Detecting “broken” $\mathcal{PT}$-symmetry

An algorithm to determine the structure of the spectrum of a pseudo-Hermitean matrix

Motivation

- $\mathbf{H} = p^2 + ix^3 \neq \mathbf{H}^\dagger$ with $[\mathbf{H}, \mathbf{PT}] = 0 \Rightarrow \{E_n,$ real and/or cc pairs$\}$

  ⋆ Is **H** diagonalizable? (StW: `quant-ph/0507202`, in finite dimensions)

  ⋆ Does **H** have a real spectrum? (P Dorey et al: JPA**34** (2001) 5679)

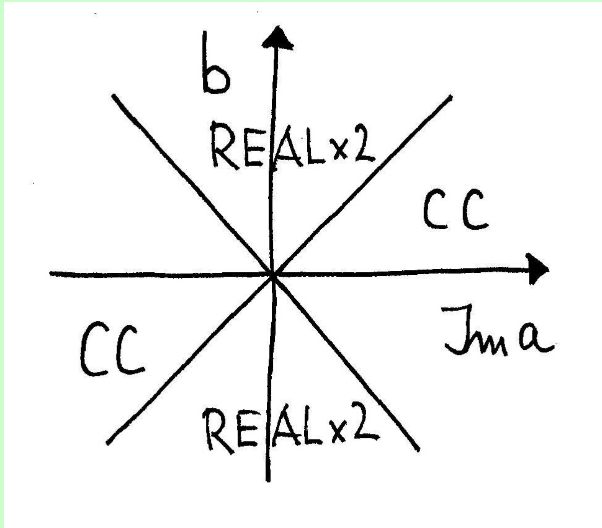
- $\mathbf{H} = \begin{pmatrix} a & b \\ b & a^* \end{pmatrix}, \; b \in \mathbb{R}$

  $\Rightarrow \; \lambda_\pm = \Re a \pm \sqrt{\boldsymbol{b^2 - (\Im a)^2}}$

  ⋆ $|b| = |\Im a|$ (**H** not diagonalizable)

  ⋆ $|b| > |\Im a|$ ("unbroken" symmetry: real E_n)

  ⋆ $|b| < |\Im a|$ ("broken" symmetry: cc pairs of E_n)



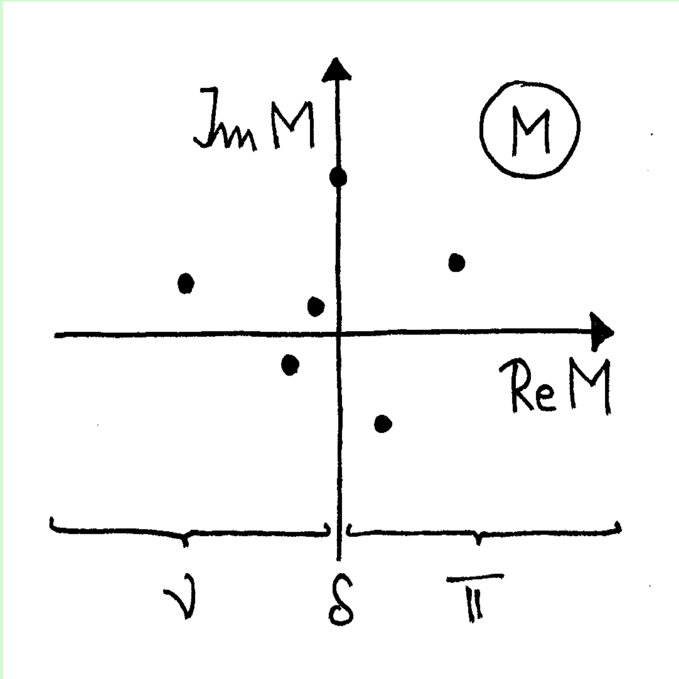
Detecting "broken" PT-symmetry

- Stability
  - ⋆ Inertia of matrices
  - ⋆ Jacobi's criterion of stability
- Stability of quasi-Hermitean matrices
  - ⋆ Inertia for "(un-) broken" PT-symmetry
  - ⋆ Zeros of real polynomials - Borhard-Jacobi theorem
- The algorithm
- Example
- Summary and Outlook

Stability and inertia of matrices

- linear systems:

$$\frac{d\mathbf{x}}{dt} = \mathbf{M} \cdot \mathbf{x}$$



$\star$ stable solutions $\mathbf{x}(t)$?

