Geodesic equation

$$\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\beta\gamma} \frac{dx^\beta}{d\lambda} \frac{dx^\gamma}{d\lambda} = 0$$

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta} \left(g_{\beta\delta,\gamma} + g_{\gamma\delta,\beta} - g_{\beta\gamma,\delta} \right)$$

$$= \frac{1}{2} \eta^{\alpha\delta} \left(h_{\beta\delta,\gamma} + h_{\gamma\delta,\beta} - h_{\beta\gamma,\delta} \right)$$

If a light ray is moving only in the x-t plane, then

$$ds^2 = 0 = -dt^2 + (1+h_{11})\, dx^2$$

For given $h_{\alpha\beta}$, $\Gamma^\alpha_{00} = 0 \quad \therefore \quad \frac{d^2 x^\alpha}{d\lambda^2} = 0$

Mirrors and base do not move!

Along x-arm, $\Delta t_1 = \sqrt{1+h_{11}}\; 2\Delta$ where $\Delta = 4$ km

y-arm $\Delta t_2 = \sqrt{1-h_{11}}\; 2\Delta$

$$\Delta t_1 - \Delta t_2 = 2\Delta\, h_{11}$$

Transverse Traceless (TT) gauge

A wave propagating in the $+z$ direction can be described as $(t-z)$

$$h_{\mu\nu} = \bar{h}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{11}(t-z) & h_{12}(t-z) & 0 \\ 0 & h_{12}(t-z) & -h_{11}(t-z) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- Satisfies wave equation $\Box h_{\mu\nu} = 0$
- Satisfies Lorentz gauge $h^{\mu\nu}{}_{,\nu} = 0$
- Is traceless ($\eta^{\alpha\beta} h_{\alpha\beta} = 0$)
- Wave affects x-y plane, transverse to direction of propagation z.

Solution represents ANY (linearized) gravitational wave. Thus gravitational radiation has TWO degrees of freedom.

PROOF omitted. Uses gauge freedom to reduce 10 independent quantities ($h_{\mu\nu}$) to 2.

Usually, to make clear that we are in TT gauge, we write

$$h_{\mu\nu}^{TT}$$

Also use $h_+^{TT} = h_{11}^{TT} = -h_{22}^{TT}$; $h_\times^{TT} = h_{12}^{TT}$

beginning at rest. Both then remain at these coordinate positions, and the proper distance between them is

$$\Delta l = \int |ds^2|^{1/2} = \int |g_{\alpha\beta}\, dx^\alpha\, dx^\beta|^{1/2}$$
$$= \int_0^\varepsilon |g_{xx}|^{1/2}\, dx = |g_{xx}(x=0)|^{1/2}\varepsilon$$
$$\approx [1 + \tfrac{1}{2} h^{TT}_{xx}(x=0)]\varepsilon. \qquad (9.24)$$

Now, since h^{TT}_{xx} is not generally zero, the *proper* distance (as opposed to the coordinate distance) does change with time. This is an illustration of the difference between computing a coordinate-dependent number (the position of a particle) and a coordinate-independent number (the proper distance between two particles). The effect of the wave is unambiguously seen in the coordinate-independent number.

Another approach to the same question involves the equation of geodesic deviation, Eq. (6.87). Between the two particles set up the connecting vector ξ^α. It obeys the equation

$$\frac{d^2}{d\tau^2}\xi^\alpha = R^\alpha{}_{\mu\nu\beta} U^\mu U^\nu \xi^\beta, \qquad (9.25)$$

where $\vec{U} = d\vec{x}/d\tau$ is the four-velocity of the two particles. In these coordinates the components of $\vec{U}$ are needed only to lowest (i.e. flat-space) order, since any corrections to U^α that depend on $h_{\mu\nu}$ will give terms second order in $h_{\mu\nu}$ in the above equation (because $R^\alpha{}_{\mu\nu\beta}$ is already first order in $h_{\mu\nu}$). Therefore $\vec{U} \to (1, 0, 0, 0)$ and, initially, $\vec{\xi} \to (0, \varepsilon, 0, 0)$. Then, to first order in $h_{\mu\nu}$ Eq. (9.25) reduces to

$$\frac{d^2}{d\tau^2}\xi^\alpha = \frac{\partial^2}{\partial t^2}\xi^\alpha = \varepsilon R^\alpha{}_{00x} = -\varepsilon R^\alpha{}_{0x0}. \qquad (9.26)$$

