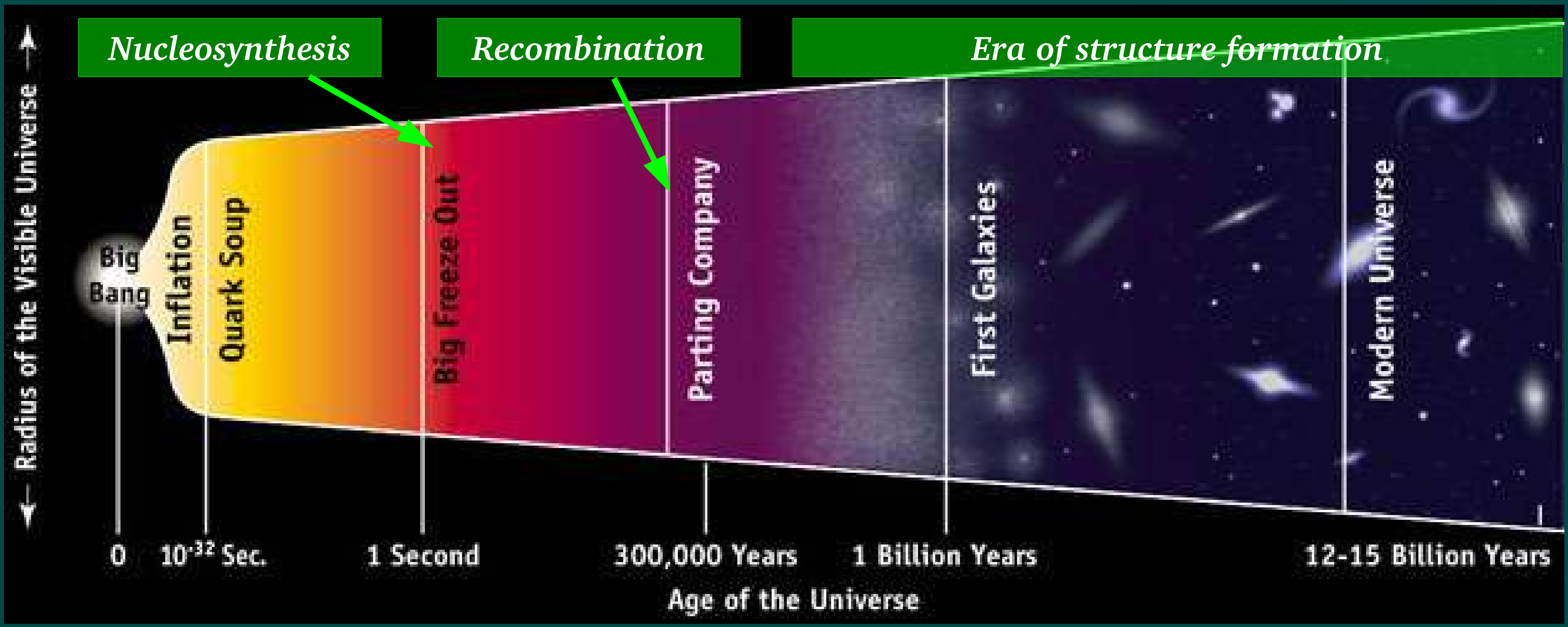
Computational Astrophysics

Lecture 5: Cosmology

Paul Ricker

University of Illinois at Urbana-Champaign
National Center for Supercomputing Applications
Urbana, Illinois USA

Context: the expanding universe



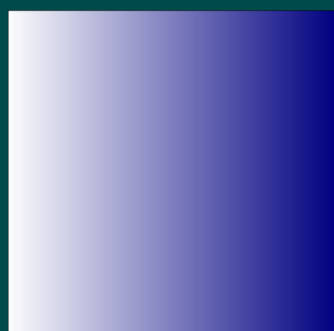
Barry Sanders, NCSA

The metric

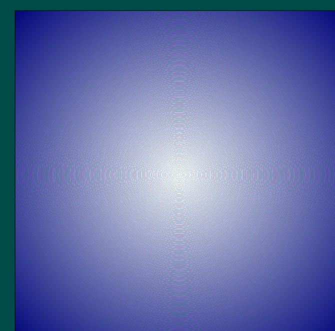
Fundamental assumptions of physical cosmology:

- Isotropy – same in all directions (testable: microwave background $\delta T/T < 10^{-5}$)
- Copernican principle – we do not occupy a privileged location

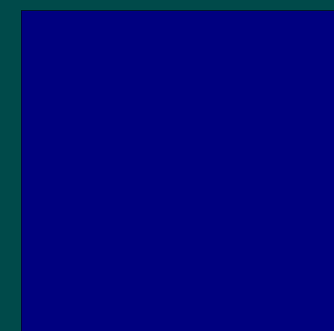
Together these imply homogeneity: on large scales properties of Universe are independent of location



Homogeneity without isotropy



Isotropy without homogeneity



Homogeneity and isotropy

The most general space-time metric consistent with homogeneity and isotropy is the Robertson-Walker metric:

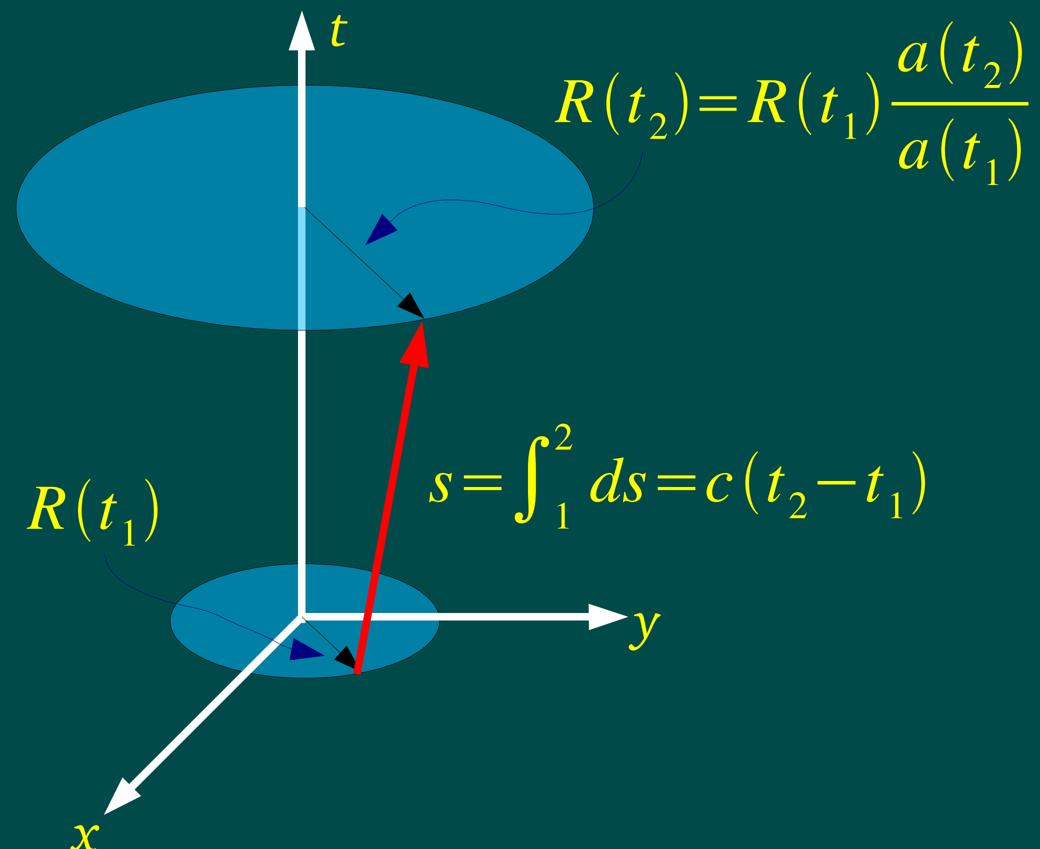
$$ds^2 = c^2 dt^2 - a(t)^2 \left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2 \right]$$

The metric – 2

If we have two events, one at (r,θ,φ,t) (spherical coordinates + time) and the other at $(r+\mathrm{d}r,\theta+\mathrm{d}\theta,\varphi+\mathrm{d}\varphi,t+\mathrm{d}t)$, the spacetime interval between them is $\mathrm{d}s$. Also:

$a(t) \equiv$ the scale factor, normalized so that today $a(t_0) \equiv a_0 = 1$

$k \equiv$ spatial curvature



Friedmann equations

If we insert the RW metric into the Einstein equations of general relativity,

$$8\pi G_{\mu\nu} = T_{\mu\nu}$$

spacetime curvature from RW metric

stress-energy of uniform, isotropic fluid

we get the Friedmann equations:

$$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$ time-time component

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3P}{c^2}\right)$$ space-space components

These equations describe the *kinematics* of the expansion. Here

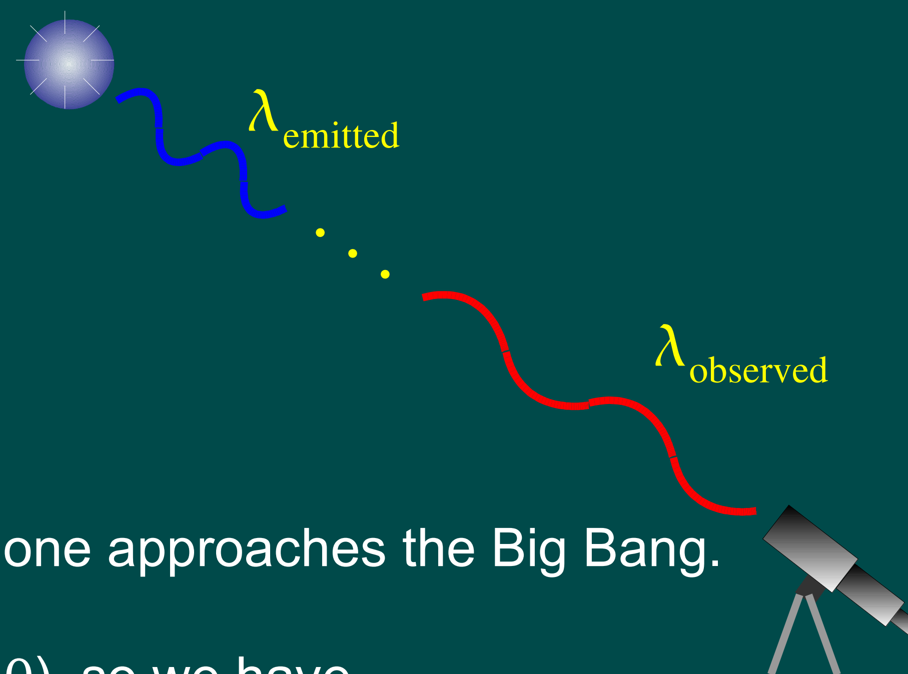
$H(t) \equiv$ the Hubble parameter; $H(t_0) \equiv H_0$ (the Hubble "constant")

The density ρ and pressure P include contributions from *all* sources of mass-energy.