need negative real parts: $\Re \, \mathrm{M}_n < 0$

- inertia of **M** wrt the imaginary axis

$\star$ $\mathtt{In}\,\mathbf{M} = \{\nu, \delta, \pi\}$

- stability of **M**:

$\star$ need $\mathtt{In}\,\mathbf{M} = \{\boldsymbol{N}, 0, 0\}$

Inertia of Hermitean matrices: Jacobi's method

- given: $(N \times N)$ matrix $\mathbf{L} = \mathbf{L}^\dagger$
  - ⋆ leading principal submatrices $\mathbf{L}_1, \mathbf{L}_2, \ldots, \mathbf{L}_N (\equiv \mathbf{L})$
  - ⋆ let $d_0 = 0$ and calculate $d_1, d_2, \ldots, d_{N-1}, d_N$

    where $d_n \equiv \det \mathbf{L}_n (\neq 0)$

$$\Rightarrow \; +, (\pm)_1, (\pm)_2, \ldots, (\pm)_{N-1}, (\pm)_N \qquad (\star)$$

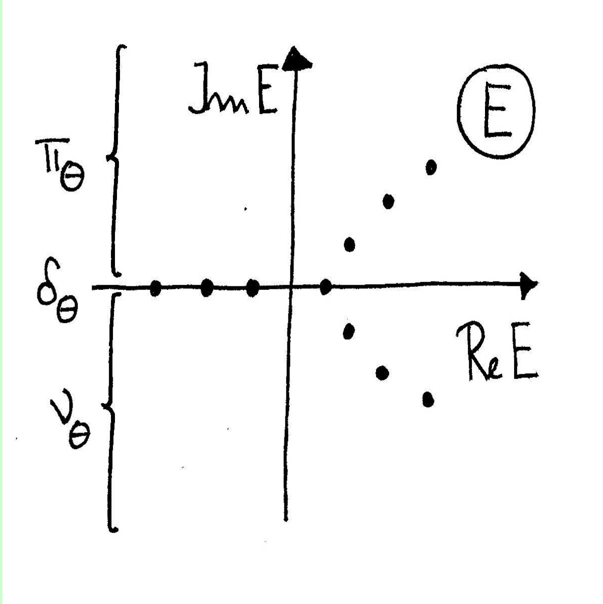
- **then, Jacobi tells us**: (G Jacobi: J Reine Angew. Math. **53** (1857) 265)

$$\left.\begin{array}{r} \#\text{ of constancies in } (\star) \equiv \pi \\ \#\text{ of alterations in } (\star) \equiv \nu \end{array}\right\} \Rightarrow \mathtt{In}\,\mathbf{L} = (\nu, 0, \pi)$$

Stability and inertia of quasi-Hermitean Matrices

- $\mathbf{H} \neq \mathbf{H}^\dagger$ with $[\mathbf{H}, \mathbf{PT}] = 0$
  - ⋆ real char. polynomial $p_{\mathbf{H}}(\lambda) = \sum_1^N h_n \lambda^n$:

    $p^*_{\mathbf{H}}(\lambda) = p_{\mathbf{H}}(\lambda^*)$



- inertia of $\mathbf{H}$ wrt the real axis
  - ⋆ $\mathtt{In}_\ominus \mathbf{H} = \{\nu_\ominus, \delta_\ominus, \pi_\ominus\}$

- inertia of quasi-Hermitean matrices $\mathbf{H}$
  - ⋆ real spectrum - "unbroken" symmetry: $\mathtt{In}_\ominus \mathbf{H} = \{\mathbf{0}, N, \mathbf{0}\}$
  - ⋆ mixed spectrum - "broken" symmetry: $\mathtt{In}_\ominus \mathbf{H} = \{\boldsymbol{m}, N-2m, \boldsymbol{m}\}$, $m > 0$

Zeros of real polynomials

- How many real zeros does a real polynomial of degree N have **?**

  $\star$ $p(\lambda) = \sum_1^N p_n \lambda^n$

  $\star$ Newton sums: $s_0 = N, s_n = \lambda_1^n + \ldots + \lambda_N^n\,, n = 1, 2, \ldots$

$$\Rightarrow \text{Hankel matrix } \mathbf{L} = \mathbf{L}_p^\dagger = \begin{pmatrix} s_0 & s_1 & s_2 & \cdots & s_{N-1} \\ s_1 & s_2 & & \cdots & s_N \\ s_2 & & & & s_{N+1} \\ \vdots & & & & \vdots \\ s_{N-1} & s_N & & \cdots & s_{2N-2} \end{pmatrix}$$