Now, it is a simple matter to use Eq. (8.25) to show that, in the TT gauge,

$$\left.\begin{aligned} R^x{}_{0x0} &= R_{x0x0} = -\tfrac{1}{2} h^{TT}_{xx,00}, \\ R^y{}_{0x0} &= R_{y0x0} = -\tfrac{1}{2} h^{TT}_{xy,00}, \\ R^y{}_{0y0} &= R_{y0y0} = -\tfrac{1}{2} h^{TT}_{yy,00} = -R^x{}_{0x0}, \end{aligned}\right\} \qquad (9.27)$$

and all other independent components vanish. This means that two particles initially separated in the x direction have a separation vector which obeys

$$\frac{\partial^2}{\partial t^2}\xi^x = \tfrac{1}{2}\varepsilon \frac{\partial^2}{\partial t^2} h^{TT}_{xx}, \quad \frac{\partial^2}{\partial t^2}\xi^y = \tfrac{1}{2}\varepsilon \frac{\partial^2}{\partial t^2} h^{TT}_{xy}, \qquad (9.28a)$$

which is clearly consistent with Eq. (9.24). Similarly, two particles initially

separated by ε in the y direction obey

$$\frac{\partial^2}{\partial t^2}\xi^y = \tfrac{1}{2}\varepsilon \frac{\partial^2}{\partial t^2} h^{TT}_{yy} = -\tfrac{1}{2}\varepsilon \frac{\partial^2}{\partial t^2} h^{TT}_{xx},$$
$$\frac{\partial^2}{\partial t^2}\xi^x = \tfrac{1}{2}\varepsilon \frac{\partial^2}{\partial t^2} h^{TT}_{xy}. \qquad (9.28b)$$

Polarization of gravitational waves. These equations help us describe the polarization of the wave. Consider a ring of particles initially at rest in the x–y plane, as in Fig. 9.1(*a*). Suppose a wave has $h^{TT}_{xx} \neq 0$, $h^{TT}_{xy} = 0$. Then the particles will be moved (in terms of proper distance relative to the one in the center) in the way shown in Fig. 9.1(*b*), first in (say), then

