3)}^{(b)}, C_{(3)}^{(c)} \right) \sim (1, 1, -2).$$

The energies stay real in the 2D domain

$$Y - Z_{crit} \leq Z \leq Z_{crit} - 2Y.$$

[vertices  $(0, \pm 4.475)$  and  $(2.98, -1.49)$ ].

VII. A bit of a technical interlude:

Numerics relevant!

Return, quickly, to  $K = 1!!$

**(a) weakly** non-Hermitian regime:

$$s = s_n = \frac{(n+1)\pi}{2} + \tau \frac{Q_n}{2}, \quad \tau = (-1)^n$$

→ solvable by **iterations**:

the first small quantity  $\varrho \equiv \frac{1}{L} = \frac{1}{(n+1)\pi}$

the second one  $\alpha = \frac{2Z_{eff}(\sigma)}{L}$  or  $\beta = \alpha\varrho$

→ a “generalized continued fraction”

$$Q = \arcsin \left( 2t \frac{\varrho}{1 + \tau Q \varrho} \sinh 2t \right),$$

$$\text{where } 2t = \frac{\alpha}{1 + \tau Q \varrho}$$

**(b) intermediate non-Hermiticities:**

→ *ad hoc* **perturbation theory:**

$$\rightarrow \arcsin(x) = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \dots$$

$$Q = Q(\alpha, \beta) = \alpha\beta \Omega(\alpha, \beta),$$

$$\begin{aligned} \rightarrow \Omega(\alpha, \beta) &= 1 + c_{10} \alpha^2 + c_{01} \beta^2 + \\ &+ c_{20} \alpha^4 + c_{11} \alpha^2 \beta^2 + c_{02} \beta^4 + \mathcal{O}(\alpha^6) \end{aligned}$$

→ equation **re-arranged:**

$$[1 + \tau \beta^2 \Omega(\alpha, \beta)] \operatorname{arcsinh}(\Lambda) = \alpha$$

$$\Lambda = [1 + \tau \beta^2 \Omega(\alpha, \beta)]^2 \frac{1}{\beta} \sin[\alpha\beta \Omega(\alpha, \beta)]$$



**(c) formulae:**

→ leading order relation

$$0 = \left(-\frac{1}{6} + c_{10} + c_{01} \varrho^2 + 3\tau \varrho^2\right) \alpha^3 + \dots$$

determines the first two coefficients,

$$c_{10} = \frac{1}{6}, \quad c_{01} = -3\tau,$$

the next-order  $O(\alpha^5)$  gives

$$c_{20} = \frac{1}{120}, \quad c_{11} = \frac{1-8\tau}{6}, \quad c_{02} = 15$$

and the  $1 + O(\alpha^4)$  formula

$$Q_n = \frac{4 Z_{eff}^2}{(n+1)^3 \pi^3} + \frac{8 Z_{eff}^4}{3 (n+1)^5 \pi^5} \left( 1 + \frac{18 (-1)^{n+1}}{(n+1)^2 \pi^2} \right).$$

**VIII.**

**FOUR channels**

$K = 4$  warning:  $\mathbf{R}_{(4,2)}$  is Hermitian,

mere six constraints upon 16 couplings.

Not enough symmetry for us.

Unique coupling-matrix left,

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & U & D & U \\ L & Z & L & D \\ D & U & Z & U \\ L & D & L & Z \end{pmatrix}, \quad L = 1, 3.$$

**solution :**

Four shifts of the effective  $Z$ ,

$$[-D, -D, D + 2\sqrt{UL}, D - 2\sqrt{UL}]$$

with respective eigenvectors

$$\{1, 0, -1, 0\}, \{0, 1, 0, -1\},$$

$$\{U, \pm\sqrt{UL}, U, \pm\sqrt{UL}\}.$$

**remark:**

from the pseudo-parity

$$\mathbf{r}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

repartitioned model

$$\mathbf{A}_{(interaction)}^{(permuted)} = \left( \begin{array}{cc|cc} Z & D & U & U \\ D & Z & U & U \\ \hline L & L & Z & D \\ L & L & D & Z \end{array} \right), \quad L = 1, 3.$$

IX.

