Stellenbosch 24/11/05

On the reality (and unreality) of PT-symmetric quantum mechanics

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Introduction:

A key question in the analysis of any PT-symmetric system:

¿ is the spectrum real (or not) ?

Bad news: • unlike in standard QM, this is in general a very nontrivial question.

Good news: • the nontriviality can result in interesting maths in cases where reality does hold;

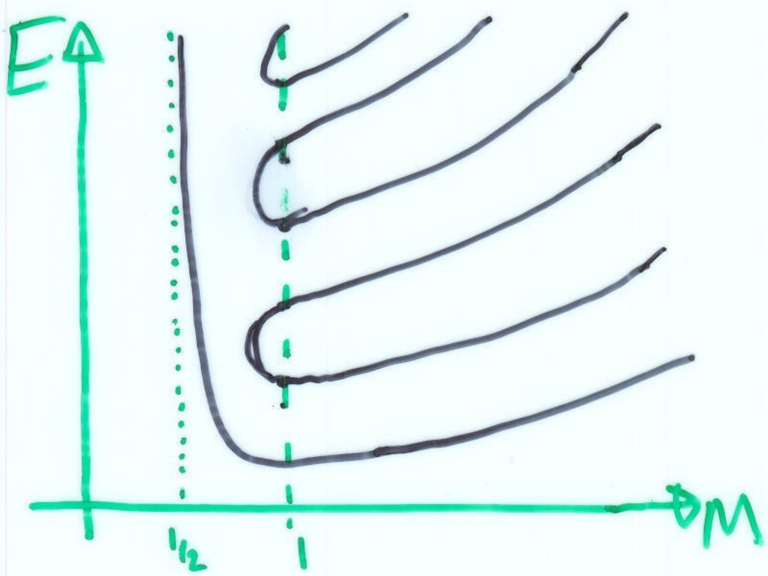
• the possibility that some levels may go complex allows for interesting new phenomena.

Examples:

1) Bender-Boettcher problem: [BB 1998]

$$\left[-\frac{d^2}{dx^2} - (ix)^{2M}\right]\psi = E\psi$$

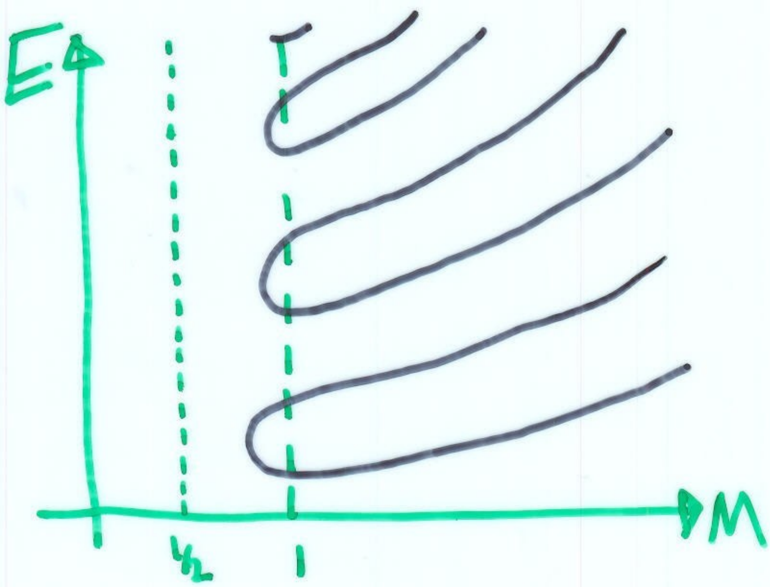
- $M=1$ SHO (exactly solved, $E=2m+1$, $m=0,1,2\ldots$)
- $M \geqslant 1$ Spectrum is real even though the problem is not Hermitian
- $M<1$ Spectrum is "infinitely complex", starting from the top



2) "Spinning" Bender-Boettcher problem: [PED&RT 1999]

$$\left[-\frac{d^2}{dx^2} - (ix)^{2M} + \frac{l(l+1)}{x^2}\right]\psi = E\psi$$

- $M=1$ SHO+AM (exactly-solved, $E = 4n+2 \pm (2l+1)$, $n=0,1,2\ldots$)
- $M \geq 1$ Spectrum <u>real</u>
- $M<1$ Spectrum infinitely complex, starting from the top.



($l = -0.025$)

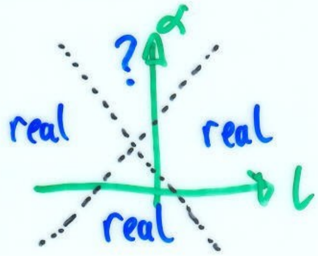
[NB: interesting difference between $l=0$ and $l=-0.025$]

3) Inhomogeneous spinning Bender-Boettcher problem:

[PED, CD, RT 2001]

$$\left[-\frac{d^2}{dx^2} - (ix)^{2M} - \alpha (ix)^{M-1} + \frac{l(l+1)}{x^2}\right]\psi = E\psi$$

- For $M>1$, the spectrum is <u>proved</u> to be real
  for $\alpha < M+1+|2l+1|$



- There are now <u>two</u> ways for the spectrum to go complex:

  (1) If M dips below 1, spectrum becomes <u>infinitely</u> complex, starting from the <u>top</u>, as before.

  (2) For $M>1$, (l,α) can venture into the forbidden zone
  $\alpha > M+1+|2l+1|$.
  Then the spectrum becomes (can become) <u>finitely</u> complex, starting from (near) the <u>bottom</u>.

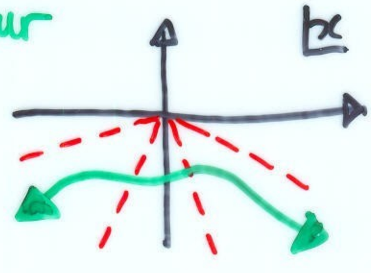
The $M=3$ case is particularly nice –

- set $\rho = \sqrt{3}(2l+1)$; then the problem is

$$\left[-\frac{d^2}{dx^2} + x^6 + \alpha x^2 + \frac{\rho^2 - 3}{12x^2}\right]\psi = E\psi$$

- NB1: why isn't the spectrum trivially real?