  $\star$ find s_n from: $p'(\lambda) = (s_0\lambda^{-1} + s_1\lambda^{-2} + \ldots)p(\lambda)$

- Borhard and Jacobi show that (J. Math. Pures Appl. **12** (1847) 50; J. Reine Angew. Math. **53** (1857) 265)

  $\star$ a real $p(\lambda)$ has $\left\{ \begin{array}{c} \boldsymbol{\nu} \text{ different pairs of cc zeros} \\ \boldsymbol{\pi} - \nu \text{ different real zeros} \end{array} \right\}$ where $\texttt{In}\ \mathbf{L}_p = \{\nu, \delta, \pi\}$

Algorithm detecting complex eigenvalues

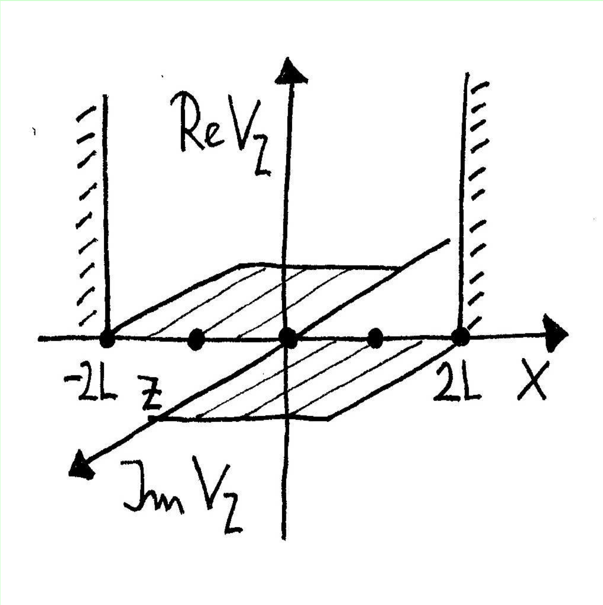
- input: $(N \times N)$ matrix **H** with $[\mathbf{H}, \mathbf{PT}] = 0$
  - $\star$ calculate $p_{\mathbf{H}}(\lambda)$
  - $\star$ determine Newton sums s_n from $p_{\mathbf{H}}(\lambda)$
  - $\star$ define Hermitean Hankel matrix $\mathbf{L_H}$
  - $\star$ calculate determinants d_n of principal minors of $\mathbf{L_H}$
  - $\star$ find $\#$ of constancies $\boldsymbol{\pi}$ and alterations $\boldsymbol{\nu}$
  - $\star \Rightarrow \mathtt{In}\,\mathbf{L_H} = \{\nu, 0, \pi\}$ with $N = \pi + \nu$
- output: $\mathtt{In}_{\ominus}\mathbf{H} = \{\nu, \pi - \nu, \nu\} \equiv \{\nu, N - \nu, \nu\}$

  $\Rightarrow$ **PT**-symmetry broken if $\boldsymbol{\nu} > 0$

Example: Discretized PT-Symmetric Square-Well

- particle in **PT**-invariant square-well potential (M Znojil: PLA **285** (2001) 7)

$$V_Z(x) = \begin{cases} -iZ\,, & -2L < x < 0 \\ 0\,, & x = 0 \\ iZ\,, & 0 < x < 2L \end{cases} \qquad (Z \in \mathbb{R})$$



- **PT**-invariant matrix model

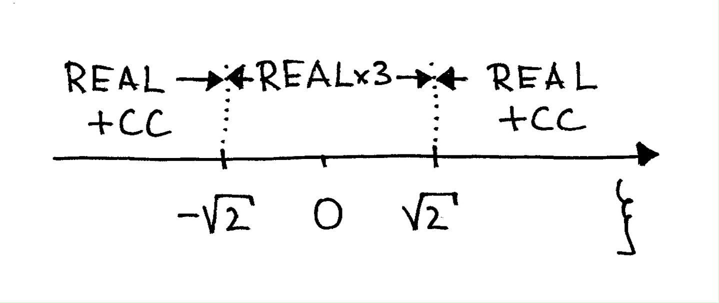
  ⋆ discretize space: $x \to kL,\ k = 0, \pm 1, \pm 2$

  $\Rightarrow (2\mathbf{I} - \mathbf{H})\boldsymbol{\psi} = E\boldsymbol{\psi}$

$$\mathbf{H} = \begin{pmatrix} i\xi & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -i\xi \end{pmatrix}, \quad \xi = 2mL^2Z/\hbar^2$$

Algorithmic solution

- input: $\mathbf{H} = \begin{pmatrix} i\xi & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -i\xi \end{pmatrix}$