**FIVE channels**

$$\mathbf{r}_{(5,1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \dots,$$

all lead to the same

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & D & D & X \\ X & Z & X & D & D \\ D & X & Z & X & D \\ D & D & X & Z & X \\ X & D & D & X & Z \end{pmatrix}.$$

→ exceptional eigenvalue  $F_0 = 2D + 2X$

giving eigenvector  $\{1, 1, 1, 1, 1\}$

→ the reduced  $Z = 0$  matrix  $A$  has the pair of the twice degenerate eigenvalues,

$$F_{\pm} = \frac{1}{2} \left[ -D - X \pm \sqrt{5} (-D + X) \right]$$

with the two respective eigenvectors

$$\{1 \mp \sqrt{5}, -1 \pm \sqrt{5}, 2, 0, -2\}$$

$$\{1 \mp \sqrt{5}, -2, 0, 2, -1 \pm \sqrt{5}\}.$$



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**X. Another important interlude  
concerning the domain where the  
energies remain real.**

(a) a numerical **algorithm**:

$$\frac{Q}{2} \Big|_{crit} \equiv \varepsilon(t_{crit}) = \pi - \frac{Z_{crit}}{2t_{crit}},$$

$$\sin [2 \varepsilon(t)] = \frac{t \sinh 2t}{\pi - \varepsilon(t)},$$

$$\varepsilon_{(lower)}(t) = \pi/4 \text{ and } \varepsilon_{(upper)}(t) = 0.$$

$$\partial_t \varepsilon(t_{crit}) = \frac{Z_{crit}}{2t_{crit}^2},$$

$$\partial_t \varepsilon(t) = \frac{\sinh 2t + 2t \cosh 2t}{2 [\pi - \varepsilon(t)] \cos 2\varepsilon(t) - \sin 2\varepsilon(t)}$$

$$\rightarrow t_{crit} \in (0.839393459, 0.839393461),$$

$$\rightarrow s_{crit} \in (2.665799044, 2.665799069),$$

$$\rightarrow E_{crit} \in (6.401903165, 6.401903294).$$

Table 1:

iteration	$Z_{crit}^{(lower)}$	$Z_{crit}^{(upper)}$
$N$		
0	4.299	4.663
2	4.4614	4.4857
4	4.47431	4.47601
6	4.475239	4.475357
8	4.47530381	4.4753119
10	4.475308262	4.475308823
12	4.475308560	4.475308614

**XI.**

**SIX channels**

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$$L = 3$$

21 free parameters

**Hermitian  $\mathbf{R}$**  and a weak symmetry,

skipped

$L = 1$  or  $L = 5$ :

$$\mathbf{r}_{(6,1)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$\mathbf{A}$  = asymmetric:

$$\mathbf{A}_{\substack{\text{(permuted)} \\ \text{(interaction)}}} = \left( \begin{array}{cc|cc|cc} Z & Y & G & B & F & B \\ X & Z & C & F & C & G \\ \hline F & B & Z & Y & G & B \\ C & G & X & Z & C & F \\ \hline G & B & F & B & Z & Y \\ C & F & C & G & X & Z \end{array} \right) \cdot$$

eigenvalues = roots of quadratic equations

two = non-degenerate

two = doubly degenerate

$L = 2$  or  $L = 4$ :

$$\mathbf{r}_{(6,2)}^{(permuted)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$



$\mathbf{A}$  = symmetric:

$$\mathbf{A}_{(interaction)}^{(permuted)} = \left( \begin{array}{ccc|ccc} Z & X & X & C & D & G \\ X & Z & X & G & C & D \\ X & X & Z & D & G & C \\ \hline C & G & D & A & B & B \\ D & C & G & B & A & B \\ G & D & C & B & B & A \end{array} \right) .$$

eigenvalues = roots of quadratic equations

two = non-degenerate

two = doubly degenerate

**XII.**

**$2M - 1$  channels with  $M = 4$  etc**

$M = 4$ : four free parameters at all  $L$ :

$$\mathbf{A}_{(interaction)} = \begin{pmatrix} Z & X & Y & D & D & Y & X \\ X & Z & X & Y & D & D & Y \\ Y & X & Z & X & Y & D & D \\ D & Y & X & Z & X & Y & D \\ D & D & Y & X & Z & X & Y \\ Y & D & D & Y & X & Z & X \\ X & Y & D & D & Y & X & Z \end{pmatrix} .$$

Cardano formulae.

## XIII.

$2M$  channels with  $M = 4$  etc

37 free parameters for  $(2M, L) = (8, 4)$

(29 in pairs),

16 free parameters for  $(2M, L) = (8, 2)$

(all in quadruplets),

8 free parameters for  $(2M, L) = (8, 1)$  etc,

(all in octuplets).

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## **XIV. The summary of the talk:**

- (1) Our particular recipe  $\mathcal{P} \rightarrow \mathcal{R}$  allowing **finite rotations** proved feasible.
- (2) Models may be useful as carrying **new nontrivial symmetries**.
- (3) Coupled-channel Hamiltonians shown **equally** appealing within PTSQM as they were in Hermitian models.
- (4) One feels encouraged to search for some **further** extensions of “quantum practice” in **quasi-Hermitian** directions.