Redshift

The wavelength of photons increases with the scale factor. We have

$$\lambda_{\text{observed}} = \lambda_{\text{emitted}}(1+z) = \lambda_{\text{emitted}}\frac{a_0}{a}$$



Thus we commonly define the redshift z via

$$a = \frac{a_0}{1+z} = \frac{1}{1+z}$$

$z = 0$ corresponds to the present; $z \to \infty$ as one approaches the Big Bang.

A photon travels along a null geodesic ($\mathrm{d}s = 0$), so we have

$$dr = c\frac{dt}{a} = c\frac{da}{\dot{a}\,a} = c\frac{da}{a^2 H} = -\frac{c}{H(z)}dz$$

The comoving distance between two observers on the same light cone is thus

$$R = \int_0^z \frac{c\ dz}{H(z)}$$

Closed-form expressions are given for a variety of models by Mattig (1958, 59, 68).

Relative velocities

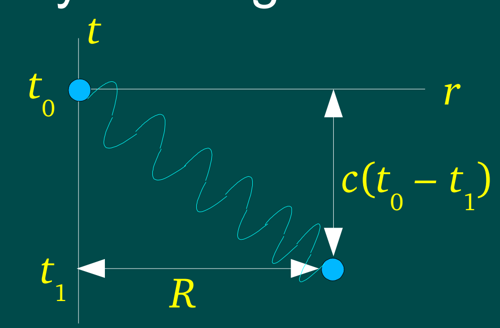
The instantaneous relative velocity of two observers moving with the expansion and separated by a comoving distance R is

$$v_{rel}(R,t)=\frac{d}{dt}\left[a(t)R\right]=H(t)a(t)R$$

We usually write

$$H_0 \equiv 100\,h\,\mathrm{km\ s^{-1}\ Mpc^{-1}}, \quad h=0.71\ \mathrm{(WMAP)}$$

Notice that v_{rel} is only the same as the *observed* relative velocity if the light travel time is short relative to the age of the Universe:

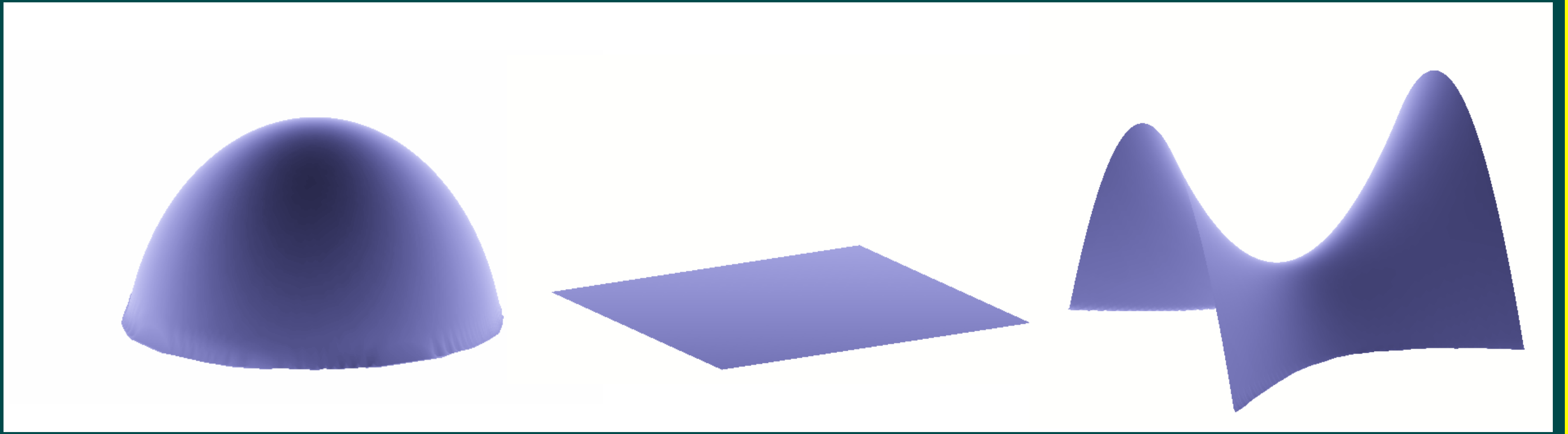


$$v_{rel,obs}=v_{rel}(R,t_1)=H(t_1)a(t_1)R$$
$$\approx H_0 R \text{ if } t_0-t_1 \ll t_0$$

If the observers have velocities relative to the spacetime expansion, the total relative velocity of the observers is the sum of their relative velocity due to expansion and their relative peculiar velocities:

Geometry

The spatial geometry of the Universe falls into one of three categories depending on the sign of k:



$k > 0$	$k = 0$	$k < 0$
closed space	flat space	open space

In reality, of course, the geometries are 3D spaces, not 2D surfaces like these.

We haven't said anything yet about the form of the expansion law (the “fate” of the Universe), just the curvature of space.

Expansion rate – limiting expressions

The expansion rate depends on how ρ and P vary with time. We can identify several limiting cases of interest:

Matter domination: $\rho > 0$ and $\rho c^2 \gg P$

The expansion dilutes the fluid but does not decrease the rest energy per particle. Thus

$$\rho \propto a^{-3} \Rightarrow a \propto t^{2/3}$$

Radiation domination: $P > 0$ and $P = \rho c^2/3$ (radiation or relativistic matter)

The expansion dilutes the fluid and decreases the energy per particle (redshift of de Broglie wavelength). Thus

$$\rho \propto a^{-4} \Rightarrow a \propto t^{1/2}$$

Curvature domination: $k/a^2 \gg G\rho$, curvature dominates density term

If k is constant and < 0 (open), we have

$$a \propto t$$

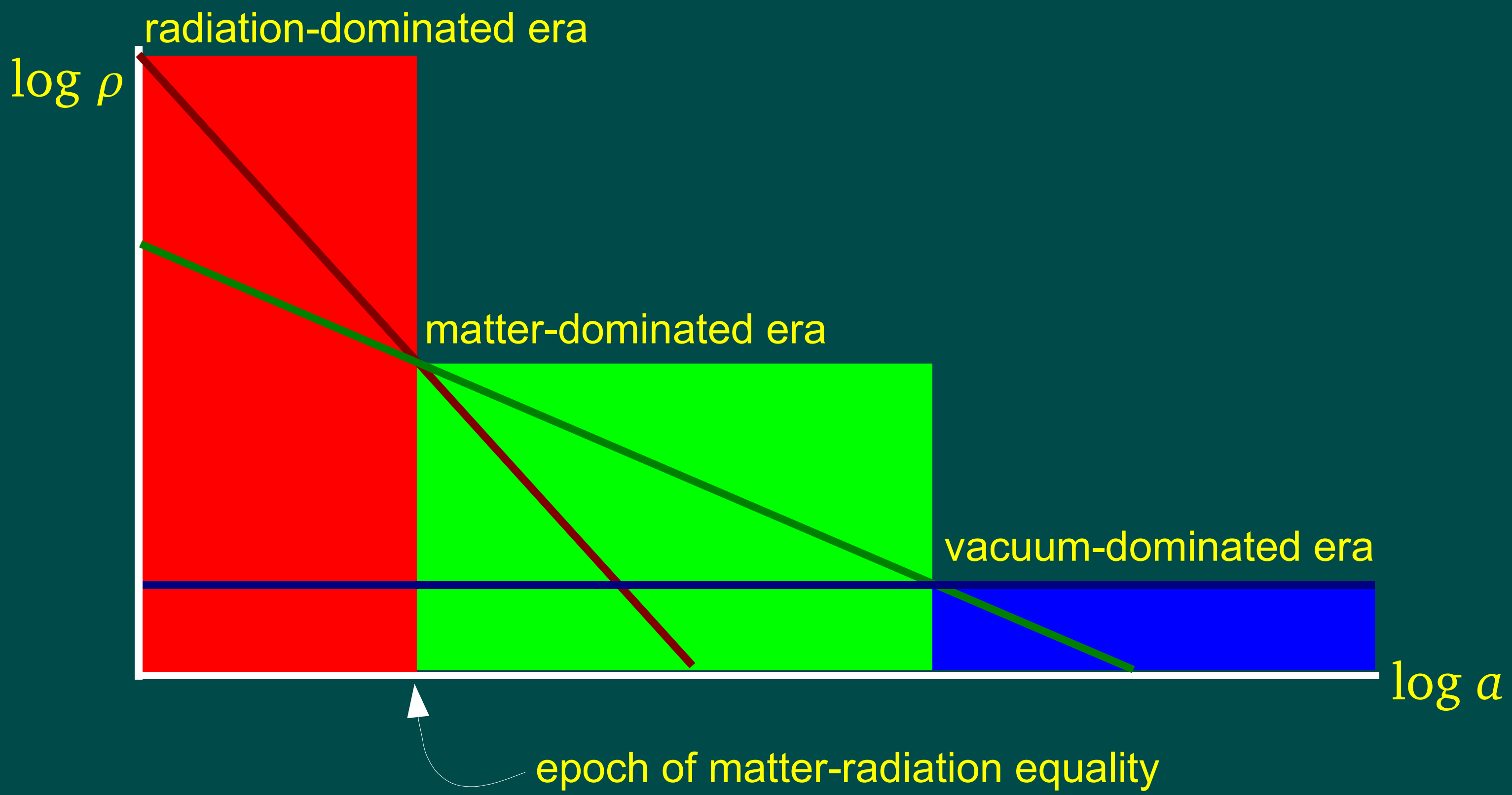
Vacuum domination: ρ = constant dominates curvature term

In this case

$$a \propto e^{Ht}, \quad H \ \text{constant}$$

Expansion rate – limiting expressions – 2

Because of the way the different fluid components' densities change with time, we can sketch out the history of the Universe according to the dominant mass-energy component:



The epoch of matter-radiation equality is very important because density fluctuations can only grow due to gravitational instability after this time.

$$1+z_{\mathrm{eq}} \approx 2.39 \times 10^{4}\, \Omega_0 h^2$$

Curvature and density

Notice that for any given expansion rate H, if the density takes on a critical value $\rho_{\rm crit} = 3H^2/8\pi G$, the required curvature $k = 0$. If the density is greater than this critical value, the curvature is positive; if it is less, the curvature is negative.