Answer: the b.c.s must continue those for SHO, & be imposed on the complex contour



- NB2: why the variable change $l \to \rho$?

Answer: a hidden relation with $su(3)$

- NB3: $\exists$ an altenative way to make a non-Hermitian problem:

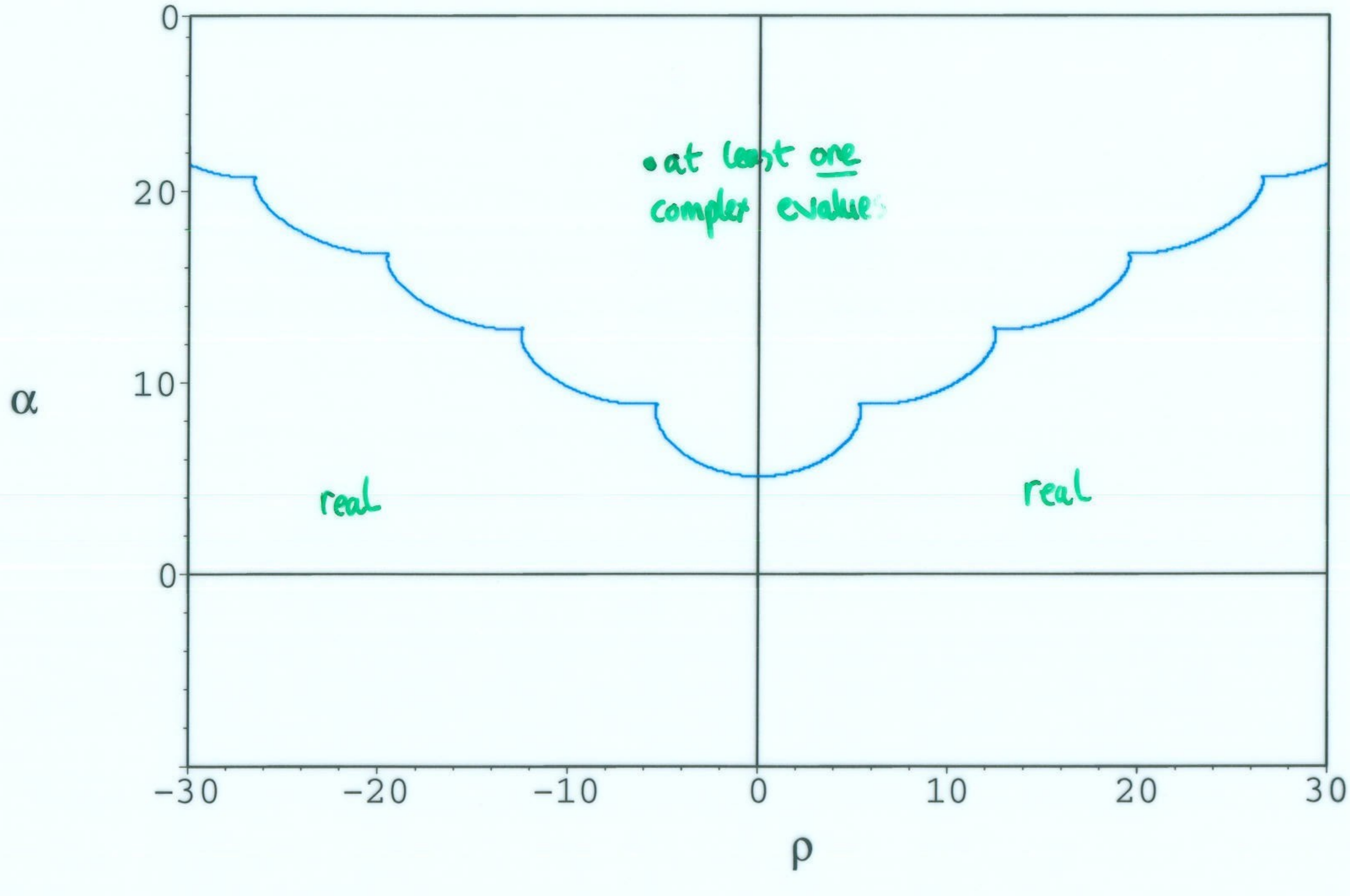
Leave x real, but impose 'radial' b.c.s

$\psi \to 0 \quad (x \to \infty)$

$\psi \sim x^{l+1} \, (x \to 0)$ and continue to negative l

(irregular b.c.)