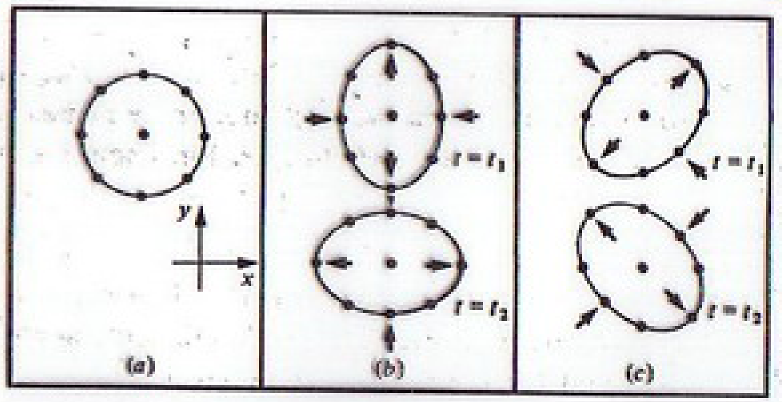


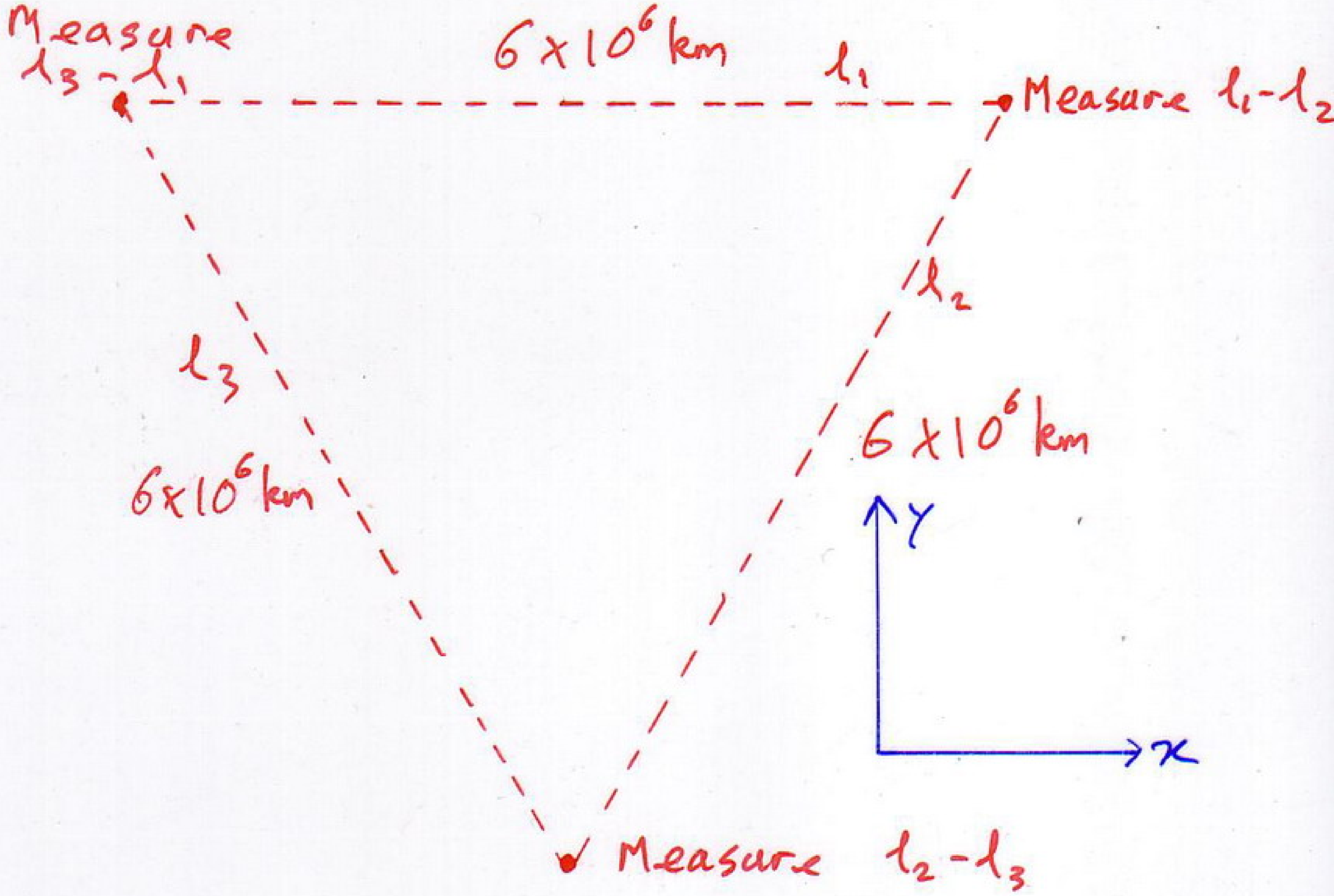
Fig. 9.1 (a) A circle of free particles before a wave travelling in the z direction reaches them. (b) Distortions of the circle produced by a wave with the '+' polarization. The two pictures represent the same wave at phases separated by 180°. Particles are positioned according to their proper distances from one another. (c) As (b) for the '×' polarization.

out, as the wave oscillates and h^{TT}_{xx} changes sign. If, instead, the wave had $h^{TT}_{xy} \neq 0$ but $h^{TT}_{xx} = h^{TT}_{yy} = 0$, then the picture would distort as in Fig. 9.1(*c*). Since h^{TT}_{xy} and h^{TT}_{xx} are independent, (*b*) and (*c*) provide a pictorial representation for two different linear polarizations. Notice that the two states are simply rotated 45° relative to one another. This constrasts with the two polarization states of an electromagnetic wave, which are 90° to each other. As Exer. 15, § 9.6, shows, this pattern of polarization is due to the fact that gravity is represented by the second-rank symmetric tensor $h_{\mu\nu}$. (By contrast, electromagnetism is represented by the vector potential A^μ of Exer. 11, § 8.6.)