Thus the *dynamical* quantity (density) sets the *kinematic* quantity (curvature).

Define the density parameter as

$$\Omega(t) \equiv \frac{\rho(t)}{\rho_{\rm crit}(t)} = \frac{8\pi G}{3H(t)^2}\rho(t)$$

Note that in general this is a function of time.

If we treat the total energy density of the Universe as approximately the sum of matter, radiation, and vacuum terms, the first Friedmann equation becomes

$$H^2(t) = \frac{8\pi G}{3}\left[\frac{\rho_{\rm m,0}}{a(t)^3} + \frac{\rho_{\rm r,0}}{a(t)^4} + \rho_{\rm v,0} - \frac{3k}{8\pi G\, a(t)^2}\right]$$

$$= H_0^2\left[\frac{\Omega_{\rm m,0}}{a(t)^3} + \frac{\Omega_{\rm r,0}}{a(t)^4} + \Omega_{\rm v,0} - \frac{\Omega_{\rm k,0}}{a(t)^2}\right]$$

Curvature and density – 2

Note that we can also divide the original equation through by $H(t)^2$ to get

$$1 = \Omega_{\mathrm{m}}(t) + \Omega_{\mathrm{r}}(t) + \Omega_{\mathrm{v}}(t) - \Omega_{\mathrm{k}}(t) \equiv \Omega(t) - \Omega_{\mathrm{k}}(t)$$

Thus in the special case $k = 0$ we always have

$$\Omega(t) \equiv 1 \; (k = 0)$$

This case is of most interest, since observations indicate that

$$\Omega_{\mathrm{m}}(t_{\mathrm{recomb}}) + \Omega_{\mathrm{r}}(t_{\mathrm{recomb}}) + \Omega_{\mathrm{v}}(t_{\mathrm{recomb}}) \approx 1.02 \quad \text{(WMAP)}$$

We can always cast the Friedmann equation in the very useful form

$$H^2(a) = H_0^2 \left[\frac{\Omega_{\mathrm{m},0}}{a^3} + \frac{\Omega_{\mathrm{r},0}}{a^4} + \Omega_{\mathrm{v},0} - \frac{\Omega_0 - 1}{a^2} \right]$$

The spatial curvature is thus set by the density parameter: models with more than critical density are closed; equal to critical, flat; and less than critical, open.

The relationship between comoving distance and redshift is then:

$$dr = \frac{c}{H_0} \left[\Omega_{\mathrm{m},0}(1+z)^3 + \Omega_{\mathrm{r},0}(1+z)^4 + \Omega_{\mathrm{v},0} + (1 - \Omega_0)(1+z)^2 \right]^{-1/2} dz$$

Evolution of matter-dominated models

In a matter-dominated universe we have $\Omega_0 = \Omega_{m,0}$ and

$$\frac{da}{dt} = \left[\frac{\Omega_0}{a} - (\Omega_0 - 1)\right]^{1/2} H_0$$

Readily integrated (though only $\Omega_0 = 1$ gives an explicit expression for $a(t)$):

For $\Omega_0 = 1$ ($k = 0$): Einstein-de Sitter model

$$a(t) = \left(\frac{3H_0 t}{2}\right)^{2/3}$$

For $\Omega_0 > 1$ ($k > 0$):

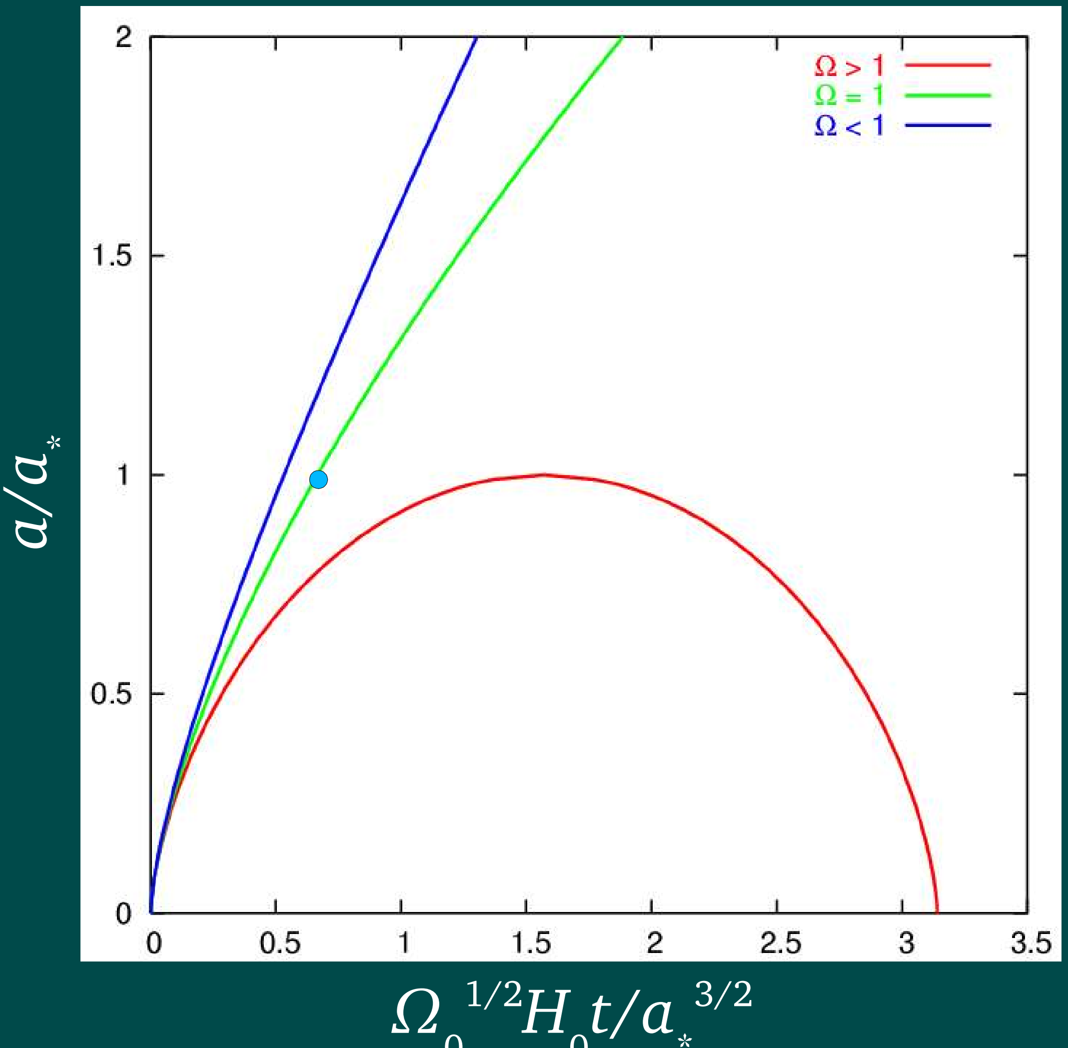
$$\sin^{-1}\left[\left(\frac{a}{a_*}\right)^{1/2}\right] - \left[\frac{a}{a_*}\left(1 - \frac{a}{a_*}\right)\right]^{1/2} = \frac{(\Omega_0 - 1)^{3/2}}{\Omega_0} H_0 t \qquad a_* \equiv \frac{\Omega_0}{\Omega_0 - 1}$$

For $\Omega_0 < 1$ ($k < 0$):

$$\left[\frac{a}{a_*}\left(1 + \frac{a}{a_*}\right)\right]^{1/2} - \sinh^{-1}\left[\left(\frac{a}{a_*}\right)^{1/2}\right] = \frac{(1 - \Omega_0)^{3/2}}{\Omega_0} H_0 t \qquad a_* \equiv \frac{\Omega_0}{1 - \Omega_0}$$

Evolution of matter-dominated models – 2

Define $a_* \equiv \begin{cases} \dfrac{\Omega_0}{|\Omega_0 - 1|} & \Omega_0 \neq 1 \\ 1 & \Omega_0 = 1 \end{cases}$



Models with:

$\Omega_0 \leq 1$ (open or flat geometry) expand forever;

$\Omega_0 > 1$ (closed geometry) eventually recollapse.

Thus there is a relationship between mass-energy density and spatial curvature on the one hand, and the evolution and "fate" of the Universe on the other.

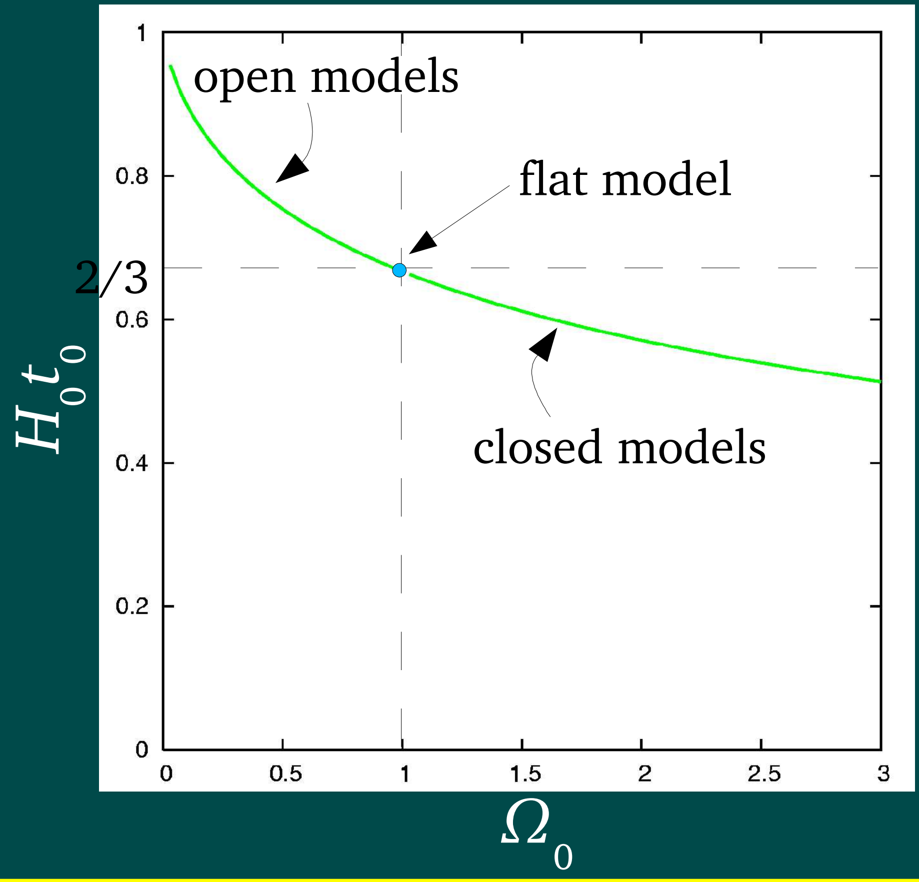
Evolution of matter-dominated models – 3