* For $M=3$, this is related to the PT problem ! *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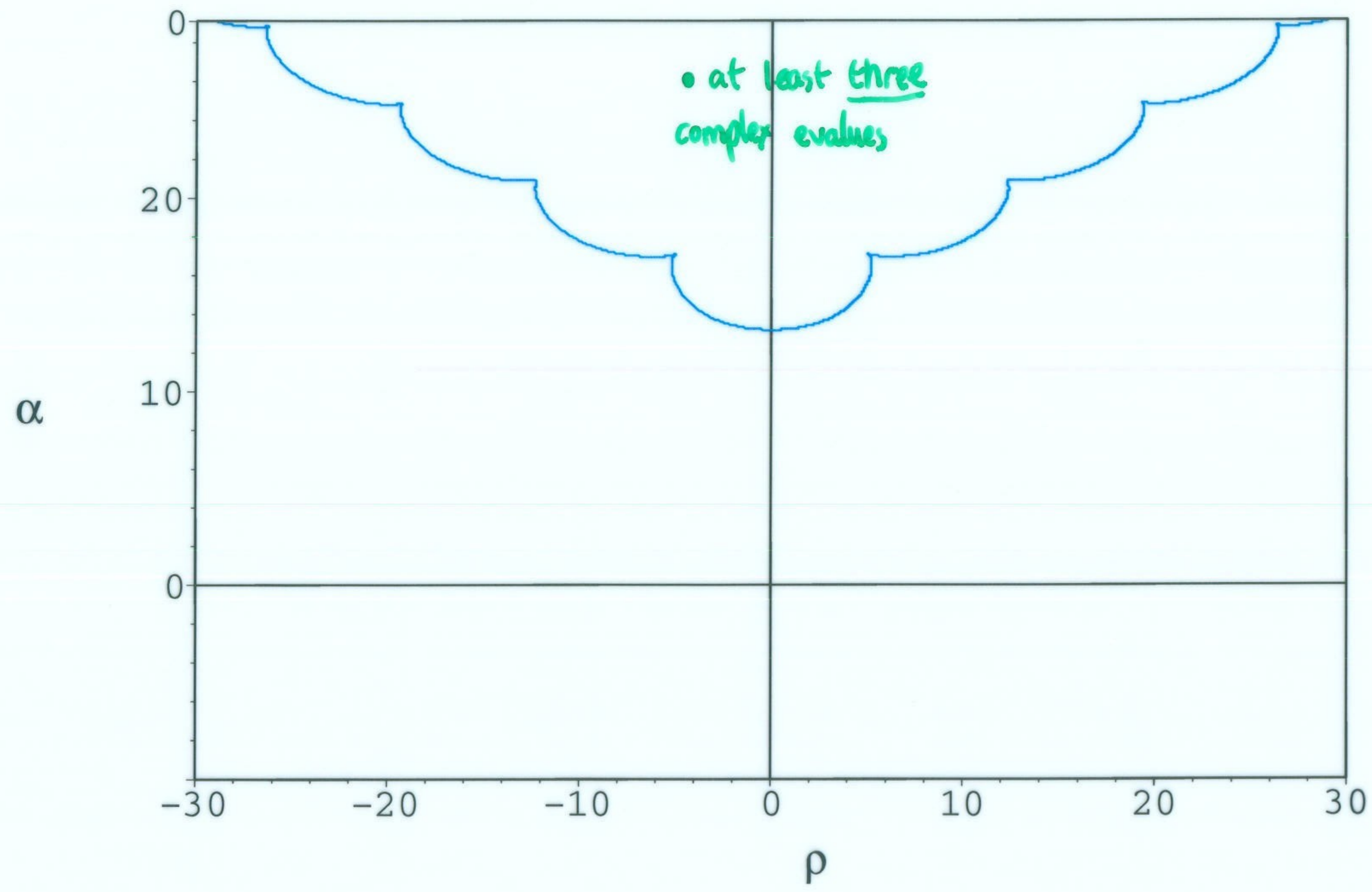

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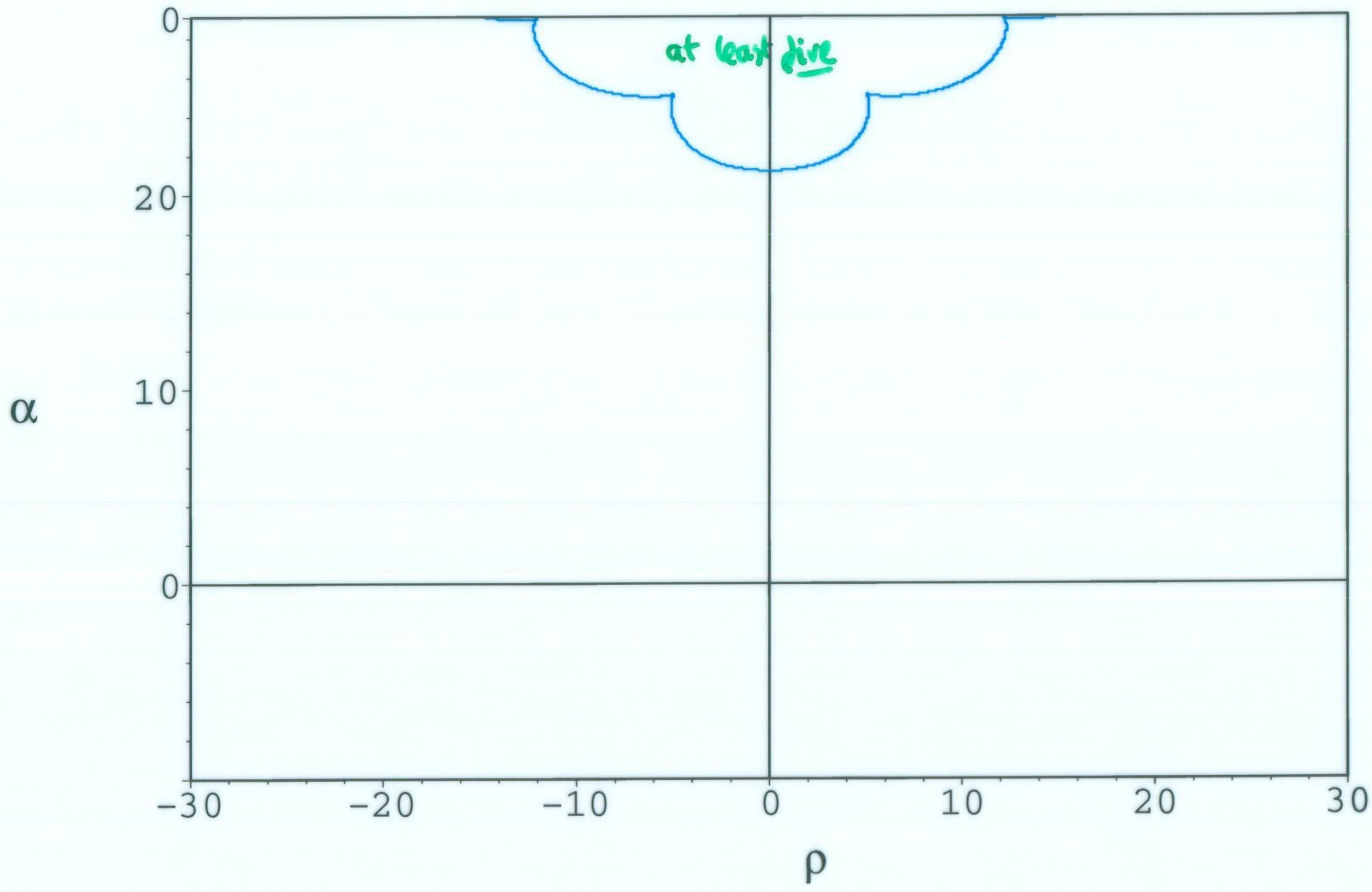