An exact plane wave. Although all waves that we can expect to detect on Earth are so weak that linearized theory ought to describe them very

LISA

3 satellites arranged in equilateral Δ of side 6×10^6 km moving along // geodesics connected by lasers



$l_2 - l_3 \rightarrow h^{TT}_{xy} = h^{TT}_{\times}$

$l_3 - l_1 \rightarrow h^{TT}_{xx} \; (= -h^{TT}_{yy}) = h^{TT}_{+}$

$l_1 - l_2 \rightarrow$ consistency check

(In practice, use all 3 measurements and find least squares best fit for h_{xx}, h_{xy} + error estimate)

Generation of gravitational waves: Quadrupole formula

Start with

$$\Box \bar{h}_{\mu\nu} = -16\pi T_{\mu\nu} \quad \text{(in Lorentz gauge)}$$

$$\bar{h}_{\mu\nu}(t, x^i) = 4 \int_V \frac{T_{\mu\nu}(t-R, y^i)}{R} d^3y$$

with $R = |x^i - y^i|$

For $|x^i| = r >> |y^i|$

$$\bar{h}_{\mu\nu}(t, x^i) = \frac{4}{r} \int_V T_{\mu\nu}(t-r, y^i) d^3y$$

Use $T^{\mu\nu}{}_{,\nu} = 0$

$T^{\mu\nu} = 0$ on $S = \partial V$ so that

$$\int_V T^{\mu k}{}_{,k} f + T^{\mu k} f_{,k} d^3y = \int_V (T^{\mu k} f)_{,k} d^3y = \oint_S T^{\mu k} f n_k dS = 0$$

where f is any scalar or tensor

$$\bar{h}_{ij}(t, x^i) = \frac{2}{r} I_{ij,00}(t-r)$$

where $I_{ij} = \int_V \rho x_i x_j dV$

Translation to TT gauge

Define $\quad \bar{I}_{jk} = I_{jk} - \frac{1}{3}\delta_{jk} I^m_{\ m}$

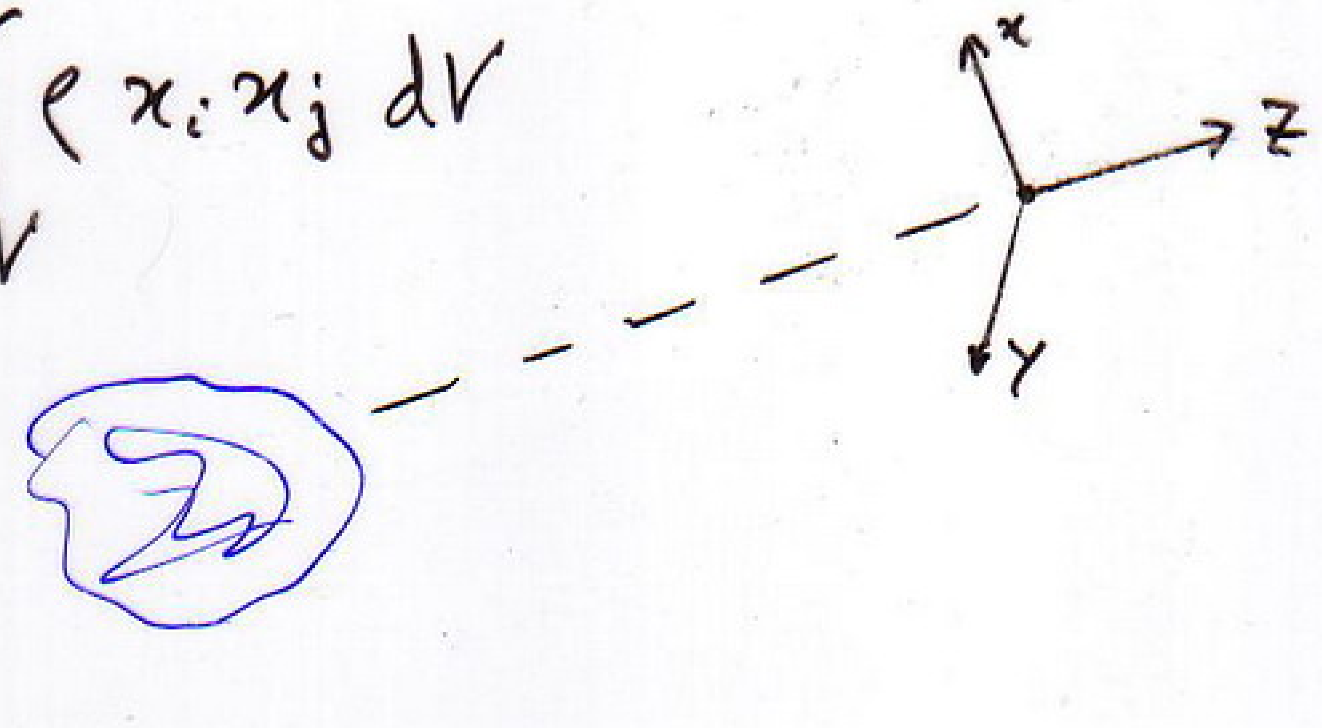
(Trace-free, or reduced, quadrupole moment tensor)