We can get the age of the Universe (t_0) in each model by setting $a=1$:

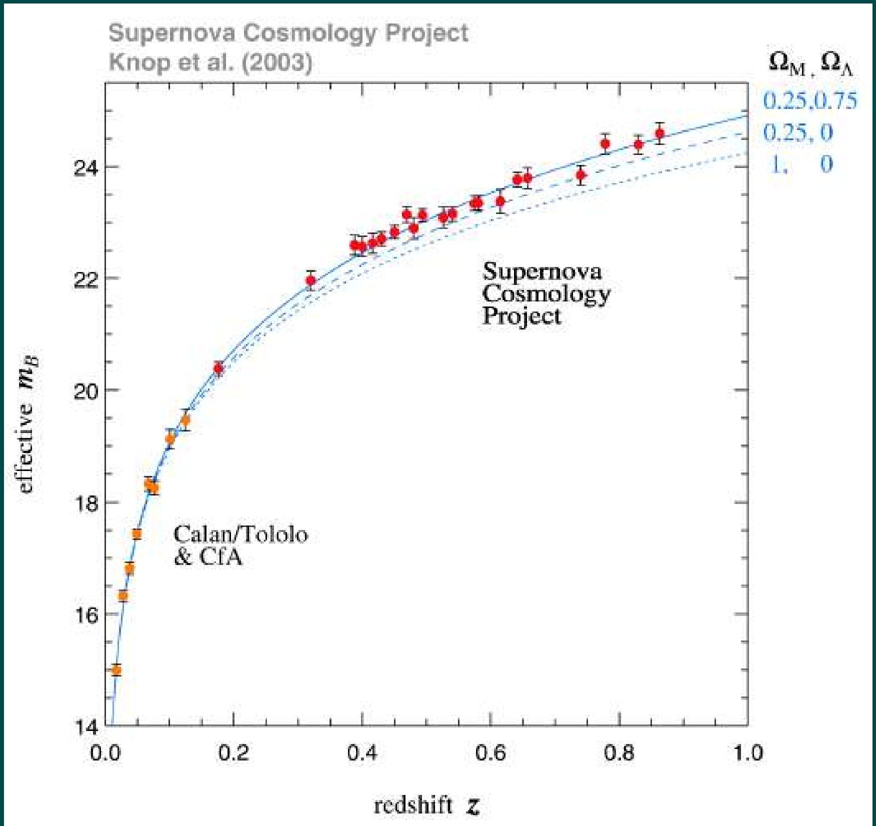
$$t_0=\begin{cases} \dfrac{2}{3H_0} & \text{if } \Omega_0=1 \\ \dfrac{a_*^{3/2}}{\Omega_0^{1/2}H_0}\left[\sin^{-1}a_*^{-1/2}-\dfrac{\sqrt{a_*-1}}{a_*}\right] & \text{if } \Omega_0>1 \\ \dfrac{a_*^{3/2}}{\Omega_0^{1/2}H_0}\left[\dfrac{\sqrt{a_*+1}}{a_*}-\sinh^{-1}a_*^{-1/2}\right] & \text{if } \Omega_0<1 \end{cases}$$

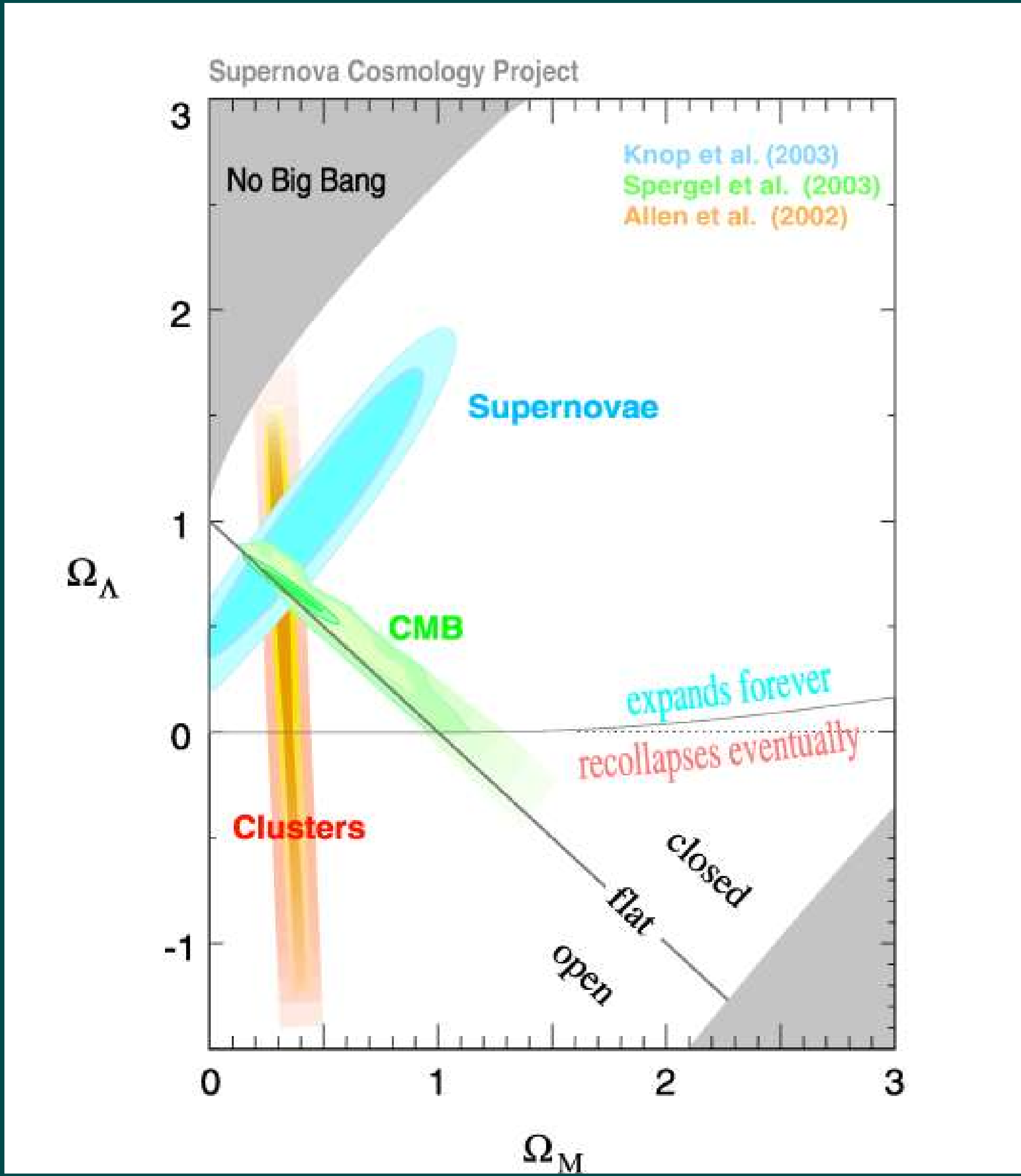
Fixing the value of H_0, we have

$$t_0\ (\text{open})>t_0\ (\text{flat})>t_0\ (\text{closed})$$



We live in a vacuum-dominated universe





Evolution of models with vacuum energy

In Einstein's original model, we have

$$\dot{a} = \ddot{a} = 0$$

which implies

$$\rho = \frac{3k}{8\pi G a^2} = -\frac{3P}{c^2}$$

Vacuum energy behaves as a fluid with negative pressure:

$$\rho_{\rm v} = -P_{\rm v}/c^2$$

If we ignore radiation and matter pressure, this is the only pressure source, so

$$\rho = \rho_{\rm m} + \rho_{\rm v} = -\frac{3P_{\rm v}}{c^2} = 3\rho_{\rm v}$$

The model is thus 2/3 matter and 1/3 vacuum energy, and $k > 0$ (geometry is closed).

Notice that if we perturb a, the matter density changes ($\propto a^{-3}$) but the vacuum energy density does not – so the model is *unstable*.

For $\Lambda > 0$, the model would explode, diluting $\rho_{\rm m}$ to 0, driving k to 0, and yielding an exponential expansion rate (de Sitter space).

Evolution of models with vacuum energy – 2

Adding Λ (or worse, dark energy with $P = w\rho$) makes the Friedmann equations much harder to solve in general. Consider non-static spatially flat models without radiation ($\Omega_0 = 1$, $\Omega_{r,0} = 0$, $\Omega_{v,0} = 1 - \Omega_{m,0}$):