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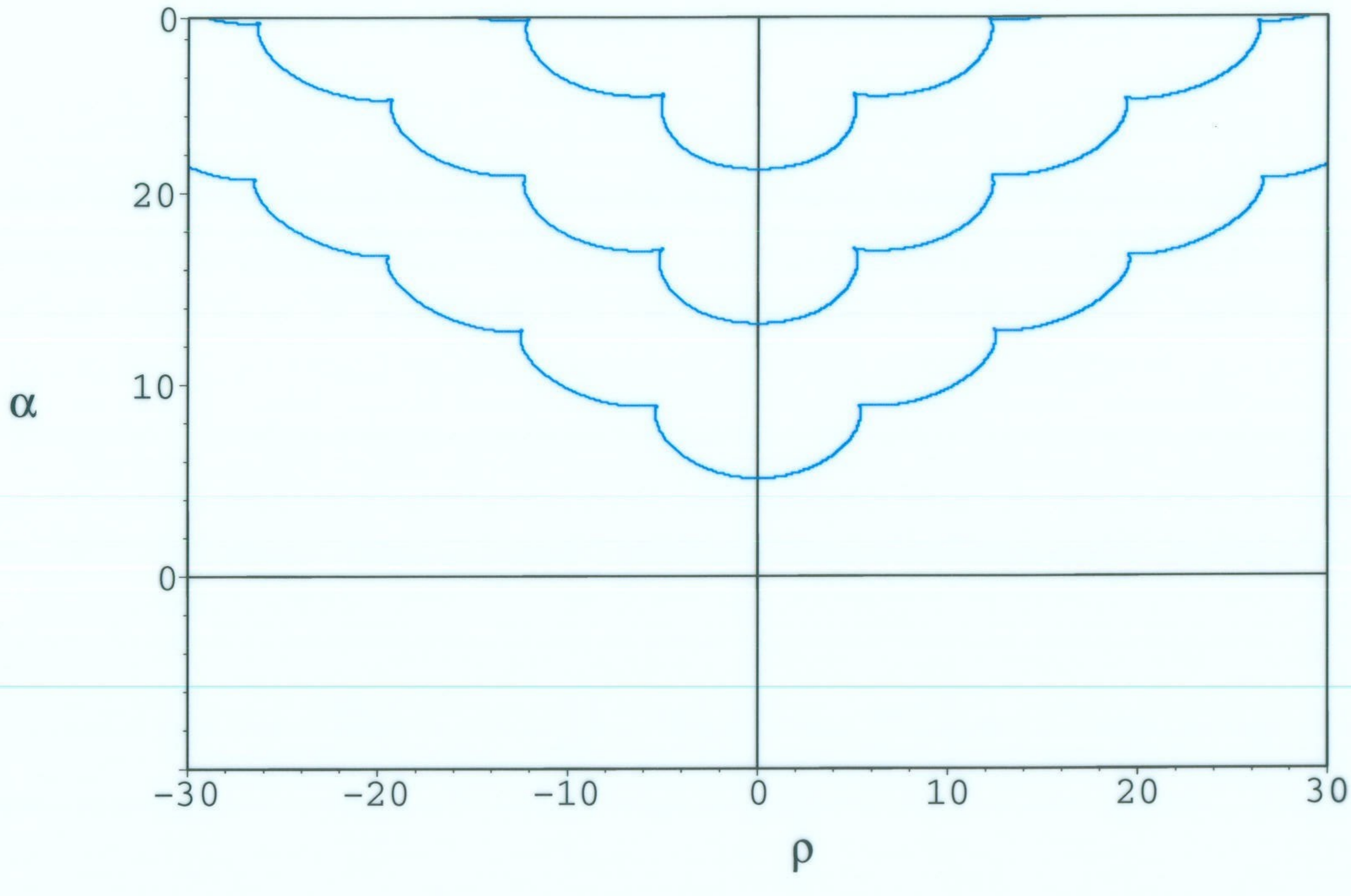

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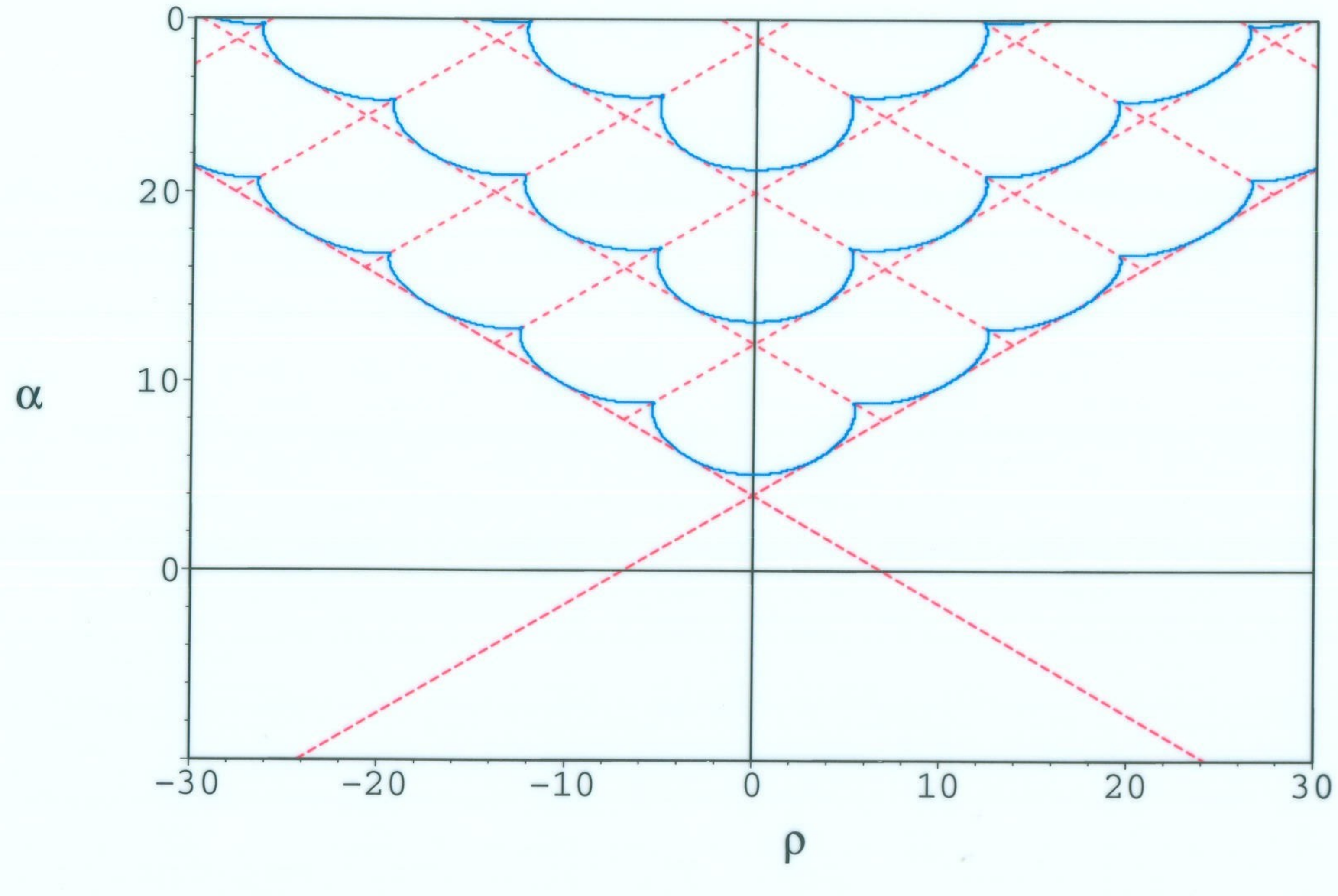