  ⋆ $p_{\mathbf{H}}(\lambda) = \lambda^3 - (\xi^2 - 2)\lambda$

  ⋆ Newton sums s_n

  ⋆ $\mathbf{L_H} = 2 \begin{pmatrix} 3/2 & 0 & (2-\xi^2) \\ 0 & (2-\xi^2) & 0 \\ (2-\xi^2) & 0 & (2-\xi^2)^2 \end{pmatrix}$

  ⋆ $d_0 = 1, d_1 = 3, d_2 = 6(2-\xi^2), d_3 = 20(2-\xi^2)^3$

  ⋆ $\begin{cases} \xi^2 < 2 & \Rightarrow & ++++ & \Rightarrow & \mathtt{In}\,\mathbf{L_H} = \{0,0,3\} \\ \xi^2 > 2 & \Rightarrow & ++-- & \Rightarrow & \mathtt{In}\,\mathbf{L_H} = \{1,0,2\} \end{cases}$

- output: $\mathtt{In}_\ominus \mathbf{H} = \{\nu, \pi - \nu, \nu\} \equiv \{\nu, N - \nu, \nu\}$

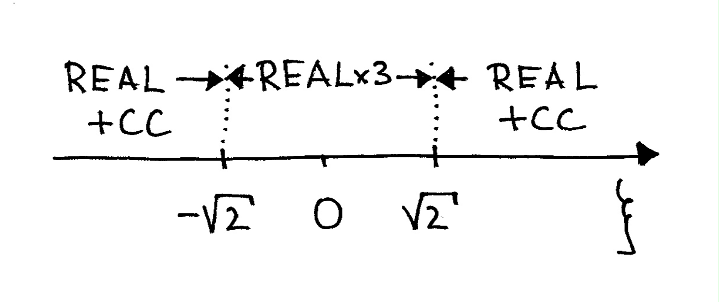
$$\mathtt{In}_\ominus \mathbf{H} = \begin{cases} \{0,3,0\} & \text{if} & |\xi| < 2 & \Rightarrow & \text{real spectrum} \\ \{1,1,1\} & \text{if} & |\xi| > 2 & \Rightarrow & \text{mixed spectrum} \end{cases}$$

Comparison with analytic solution

- matrix model

$$\mathbf{H} = \begin{pmatrix} i\xi & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & -i\xi \end{pmatrix}$$

$$\Rightarrow p_{\mathbf{H}}(\lambda) = (\lambda^2 + (\xi^2 - 2))\lambda$$

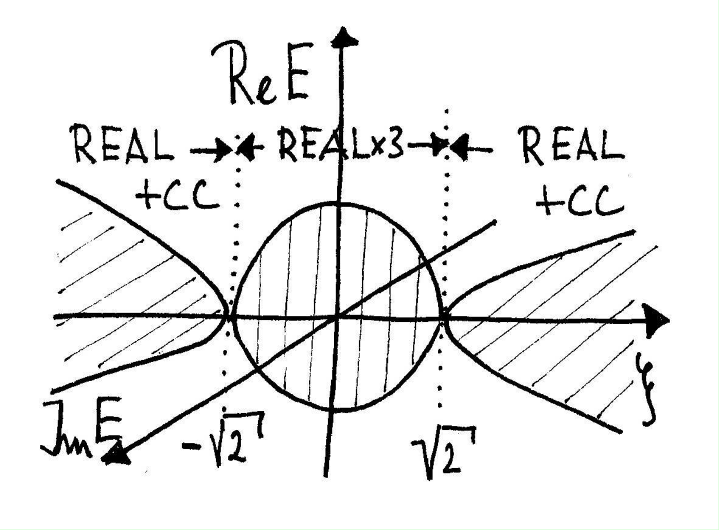


- eigenvalues:

$$E_0 = 0$$

$$E_{\pm} = \pm\sqrt{2 - \xi^2} \in \begin{cases} \mathbb{R} & \text{if } \xi^2 < 2 \\ i\mathbb{R} & \text{if } \xi^2 > 2 \end{cases}$$

- **not** diagonalizable if $\xi = \pm\sqrt{2}$



Discussion and Outlook

- given a **PT**-invariant matrix there are algorithmic tests to decide

  whether { **H** has a complete set of eigenfunctions
PT-symmetry of **H** is broken

- **PT**-invariant families of matrices **H**(ε)

  $\star$ global structure of solutions in parameter space

- generalization to operators such as **H** $= p^2 + ix^3$ **?**