$$h^{TT}_{xx} = -h^{TT}_{yy} = \frac{1}{r}\left[\bar{I}_{xx,00}(t-r) - \bar{I}_{yy,00}(t-r)\right]$$

$$h^{TT}_{xy} = \frac{2}{r}\bar{I}_{xy,00}(t-r)$$

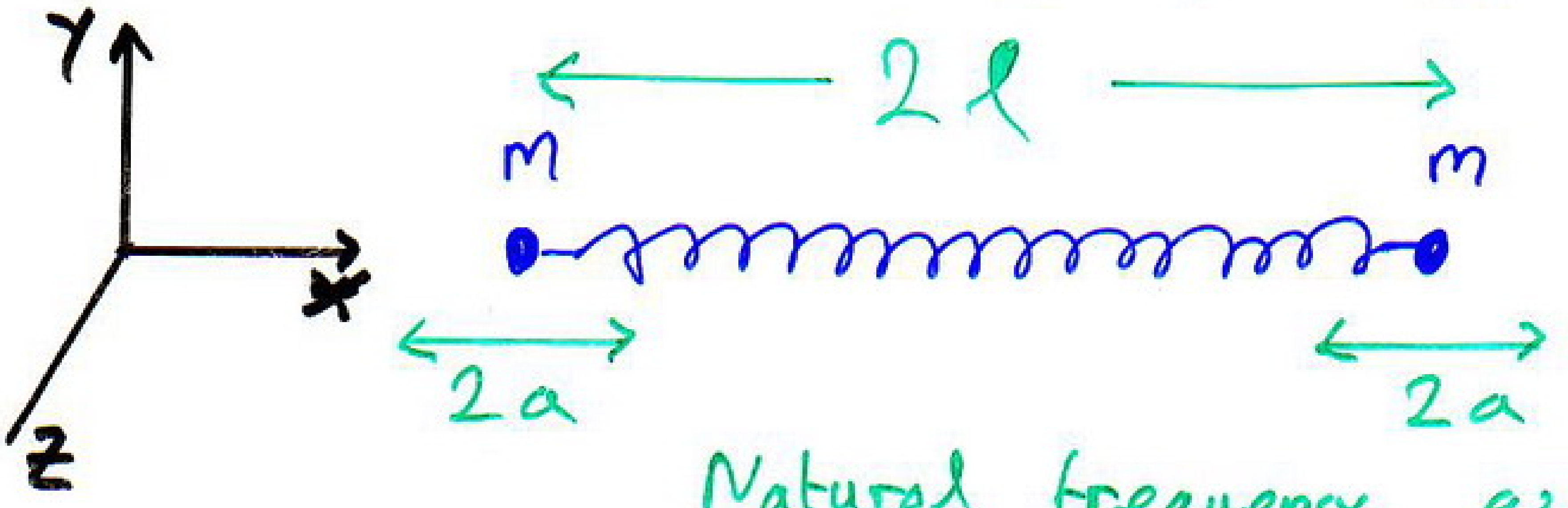
Of course, this is useful only if we are looking at the radiation on the z-axis.