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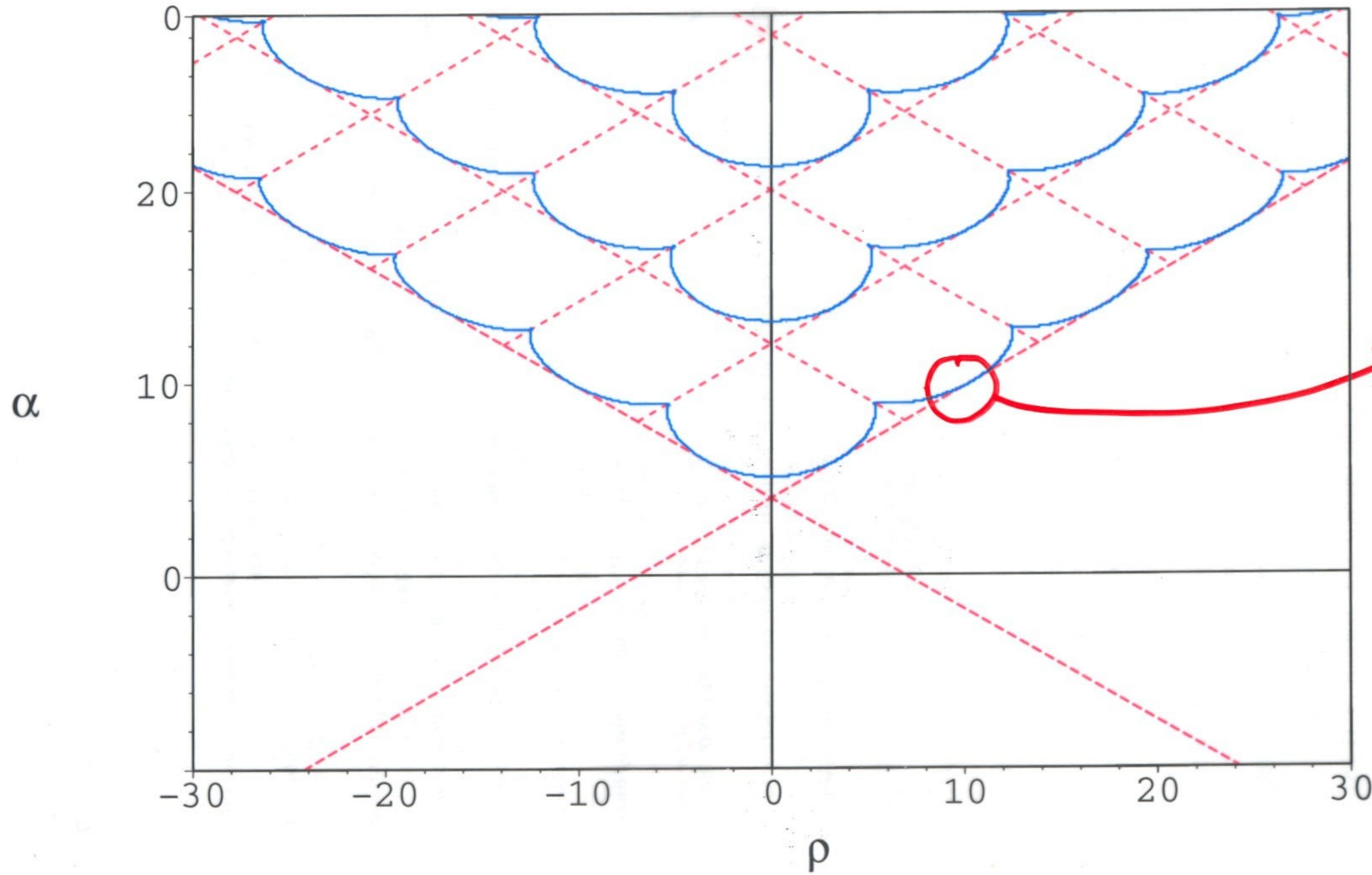
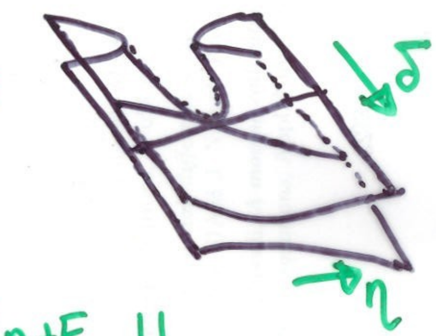
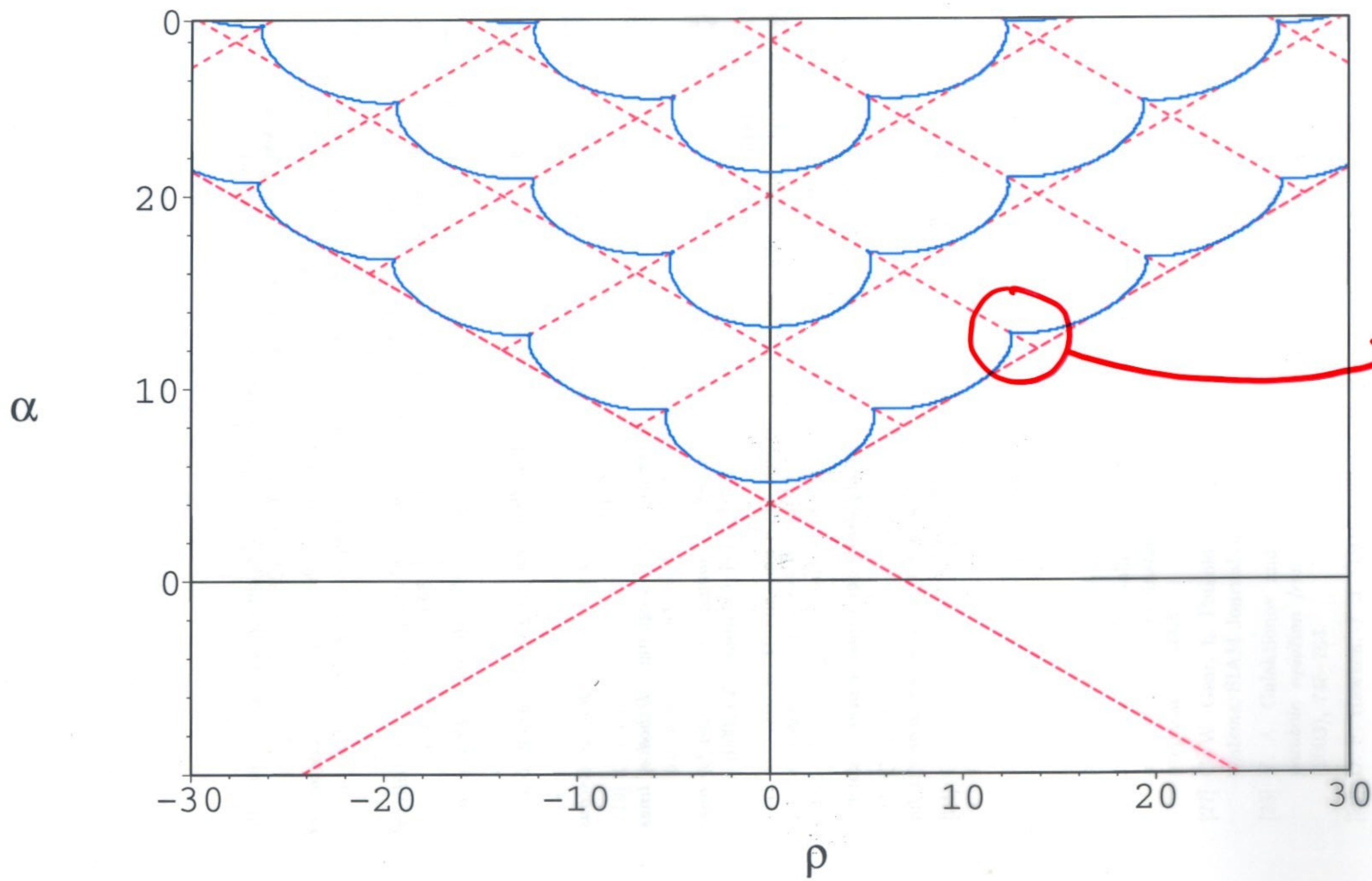
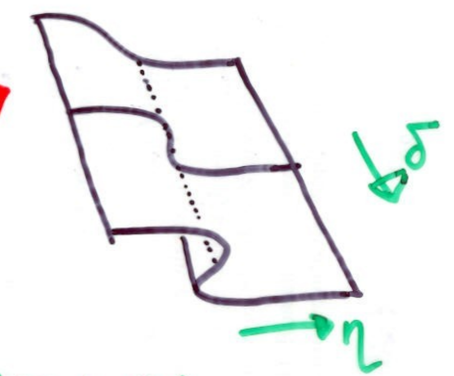