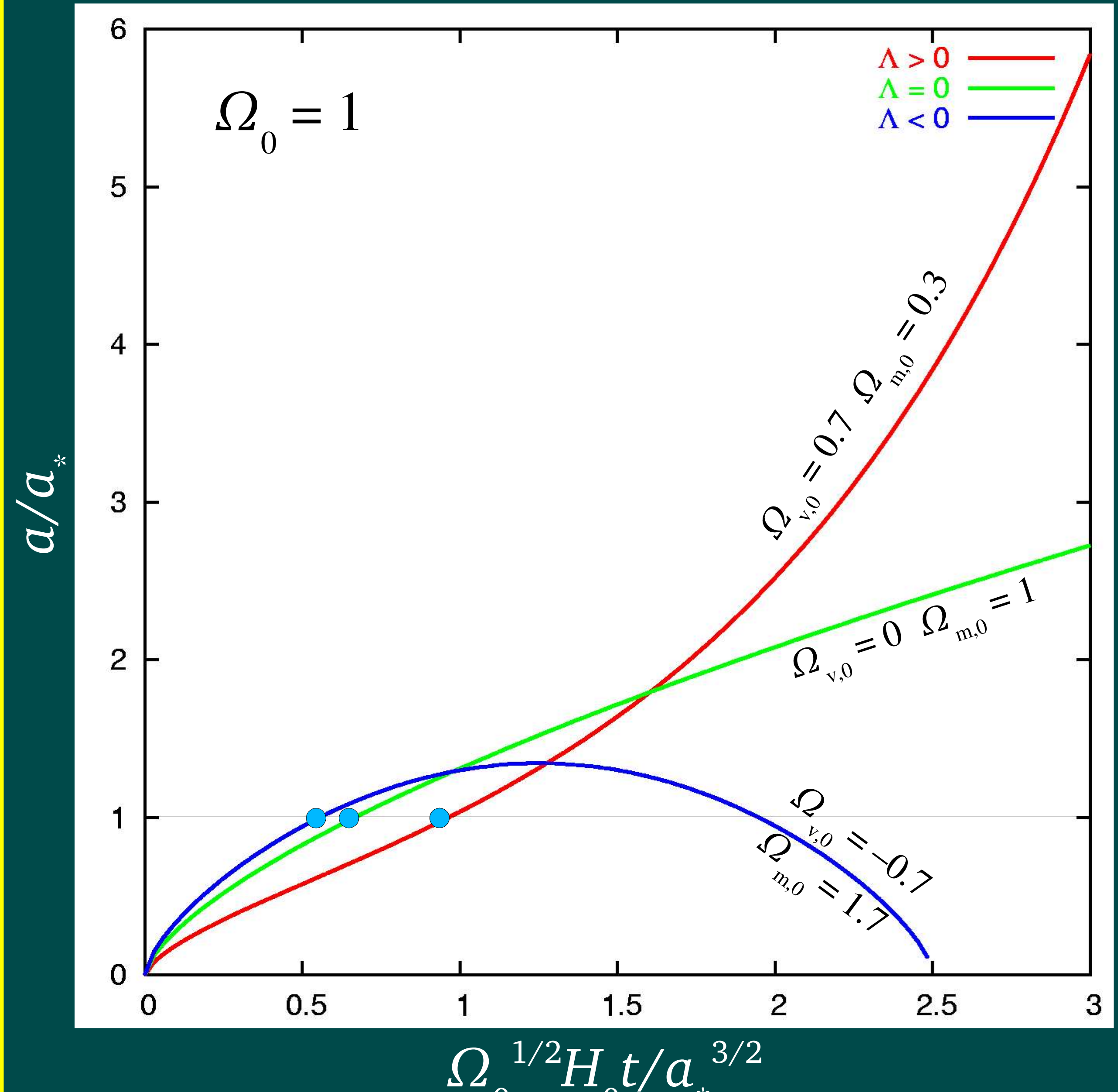
$$\frac{da}{dt} = H_0 \left[\frac{\Omega_{m,0}}{a} + \left(1 - \Omega_{m,0}\right) a^2 \right]^{1/2}$$

Then

$$a(t) = \begin{cases} \left(\dfrac{\Omega_{m,0}}{1-\Omega_{m,0}} \right)^{1/3} \sinh^{2/3}\left(\dfrac{3}{2} \sqrt{1-\Omega_{m,0}}\, H_0 t \right) & \Omega_{v,0} > 0,\ \Omega_{m,0} < 1 \\ \left(\dfrac{\Omega_{m,0}}{\Omega_{m,0}-1} \right)^{1/3} \sin^{2/3}\left(\dfrac{3}{2} \sqrt{\Omega_{m,0}-1}\, H_0 t \right) & \Omega_{v,0} < 0,\ \Omega_{m,0} > 1 \end{cases}$$

So if $\Lambda < 0$ (*attractive*), we always get recollapse, and if $\Lambda \geq 0$ (*repulsive*), we always get expansion.

Evolution of models with vacuum energy – 3



Current evidence from CMB and Type Ia supernova data suggests

$\Omega_{m,0} \cong 0.3$

$\Omega_{v,0} \cong 0.7$

$\Omega_0 \cong 1.0$

The age of the Universe for this model is

$$t_0 = \left[\frac{2}{3\sqrt{0.7}} \sinh^{-1} \sqrt{\frac{0.7}{0.3}} \right] H_0^{-1} \approx 0.96\, H_0^{-1}$$

Some remaining definitions

Hubble time:

$$t_{\text{Hubble}} \equiv H_0^{-1} \approx 9.78\, h^{-1}\ \text{Gyr}$$

Hubble radius:

$$R_{\text{Hubble}} \equiv c\, H_0^{-1} \approx 3.00\, h^{-1}\ \text{Gpc}$$

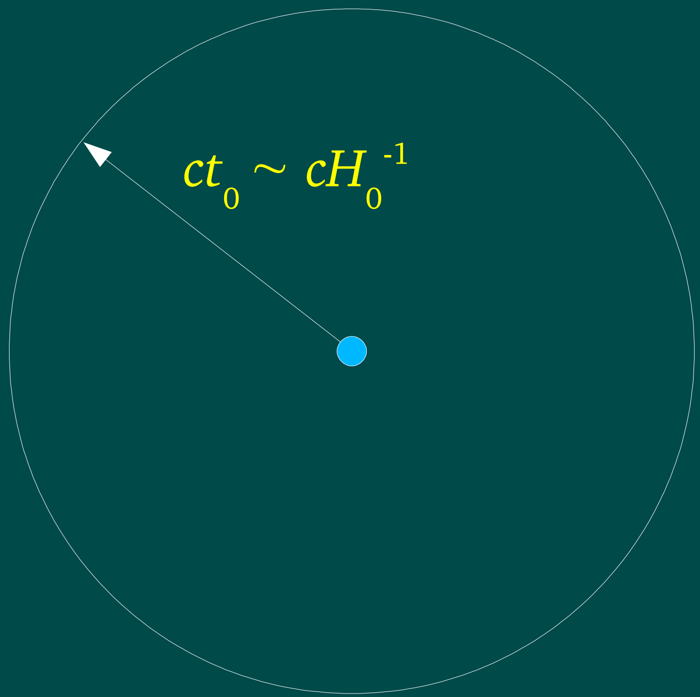
Horizon scale:

$$R_{\text{horizon}} \equiv c t_0 = \frac{2}{3} R_{\text{Hubble}}\ \text{for}\ \Omega_0 = 1,\ \Lambda = 0$$

Curvature scale:

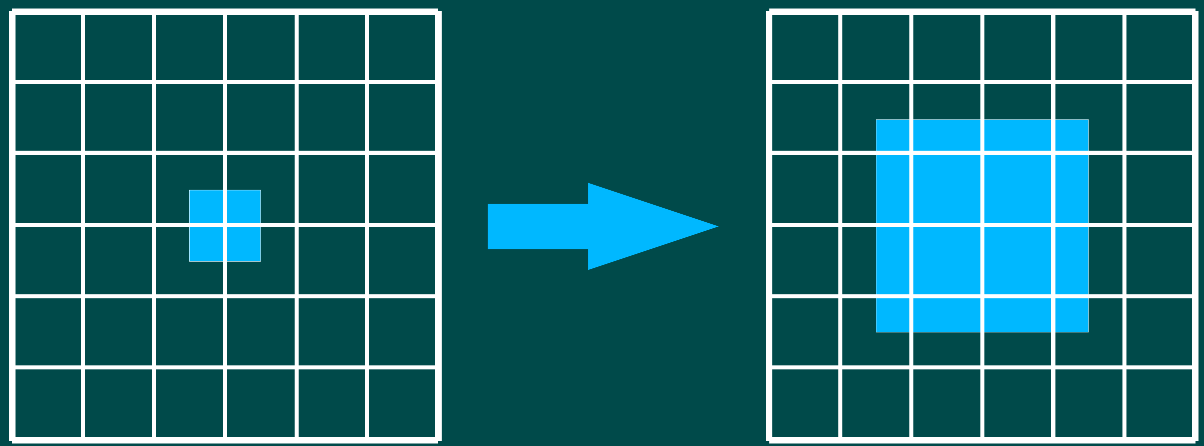
$$R_{\text{curv}} = \frac{c}{H_0}\left[\frac{\Omega_0 - 1}{k}\right]^{-1/2} \rightarrow \infty\ \text{as}\ \Omega_0 \rightarrow 1$$

horizon scale

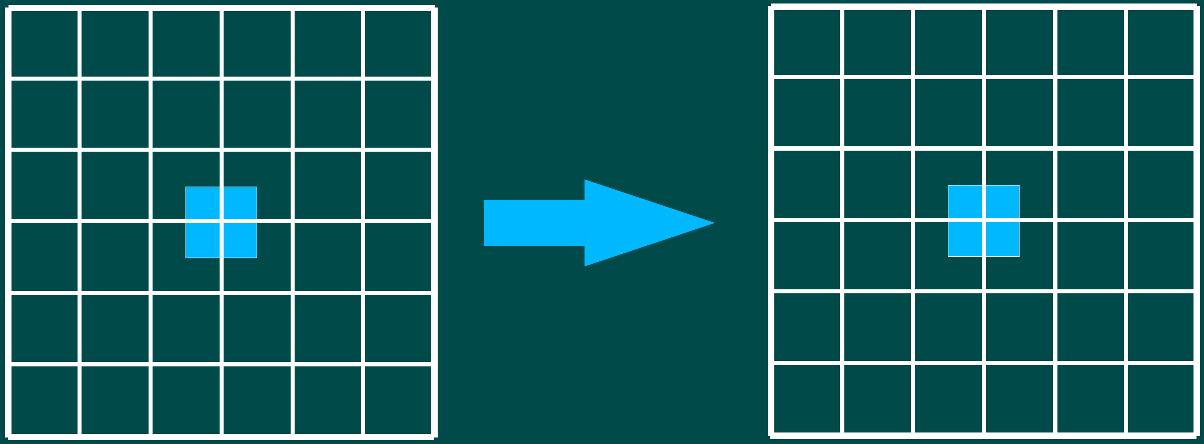


Comoving coordinates

Proper coordinates $\mathbf{r}$ are Eulerian: $\mathbf{u} = \dot{\mathbf{r}}$



Comoving coordinates $\mathbf{x}$ divide out the expansion: $\mathbf{v} = \dot{\mathbf{x}}$



$$\mathbf{r} = a\,\mathbf{x} \Rightarrow \mathbf{u} = a\,\dot{\mathbf{x}} + \dot{a}\,\mathbf{x} = a\,\mathbf{v} + H\,\mathbf{r}$$

Linear perturbation theory

On length scales small relative to the horizon scale or curvature radius, we can treat small deviations from perfect homogeneity using Newtonian physics:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} = -\frac{1}{\rho} \nabla P - \nabla \Phi$$

$$\nabla^2 \Phi = 4 \pi G \rho$$

in proper coordinates.