$$I_{ij} = \int_V \rho\, x_i x_j\, dV$$



Exercises

① Consider an harmonic oscillator



Natural frequency ω

$$I_{xx} = m(x_1^2 + x_2^2) = \text{const.} + m a^2 \cos 2\omega t + 4 m l a \cos \omega t$$

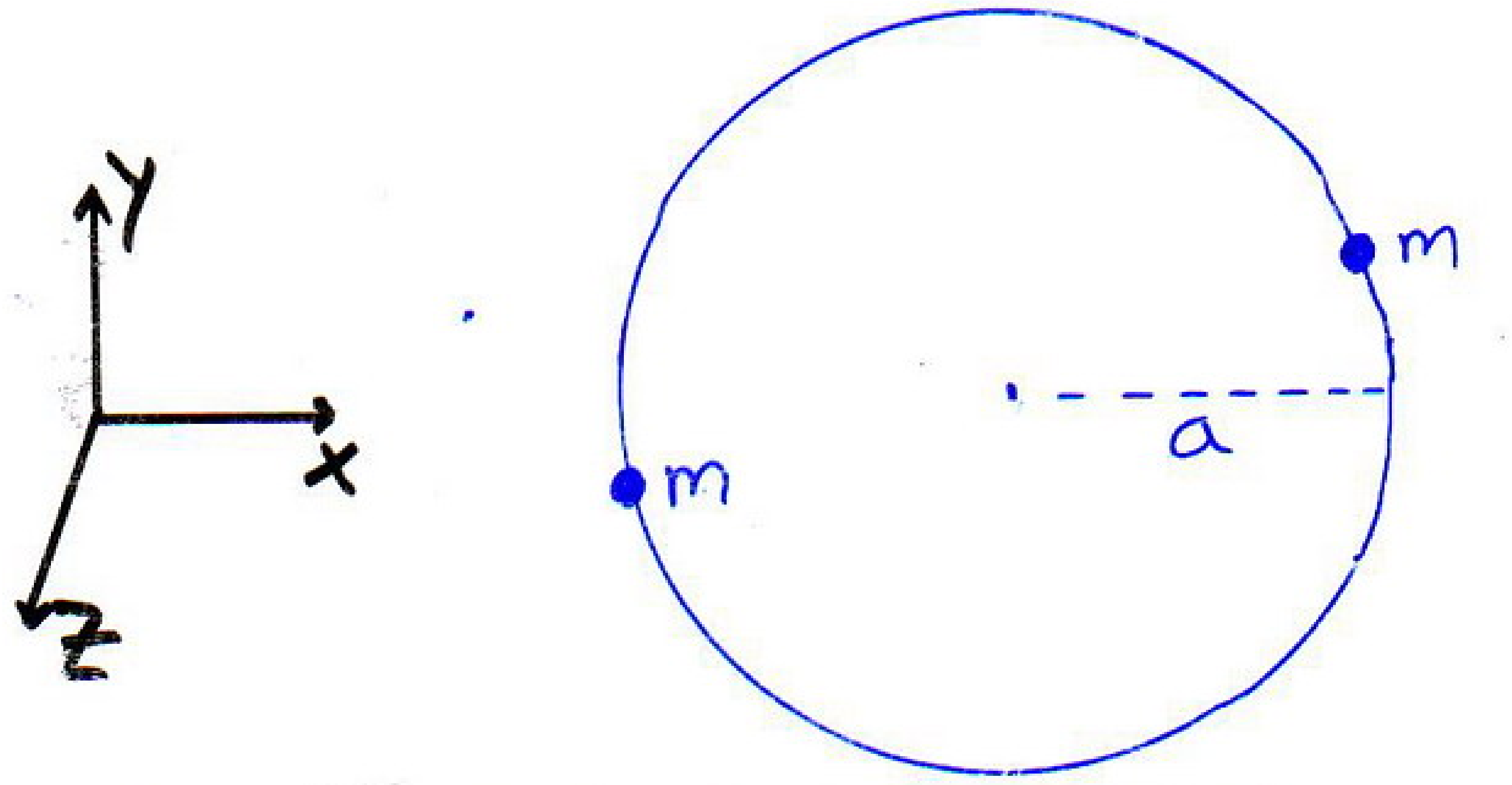
$$\mathcal{I}_{xx} = \tfrac{2}{3} I_{xx}, \quad \mathcal{I}_{yy} = \mathcal{I}_{zz} = -\tfrac{1}{3} I_{xx}$$

$$h_+^{TT} = h_{xx}^{TT} = -h_{yy}^{TT} = \frac{ma}{r}\left(a e^{2i\omega(t-r)} + 4 l e^{i\omega(t-r)}\right)_{,tt}$$

$$= -\frac{m a \omega^2 4}{r}\left(a e^{2i\omega(t-r)} + l e^{i\omega(t-r)}\right), \quad h_{xy}^{TT} = 0 = h_x^{TT}$$