$$\begin{vmatrix}\eta+E & 1\\ \delta & E\end{vmatrix} = E^2+\eta E-\delta$$





$$\begin{vmatrix} E & 1 & 0 \\ 0 & E & 1 \\ \eta & \delta & E \end{vmatrix} = E^3 - E\delta + \eta$$

Other special features at $M=3$:

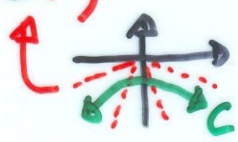
$$\left[-\frac{d^2}{dx^2} + x^6 + \alpha x^2 + \frac{\rho^2-3}{12x^2}\right]\psi(x) = E\psi(x) \quad (*)$$

$$(\rho = \sqrt{3}(2l+1))$$

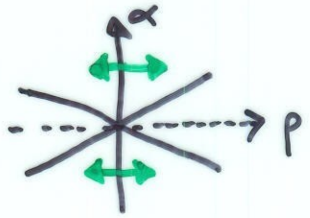
Set $\underset{\sim}{\alpha} = \begin{pmatrix}\alpha \\ \rho\end{pmatrix}$ and define

$$R(\underset{\sim}{\alpha}) = \text{Spect}\left((*), \text{ with 'radial' b.c.s} \begin{Bmatrix}\psi(x\to\infty)=0 \\ \psi(x\to 0)\sim x^{l+1}\end{Bmatrix}\right)$$

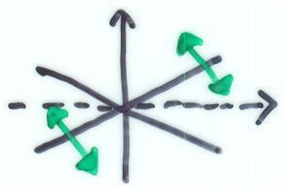
$$L(\underset{\sim}{\alpha}) = \text{Spect}\left((*), \text{ with 'lateral' (PT) b.c.s } \psi \in L^2(C)\right)$$