Perturb these equations about the uniformly expanding homogeneous state:

$$\rho = \bar{\rho}(1+\delta) \quad P = \bar{P} + \delta P \quad \mathbf{u} = H \mathbf{r} + \delta \mathbf{u}$$

The base state satisfies

$$\frac{\partial \bar{\rho}}{\partial t} + \bar{\rho} H \nabla \cdot \mathbf{r} = 0 \rightarrow \frac{\partial \bar{\rho}}{\partial t} = -3 H \bar{\rho}$$

No need for a “Jeans swindle!”

Linear perturbation theory – 2

Now substitute into the continuity equation and use the base state equation:

$$\frac{\partial \bar{\rho}}{\partial t}(1+\delta)+\bar{\rho}\frac{\partial \delta}{\partial t}+\bar{\rho}(1+\delta)\nabla\cdot(H\mathbf{r}+\delta\mathbf{u})+(H\mathbf{r}+\delta\mathbf{u})\cdot\nabla[\bar{\rho}(1+\delta)]=0$$

This gives

$$-3H\bar{\rho}(1+\delta)+\bar{\rho}\frac{\partial \delta}{\partial t}+\bar{\rho}(1+\delta)(3H+\nabla\cdot\delta\mathbf{u})+\bar{\rho}(H\mathbf{r}+\delta\mathbf{u})\cdot\nabla\delta=0$$

Cancelling terms and dropping terms of higher than linear order, we get

$$\frac{\partial \delta}{\partial t}+H\mathbf{r}\cdot\nabla\delta=-\nabla\cdot\delta\mathbf{u}$$

Similiarly, the perturbed Euler and Poisson equations become

$$\frac{\partial \delta\mathbf{u}}{\partial t}+H\mathbf{r}\cdot\nabla\delta\mathbf{u}=-\frac{1}{\bar{\rho}}\nabla\delta P-\nabla\delta\Phi-(\delta\mathbf{u}\cdot\nabla)H\mathbf{r}$$

$$\nabla^2\delta\Phi=4\pi G\bar{\rho}\delta$$

The last term in the Euler equation is just $-H\delta\mathbf{u}$.

Linear perturbation theory – 3

Convert to comoving coordinates $\mathbf{x}$: $\nabla_{\mathbf{x}} = a\nabla_{\mathbf{r}}$, and $\delta\mathbf{u} = a\mathbf{v}$, so

$$\frac{d\delta}{dt} = -\nabla\cdot\mathbf{v}$$

$$\frac{d\mathbf{v}}{dt} = -2\frac{\dot{a}}{a}\mathbf{v} - \frac{1}{a^2\bar{\rho}}\nabla\delta P - \frac{1}{a^2}\nabla\delta\Phi$$

$$\nabla^2\delta\Phi = 4\pi G\bar{\rho}\delta a^2$$

where $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + H\,\mathbf{x}\cdot\nabla$ and we drop the subscript $\mathbf{x}$ on $\nabla_{\mathbf{x}}$. Take $\mathrm{d}/\mathrm{d}t$ of the continuity equation and $\nabla\cdot$ of the Euler equation to obtain

$$\ddot{\delta} = -\nabla\cdot\dot{\mathbf{v}}$$

$$\nabla\cdot\dot{\mathbf{v}} = -2\frac{\dot{a}}{a}\nabla\cdot\mathbf{v} - \frac{c_s^2}{a^2}\nabla^2\delta - \frac{1}{a^2}\nabla^2\delta\Phi$$

where we have used

$$c_s^2 = \frac{\partial P}{\partial\rho} \Rightarrow \delta P = c_s^2\bar{\rho}\delta \Rightarrow \nabla\delta P = c_s^2\bar{\rho}\nabla\delta$$

Linear perturbation theory – 4

Finally, combine the continuity and Euler equations and use the Poisson equation to eliminate Φ, yielding

$$\ddot{\delta}+2\frac{\dot{a}}{a}\dot{\delta}-\frac{c_s^2}{a^2}\nabla^2\delta-4\pi G\bar{\rho}\delta=0$$

This is the cosmological (matter-dominated) version of the Jeans perturbation equation. As before, we can take a spatial Fourier transform (comoving) to obtain

$$\ddot{\delta}_k+2\frac{\dot{a}}{a}\dot{\delta}_k=\left(4\pi G\bar{\rho}-\frac{c_s^2 k^2}{a^2}\right)\delta_k$$

(matter domination, sub-horizon-scale)

Again we have instability growth if the RHS is > 0; but the critical wavenumber k_J now changes with time:

$$k_J=\frac{2}{c_s}\sqrt{\pi G\bar{\rho}}=\frac{2}{c_s}\sqrt{\frac{\pi G\rho_0}{a(t)^3}}$$

Modes with $k > k_J$ oscillate (sound waves); modes with $k < k_J$ are unstable.

We are interested in unstable modes... so drop the pressure term...

Linear perturbation theory – 5

If we consider only unstable modes and drop the pressure term, we can use the resulting equation for all nonrelativistic matter on sub-horizon scales, even during the radiation-dominated era:

$$\ddot{\delta}_k + 2\frac{\dot{a}}{a}\dot{\delta}_k = 4\pi G\bar{\rho}\delta_k$$

Consider the case $\Omega_0 = 1$. We have

Matter domination: $a \propto t^{2/3}, \;\; \rho \propto a^{-3}$

$$\ddot{\delta}_k + \frac{4}{3t}\dot{\delta}_k = \frac{2}{3t^2}\delta_k \Rightarrow \delta_k \propto t^{2/3} \text{ (growing) } t^{-1} \text{ (decaying)}$$

Radiation domination: $a \propto t^{1/2}, \;\; \rho_{\mathrm{m}} \propto a^{-3}, \;\; \rho_{\mathrm{r}} \propto a^{-4}$

Radiation Jeans length is comparable to the horizon size, so treat it as smooth.

$$\ddot{\delta}_k + \frac{1}{t}\dot{\delta}_k \approx 0 \Rightarrow \delta_k \propto \alpha + \beta \ln t \qquad \alpha, \beta \text{ constants}$$

The Universe expands too rapidly for dark matter fluctuations to grow.
For baryons, radiation pressure keeps sub-horizon-scale fluctuations from growing.

Linear perturbation theory – 6

Some additional points regarding sub-horizon-scale matter fluctuations:

- Growing mode corresponds to vorticity-free flow.

$$\nabla \cdot \mathbf{v} = -\dot{\delta} \Rightarrow \text{ mode with } \nabla \cdot \mathbf{v} = 0 \text{ has } \dot{\delta} = 0 \Rightarrow \text{ zero amplitude}$$

Take Fourier transform to get growing-mode peculiar velocity

$$\mathbf{v}_k = -\frac{i}{k} H\, f(\Omega)\, \delta_k \mathbf{k} \qquad f(\Omega) \equiv \frac{a}{\delta_k}\frac{d\,\delta_k}{da} \approx \Omega^{0.6} \text{ (Lahav et al. 1991)}$$

- In open universes, fluctuation growth "freezes out."

$$\delta_k \propto \begin{cases} \dfrac{1}{1+z} \propto t^{2/3} & 1+z > \Omega_0^{-1} \\ \text{constant} & 1+z < \Omega_0^{-1} \end{cases}$$

- In universes with vacuum energy, fluctuation growth is diminished.

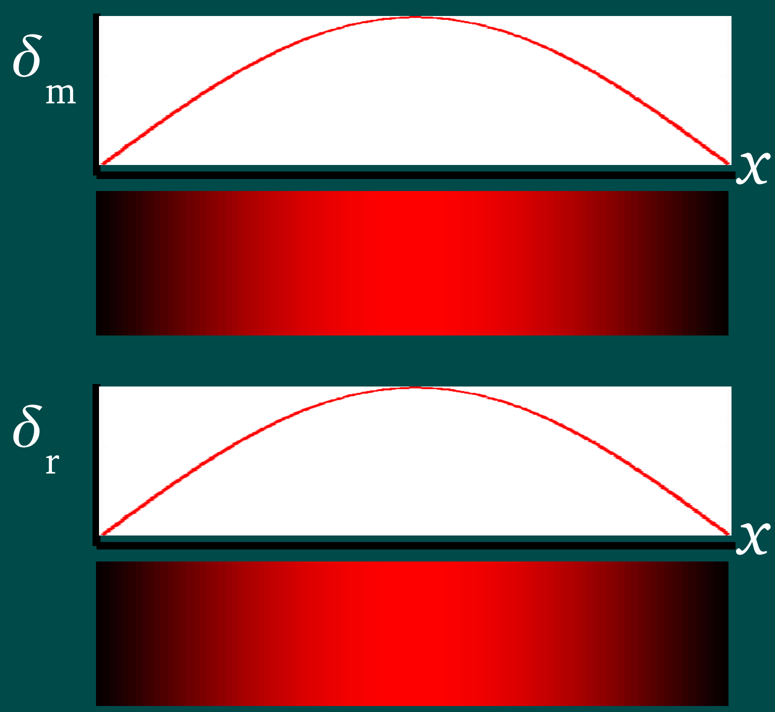
Freeze-out at $1+z \approx \Omega_{\mathrm{m},0}^{-1/3}$ for $\Omega_{\mathrm{m},0} < 1, \;\; \Omega_{\mathrm{m},0} + \Omega_{\mathrm{v},0} = 1$

Linear perturbation theory – 7

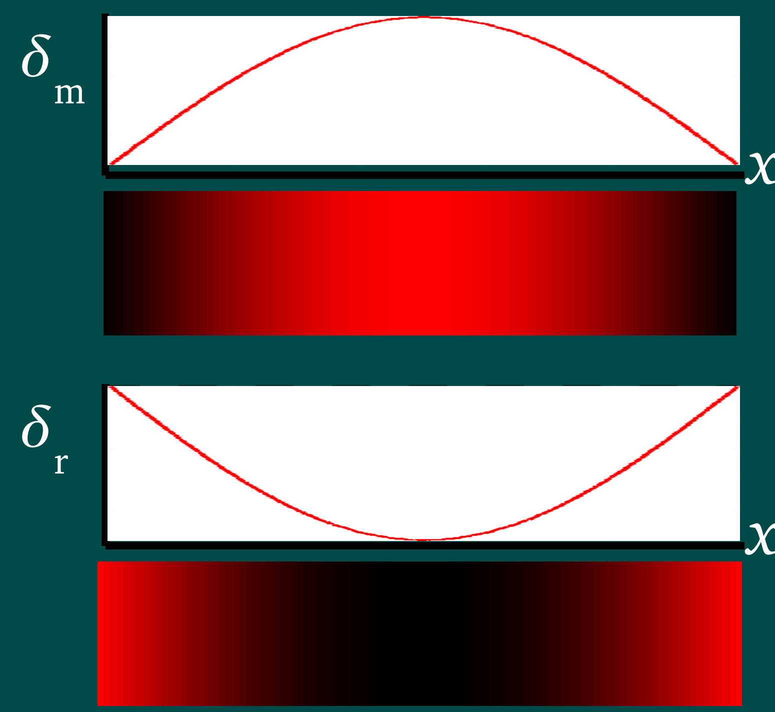
On super-horizon scales the analysis requires linearized general relativity... ignoring pressure, the matter and radiation perturbations evolve according to

$$\left[\frac{d^2}{dt^2}+2\frac{\dot{a}}{a}\frac{d}{dt}\right]\begin{bmatrix}\delta_{\mathrm{m}}\\ \delta_{\mathrm{r}}\end{bmatrix}=4\pi G\begin{bmatrix}\bar{\rho}_{\mathrm{m}} & 2\bar{\rho}_{\mathrm{r}}\\ 4\bar{\rho}_{\mathrm{m}}/3 & 8\bar{\rho}_{\mathrm{r}}/3\end{bmatrix}\begin{bmatrix}\delta_{\mathrm{m}}\\ \delta_{\mathrm{r}}\end{bmatrix}$$

This equation has two eigenmodes:



Adiabatic: matter and photons compressed together



Isocurvature: constant curvature, matter/radiation ratio perturbed

Linear perturbation theory – 8

Adiabatic modes are "honest-to-God" curvature fluctuations.