On x-axis, $h_+^{TT} = h_x^{TT} = 0$

② Consider two equal mass stars in circular orbit around each other



$$I_{xx} = m a^2 (\cos^2 \omega t + (-\cos \omega t)^2) = 2 m a^2 \cos^2 \omega t = m a^2 \cos 2\omega t + c$$

$$I_{yy} = -m a^2 \cos 2\omega t + c_2 \qquad I_{xy} = m a^2 \sin 2\omega t$$

$$I^i{}_i = c_3 \quad \therefore \quad \mathcal{I}_{ij} = I_{ij} + \text{const.}$$

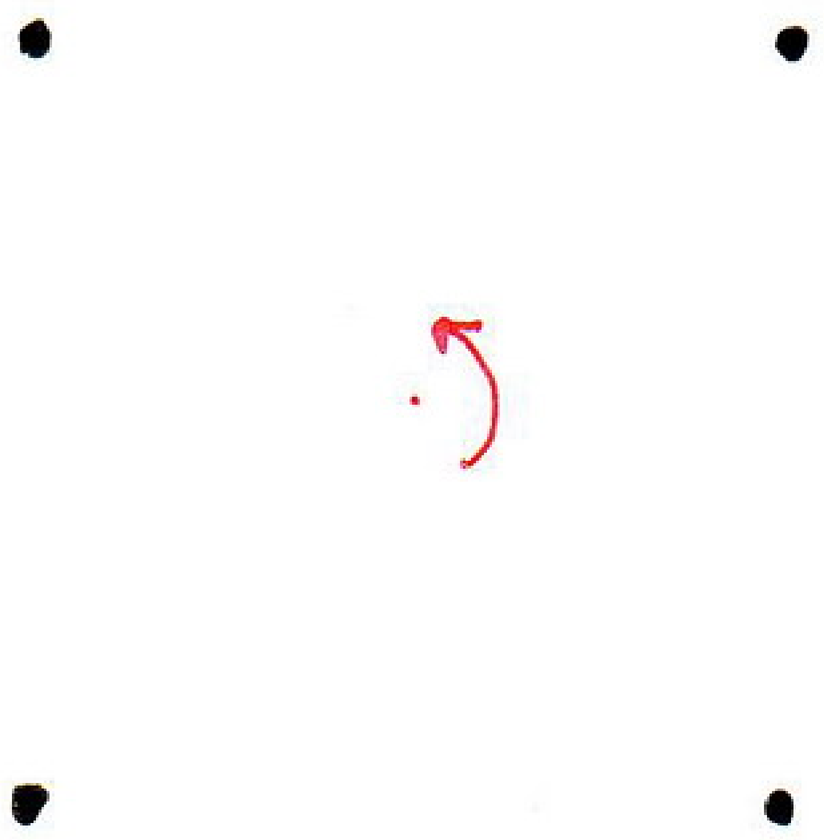
$$h_+^{TT} = -\frac{8 m a^2 \omega^2}{r} e^{2i\omega(t-r)}, \quad h_\times^{TT} = -\frac{8 i m a^2 \omega^2}{r} e^{2i\omega(t-r)}$$

In the plane of the orbit (wave propagating in x-direction)

$$h_+^{TT} = h_{yy}^{TT} = -h_{zz}^{TT} = \frac{1}{r}\left(I_{yy,00} - I_{zz,00}\right) = \frac{4 m a^2 \omega^2}{r} e^{2i\omega(t-r)}$$

$$h_\times^{TT} = h_{yz}^{TT} = 0$$

(3) Four equal mass particles are located at the corners of a square, that rotates uniformly about an axis through its centre and normal to its plane.