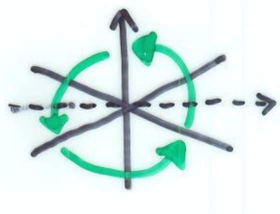
$$\mathbb{L}\begin{pmatrix}\alpha \\ \rho\end{pmatrix} = \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\alpha \\ \rho\end{pmatrix}$$



$$\mathbb{T}\begin{pmatrix}\alpha \\ \rho\end{pmatrix} = \begin{pmatrix}-1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2\end{pmatrix}\begin{pmatrix}\alpha \\ \rho\end{pmatrix}$$



$$\mathbb{L}\mathbb{T}\begin{pmatrix}\alpha \\ \rho\end{pmatrix} = \begin{pmatrix}-1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2\end{pmatrix}\begin{pmatrix}\alpha \\ \rho\end{pmatrix}$$



Then

(a) $R(\underset{\sim}{\alpha}) = R(\mathbb{T}\underset{\sim}{\alpha})$ (b) $L(\underset{\sim}{\alpha}) = L(\mathbb{L}\underset{\sim}{\alpha})$ (c) $L(\underset{\sim}{\alpha}) = R(\mathbb{L}\mathbb{T}\underset{\sim}{\alpha})$

NB: (b) is trivial, but the others aren't!

What about the "infinitely complex" transition at $M=1$?

Return to the homogeneous potential:

$$\left[-\frac{d^2}{dx^2}-(ix)^{2M}+\frac{l(l+1)}{x^2}\right]\psi=E\psi$$

<u>Aim</u>: (i) predict the transition to ∞'ly many complex eigenvalues for $M<1$, and in particular:

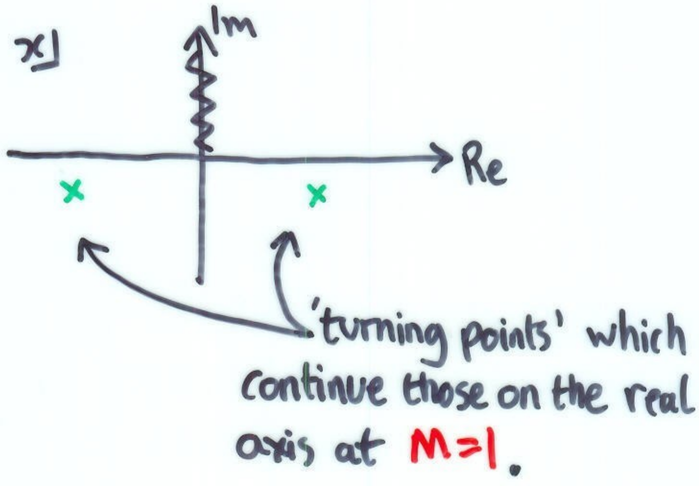
(ii) capture the way that the <u>connectivity</u> of levels depends on l.

<u>Remarks</u>:

- PT symmetry already implies that evalues are either real or complex-conjugate pairs;
- Truncation to 2×2 blocks in SHO basis [Bender-Boettcher] gives insight into (i) but <u>fails</u> for (ii).

- What about WKB? [Bender. Boettcher]

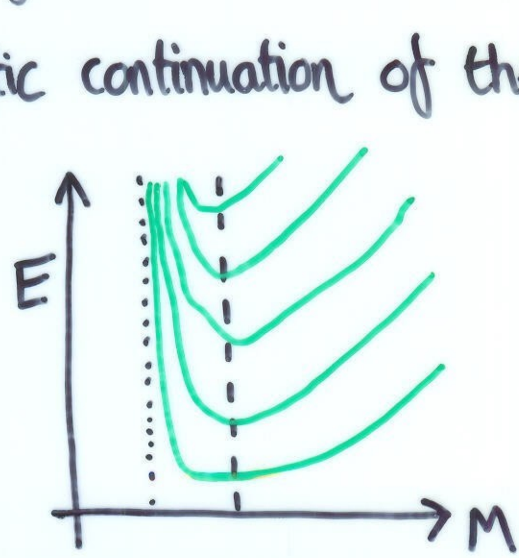
  - For $M \geq 1$ the method (also to 'all orders') works well.



Leading WKB:

$$E_n \sim \left(\frac{\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+\frac{1}{2M}\right)}{\sin\left(\frac{\pi}{2M}\right)\Gamma\left(1+\frac{1}{2M}\right)} \left(n+\frac{1}{2}\right) \right)^{\frac{2M}{M+1}}$$

  - But for $M < 1$ the method <u>fails</u>, as does the analytic continuation of the $M \geq 1$ formula:



Idea: Use the ODE/IM correspondence to extract a more general result.

Key ingredient:

- certain very powerful functional equations ['TQ relations' & 'quantum Wronskians'] apply both to (a) certain objects in integrable quantum field theories and (b) spectral determinants of certain ordinary differential equations.
- this allows (a) and (b) to be identified,

IM ODE

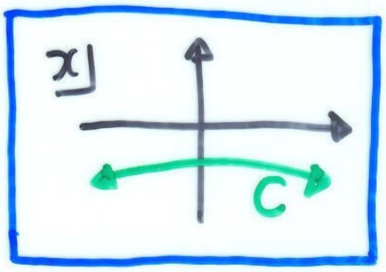
and lets us borrow ideas from integrable models to study the ODEs.

...a long story!

Spectral Determinants

$$\left(-\frac{d^2}{dx^2} - (ix)^{2M} + \frac{l(l+1)}{x^2}\right]\psi = E\psi \qquad (*)$$

① Let $\{E_i\}$ be the eigenvalues of the 'lateral' (PT) problem for $(*)$



and define

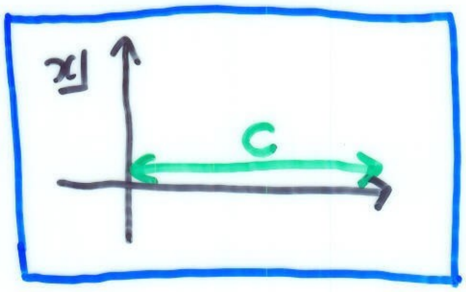
$$T(E) = \prod_{i=0}^{\infty}\left(1 + \frac{E}{E_i}\right)$$

T is a <u>spectral</u> <u>determinant</u> for the problem.