- On sub-horizon scales: radiation free-streams out of its perturbation.
  - During radiation-dominated era: matter fluctuations frozen (dark matter because of expansion rate, baryons because of radiation pressure)
  - During matter-dominated era: dark matter fluctuations grow; baryon fluctuations continue to feel radiation pressure (*Silk damping*) until recombination, then begin to grow

- On super-horizon scales: radiation *cannot* free-stream.
  - The adiabatic mode corresponds to the eigenvector (1, 4/3): $\delta_{\mathrm{r}} = 4\delta_{\mathrm{m}}/3$ and both perturbations evolve via

$$\ddot{\delta} + 2\frac{\dot{a}}{a}\dot{\delta} = 4\pi G\left(\bar{\rho}_{\mathrm{m}} + \frac{8}{3}\bar{\rho}_{\mathrm{r}}\right)\delta$$

For $\Omega = 1$ the growing mode solution is $\delta \propto a^2$ (radiation-dominated) or $\delta \propto a$ (matter-dominated). (Super-horizon-scale adiabatic perturbations evolve keeping the gauge-invariant quantity $\delta\rho/(\rho + P/c^2)$ constant.)

Linear perturbation theory – 9

Isocurvature modes (sometimes called *isothermal* modes) are fluctuations in the local equation of state.

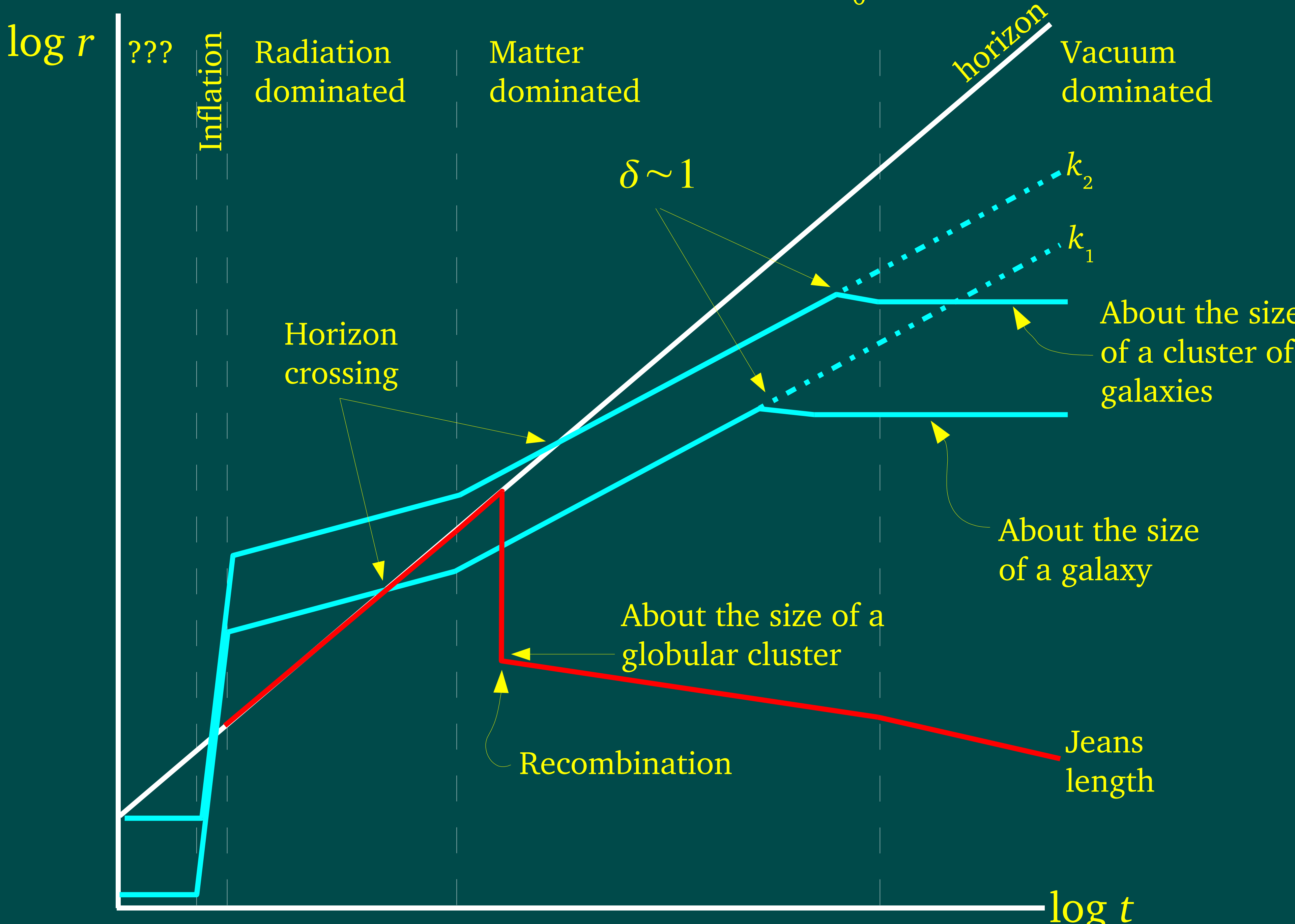
- On sub-horizon scales: isocurvature modes behave essentially like adiabatic modes: pressure variations turn into density variations.
  - During radiation domination, modes do not grow, but oscillate as sound waves.
  - After matter becomes dominant, isocurvature modes behave the same way as adiabatic modes.
- On super-horizon scales: through a suitable choice of *gauge* in the Einstein equations (which we are free to do because a unique gauge cannot be established outside the horizon), $\delta\rho = \delta\rho_{\mathrm{m}} + \delta\rho_{\mathrm{r}} = 0$.
  - Modes do not grow because pressure variations cannot drive material over super-horizon-scale distances.
  - Compensating temperature fluctuation in radiation field:

$$\delta\rho = 0 \Rightarrow \delta\left(\rho_m + \mathrm{constant} \times T^4\right) = 0$$

$$\Rightarrow \frac{\delta T}{T} = -\frac{1}{4}\frac{\rho_{\mathrm{m}}}{\rho_{\mathrm{r}}}\frac{\delta\rho_{\mathrm{m}}}{\rho_{\mathrm{m}}}$$

During radiation domination, $\delta T/T \ll 1$, hence the name “isothermal.”

Evolution of important proper length scales ($\Omega_0 = 1$)



Supercomoving coordinates

Define the comoving density, pressure, temperature, and internal energy for the gas via

$$\rho_{\rm g} = a^3 \tilde{\rho}_{\rm g} \qquad p = a\tilde{p} \qquad T = \frac{\tilde{T}}{a^2} \qquad \rho_{\rm g}\epsilon = a\tilde{\rho}_{\rm g}\tilde{\epsilon}$$

The "tilde" quantities are the *proper* density, pressure, etc. The comoving quantities correspond to the proper ones at $a = 1$ (today). Also use comoving coordinates $\mathbf{x}$.

With these definitions, the Euler equations with self-gravity and cooling are

$$\frac{\partial}{\partial t}\rho_{\rm g} + \nabla \cdot [\rho_{\rm g}\mathbf{v}_{\rm g}] = 0$$

$$\frac{\partial}{\partial t}(\rho_{\rm g}\mathbf{v}_{\rm g}) + \nabla \cdot [\rho_{\rm g}\mathbf{v}_{\rm g}\mathbf{v}_{\rm g}] + \nabla p + 2\frac{\dot{a}}{a}\rho_{\rm g}\mathbf{v}_{\rm g} + \rho_{\rm g}\nabla\phi = 0$$

$$\frac{\partial}{\partial t}(\rho_{\rm g}E) + \nabla \cdot [(\rho_{\rm g}E + p)\,\mathbf{v}_{\rm g}] + \frac{\dot{a}}{a}\left[(3\gamma - 1)\rho_{\rm g}\epsilon + 2\rho_{\rm g}v_{\rm g}^2\right] - \rho_{\rm g}\frac{\partial\phi}{\partial t} + \Lambda = 0$$

$$\frac{\partial}{\partial t}(\rho_{\rm g}\epsilon) + \nabla \cdot [(\rho_{\rm g}\epsilon + p)\,\mathbf{v}_{\rm g}] - \mathbf{v}_{\rm g}\cdot\nabla p + \frac{\dot{a}}{a}(3\gamma - 1)\rho_{\rm g}\epsilon + \Lambda = 0$$

Redshift terms

$$\rho_{\rm g}E \equiv \rho_{\rm g}\epsilon + \frac{1}{2}\rho_{\rm g}v_{\rm g}^2 + \rho_{\rm g}\phi$$

$$\rho_{\rm g}\epsilon = \frac{p}{\gamma - 1} = \frac{\rho_{\rm g}kT}{(\gamma - 1)\mu}$$

Supercomoving coordinates – 2

The redshift terms can be treated using operator splitting; they look like

$$\frac{d\,\rho\,\mathbf{v}}{dt} = -2\,\frac{\dot{a}}{a}\,\rho\,\mathbf{v}$$

$$\frac{d\,\rho\,E}{dt} = -\frac{\dot{a}}{a}\left[(3\,\gamma-1)\,\rho\,\epsilon + 2\,\rho\,v^2\right]$$

$$\frac{d\,\rho\,\epsilon}{dt} = -\frac{\dot{a}}{a}(3\,\gamma-1)\,\rho\,\epsilon$$

Solution is straightforward –

$$\frac{dX}{dt} = -\alpha(t)\,X \Rightarrow X = \text{constant} \times e^{-\int \alpha\,dt}$$

$$\Rightarrow X_{n+1} = X_n\,e^{-\int_n^{n+1} \alpha\,dt} = X_n\,e^{-\alpha_{n+1/2}\Delta t + O(\Delta t^2)}$$

Supercomoving coordinates – 3

The Poisson equation for the comoving potential is

$$\nabla^2\phi = \frac{4\pi G}{a^3}\left[(\rho_{\rm g}+\rho_{\rm dm}) - \overline{(\rho_{\rm g}+\rho_{\rm dm})}\right]$$

The equations of motion for particles (dark matter, stars, etc.) are

$$\frac{d\mathbf{x}_{\rm dm}}{dt} = \mathbf{v}_{\rm dm}$$

$$\frac{d\mathbf{v}_{\rm dm}}{dt} + 2\frac{\dot{a}}{a}\mathbf{v}_{\rm dm} = -\nabla\phi$$

And of course the Friedmann equation can be solved numerically with an ODE integrator.

$$H^2(t) \equiv \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\left(\frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} + \Omega_\Lambda - \frac{\Omega_c}{a^2}\right)$$

Initializing cosmological simulations (grid)

1. Compute the Fourier transform of the density fluctuation field, $\delta_{\mathbf{k}} = |\delta_{\mathbf{k}}| \exp(i\theta_{\mathbf{k}})$:

   For each k-space zone pqr, $|\delta_{pqr}| = D_+(z)\sqrt{P(k_{pqr}, z=0)\,\eta}$

   η an exponential deviate.

   phase $\theta_{pqr} = 2\pi\zeta$ ζ a uniform deviate in [0,1).

   Note 1: must have $\delta_{N-p,N-q,N-r} = \delta_{pqr}$, $\theta_{N-p,N-q,N-r} = -\theta_{pqr}$ since $\delta(\mathbf{x})$ is real-valued.

   Note 2: usually choose initial redshift z so that $\max[\delta(\mathbf{x})] = 1$.

2. Inverse Fourier transform to get the real-space density fluctuation $\delta_{ijk} = \delta(\mathbf{x}_{ijk})$.

3. To get the velocity field, use $\nabla \cdot \mathbf{v} = -\dot{\delta}$ and the fact that the velocity is potential:

$$\mathbf{v}_{pqr} = \frac{i\,\mathbf{k}_{pqr}}{k_{pqr}^2}\frac{\dot{D}_+}{D_+}\delta_{pqr}$$

   then inverse Fourier transform to get $\mathbf{v}_{ijk} = \mathbf{v}(\mathbf{x}_{ijk})$.

Initializing cosmological simulations (particles)

1. Take unperturbed positions $\mathbf{q}$ to lie on a grid: $\mathbf{q}_{ijk} = (i\Delta x, j\Delta y, k\Delta z)$

2. Compute the Fourier transform of the velocity potential ψ and velocity $\mathbf{v}$:

$$\mathbf{v} = \nabla\psi \Rightarrow \mathbf{v}_{pqr} = i\mathbf{k}_{pqr}\psi_{pqr}$$

$$\nabla\cdot\mathbf{v} = -\dot{\delta} \Rightarrow \nabla^2\psi = -\dot{\delta}$$

$$\psi_{pqr} = \frac{\dot{\delta}_{pqr}}{k_{pqr}^2}$$

$$\mathbf{v}_{pqr} = \frac{i\mathbf{k}_{pqr}}{k_{pqr}^2}\dot{\delta}_{pqr}$$

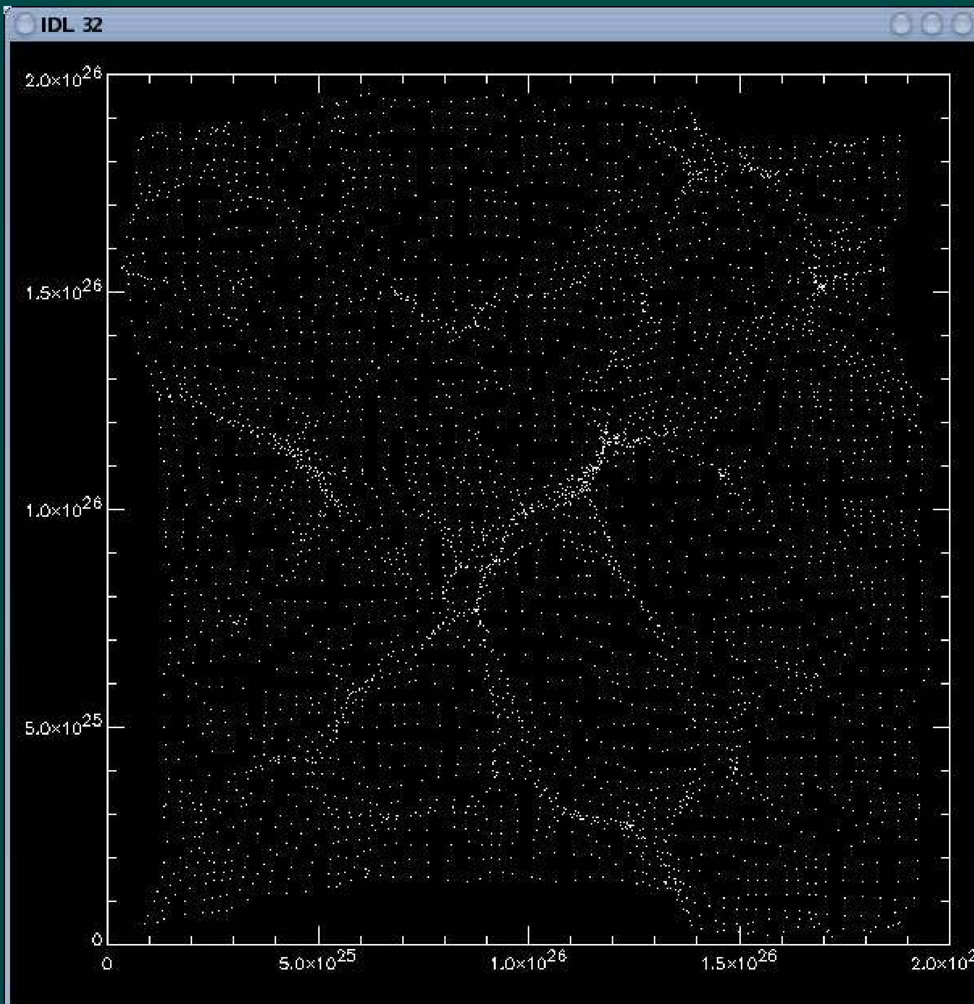
   The δ_{pqr} are computed as for grid-based initialization.

3. Inverse Fourier transform to get the particle velocities. The displaced particle positions are then

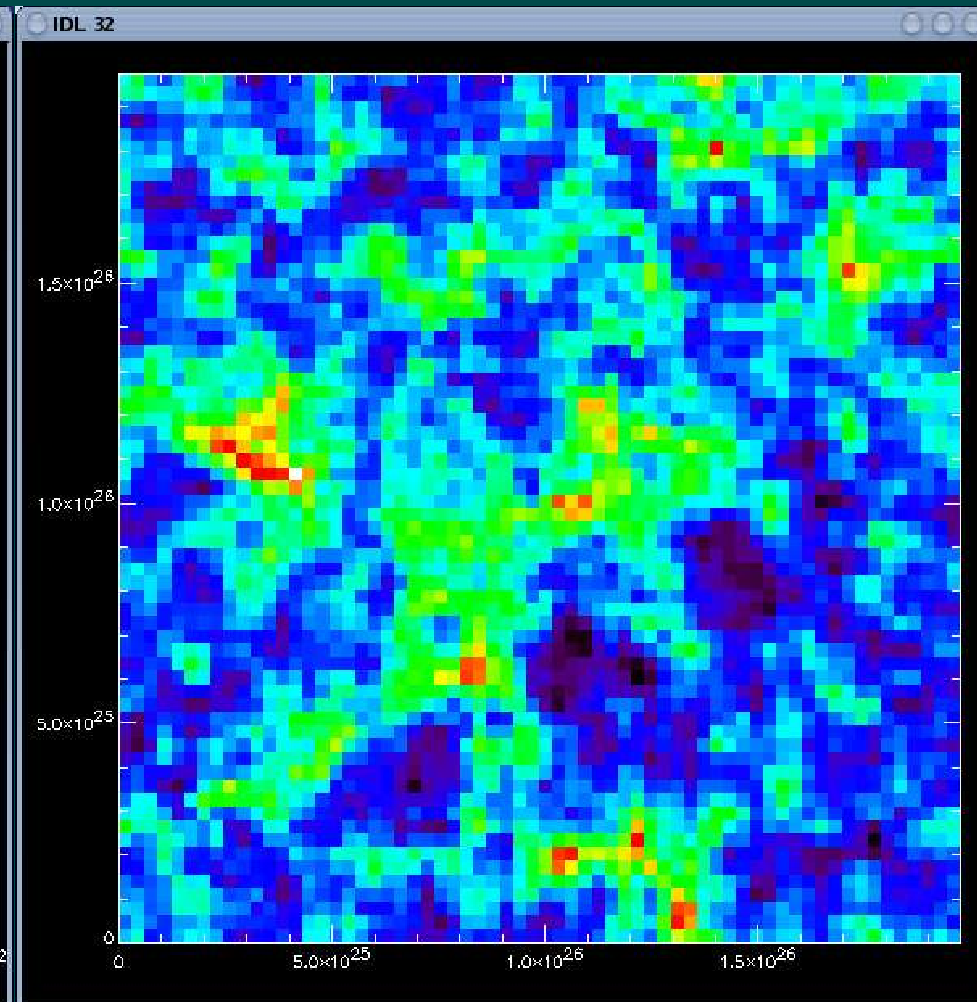
$$\mathbf{x}_{ijk} = \mathbf{q}_{ijk} + \frac{D_+}{\dot{D}_+}\mathbf{v}_{ijk}$$

Zel'dovich approximation example

Dark matter particle positions



Mesh gas overdensities



(displacements multiplied by 7)