(zeros of T are the (negated) eigenvalues,

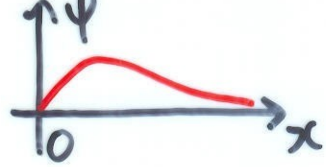
cf. $\det(M + \lambda I)$ for a finite matrix M)

② Now shift $x \to x/i$ and replace C by a radial contour:



$$\left(-\frac{d^2}{dx^2} + x^{2M} + \frac{l(l+1)}{x^2}\right)\psi = E\psi$$

$$\left\{\begin{array}{l} \psi(x) \to 0 \text{ as } x \to \infty \\ \psi(0) \sim x^{l+1} \end{array}\right\}$$



Write the eigenvalues of this problem as $\{e_i\}$

and set
$$Q(E) = \prod_{i=0}^{\infty}\left(1 - \frac{E}{e_i}\right)$$

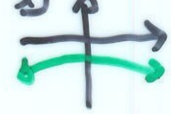
(another spect. det.)

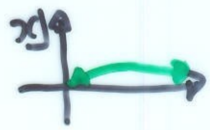
Notes:

- The products fail to converge for $M \leq 1$ (& reality is lost for T for $M < 1$) — not a coincidence!
- Problems ① and ② are related by $\{x \to x/i\}$ and {lateral b.c. $\to$ radial b.c.}

But there's a much deeper connection...

Functional Relations

$T(E)$: lateral (PT) problem 

$Q(E)$: radial (Hermitian) problem 

Considering solutions in various Stokes sectors shows

$$\boxed{T(E)\,Q(E) = \omega^{-l-1/2}\,Q(\omega^{-2}E) + \omega^{l+1/2}\,Q(\omega^{2}E) \qquad \omega = e^{i\pi/(M+1)}}$$

- This is Baxter's TQ relation from integrable models.

- It holds for <u>all</u> values of M.

- For $M > 1$, when the simple ∞ products converge, it leads to a reality proof.

- For $M < 1$ it can be used to understand unreality instead...

Asymptotics from ODE/IM (sketch)

- TQ relation:

$$T(E)Q(E) = \omega^{-l-1/2}Q(\omega^{-2}E) + \omega^{l+1/2}Q(\omega^{2}E)$$

- Define

$$a(E) = \omega^{2l+1}\frac{Q(\omega^{2}E)}{Q(\omega^{-2}E)}$$

- By TQ,

$$a(E) = -1$$

$$\Updownarrow$$

$$\text{either of}\begin{cases} T(E)=0,\ E=-E_i & \text{(PT)} \\ Q(E)=0,\ E=e_i & \text{(Hermitian)} \end{cases}$$

- Set $f(E) = \log a(E)$

Then the set of points at which $f(\theta) = (2n+1)\pi i$ is precisely $\{-E_i\} \cup \{e_i\}$.

$\rightarrow$ f is a "counting function" for the spectral problems.

Furthermore, $f(E)$ satisfies a nonlinear integral equation from which asymptotics can be extracted and matched with conservation laws in integrable models.

Results:

Hermitian evals

- For $E>0$ (in fact, in a sector containing the +ve real E axis) $f(E)$ is a double series in $E^{-(M+1)/2M}$ and $E^{-(M+1)}$ (plus a non-pert bit)

PT evals

- For $E<0$, $f(E)$ is a single series, which is different for $M>1$ and $M<1$.

For $M>1$, you get the WKB series, but for $M<1$ WKB terms vanish & are replaced by a Born-like series.

Leading approx for $E<0$:

(a) $M>1$: $$f(E) \sim 2i\sin\left(\frac{\pi}{2M}\right)\sqrt{\pi}\,\frac{\Gamma\left(1+\frac{1}{2M}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2M}\right)}(-E)^{(M+1)/2M}$$

(reproduces WKB)

(b) $M=1$: $$f(E) = i\pi\left(l+\tfrac{1}{2}\right) + \frac{i\pi}{2}(-E)$$

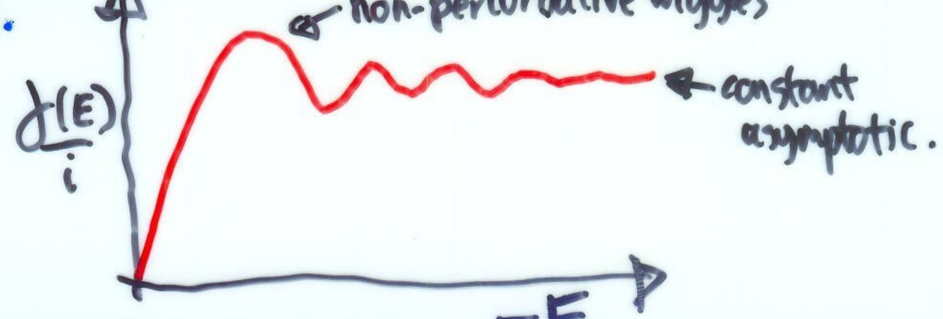
(SHO)

(c) $M<1$: $$f(E) \sim 2\pi i\left(l+\tfrac{1}{2}\right)$$

(useless)

NB: the interpolation (a) → (b) → (c) is subtle, & involves a non-perturbative term in a crucial way.

Adding extra Born-like terms to (c) only captures <u>one</u> eigenvalue; but there is a non-perturbative term arising from the integral equation which rescues the story.



Together, <u>all</u> features of the $M<1$ story are recovered!

Further reading:

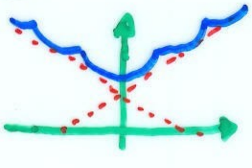
Reality conjectures and the infinitely-many level mergings:

- C. Bender & S. Boettcher PRL 80 (1998) 5243
- P. Dorey & R. Tateo NPB 563 (1999) 573

Reality proof:

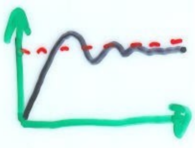
- P. Dorey, C. Dunning & R. Tateo
  J. Phys A 34 (2001) 5679

Finitely-many mergings for inhomogeneous potentials:



- P. Dorey, C. Dunning & R. Tateo
  J. Phys A 34 (2001) L391

Analytic treatment of the infinite level-mergings:



- P. Dorey, A. Millican-Slater & R. Tateo
  J. Phys A 38 (2005) 1305