I_{ij} = constant $\rightarrow h_+^{TT} = h_\times^{TT} = 0$

For weak gravitational fields with slowly moving sources, the quadrupole formula leads to

$$L = \frac{1}{5} \langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle$$

for the energy output (= mass loss) per unit time.

The energy flux in a wave is the T^{03} component of

$$T_{\alpha\beta} = \langle\langle h^{TT}_{\mu\nu,\alpha} h^{TT\mu\nu}_{,\beta} \rangle\rangle$$

(assuming propagation in the z-direction)

Particle mass m in circular orbit around Schwarzschild black hole mass M

Orbit period $T = 2\pi \sqrt{\frac{a^3}{M}}$

Energy $E = m\left(1 - \frac{2M}{a}\right)\left(1 - \frac{3M}{a}\right)^{-\frac{1}{2}}$

$\sim m - \frac{Mm}{2a}$

A Schwarzschild black hole that experiences a small perturbation will radiate with

$$h_{TT} \sim \frac{1}{r} \exp\left(-\frac{\omega_1 t}{M} + \frac{i \omega_2 t}{M}\right)$$

The lowest mode has $\omega_1 = 0.0889$,

$\omega_2 = 0.373$

Investigate the waveform produced by the inspiral of a $10\,M_\odot$ black hole into a $10^6\,M_\odot$ Schwarzschild black hole

starting at $a = 10 \times 10^6\,M_\odot$

Suppose system is at 1 Mpc from detect

$10^6 M_\odot$ black hole moves so slowly about centre of mass that its contribution to I_{ij} is negligible. So, values are $\frac{1}{2}$ those obtained in Ex. 2

$$I_{xx} = \tfrac{1}{2} ma^2 \cos 2\omega t + C_1, \quad I_{yy} = -\tfrac{1}{2} ma^2 \cos 2\omega t + C_2$$
$$I_{xy} = \tfrac{1}{2} ma^2 \sin 2\omega t + C_3$$

$$\rightarrow L = \frac{32}{5}\frac{m^2 M^3}{a^5} = \frac{32}{5} \times 10^{-15}$$

$$T = 2\pi\sqrt{\frac{a^3}{M}} = 2\pi\sqrt{10}\ 10^7\ M_\odot$$

Conversions: $2\times 10^5 M_\odot \simeq 300000\,\text{km} \simeq 1\,\text{s}$

so $T \simeq 1000\,\text{s}.$

Energy emitted per orbit $= LT = \frac{64\pi\sqrt{10}}{5} \times 10^{-8}$

$\simeq 1.27\times 10^{-6} M_\odot$

This energy can only come from orbital decay

$$\frac{dE}{da} \simeq \frac{Mm}{2a^2} = 0.5\times 10^{-7}$$

Thus, orbital decay per orbit is

$$\Delta a = \frac{1.27\times 10^{-6}}{0.5\times 10^{-7}} \simeq 25.4\ M_\odot$$

For orbit to decay from $a = 10.5\times 10^6 M_\odot$ to $9.5\times 10^6 M_\odot$ requires $10^6/25.4$ orbits $\simeq 40\,000$ orbits taking $40000T = 4\times 10^7\,\text{s}$ ($\simeq 1.3$ years)

$$h^{TT}_{+,\times} \sim \frac{4 m a^2 \omega^2}{r}$$

$m = 10 M_\odot = 15\,\text{km}, \quad r = 1\,\text{Mpc} = 3 \times 10^{19}\,\text{km}$

$\omega = \dfrac{1}{\sqrt{10}\, 10^7}$

$$h^{TT}_{+,\times} \sim 2 \times 10^{-18}$$

E has minimum at $a = 6$

$a \leq 6$: No stable circular orbit

$a = 6$ is ISCO (Innermost Stable Circular Orbit)

For $a < 6$, plunge towards black hole $10^6 M_\odot$.

Waveform is maximal, and need numerical relativity to calculate it.

Wave signal grows in amplitude, and frequency increases, until plunge and merger phase starts

N.B. - We have applied quadrupole formula outside its domain of applicability ($v, \phi \ll 1$), so these results are only approximate

After merger, apply theory of linear perturbations of a black hole.

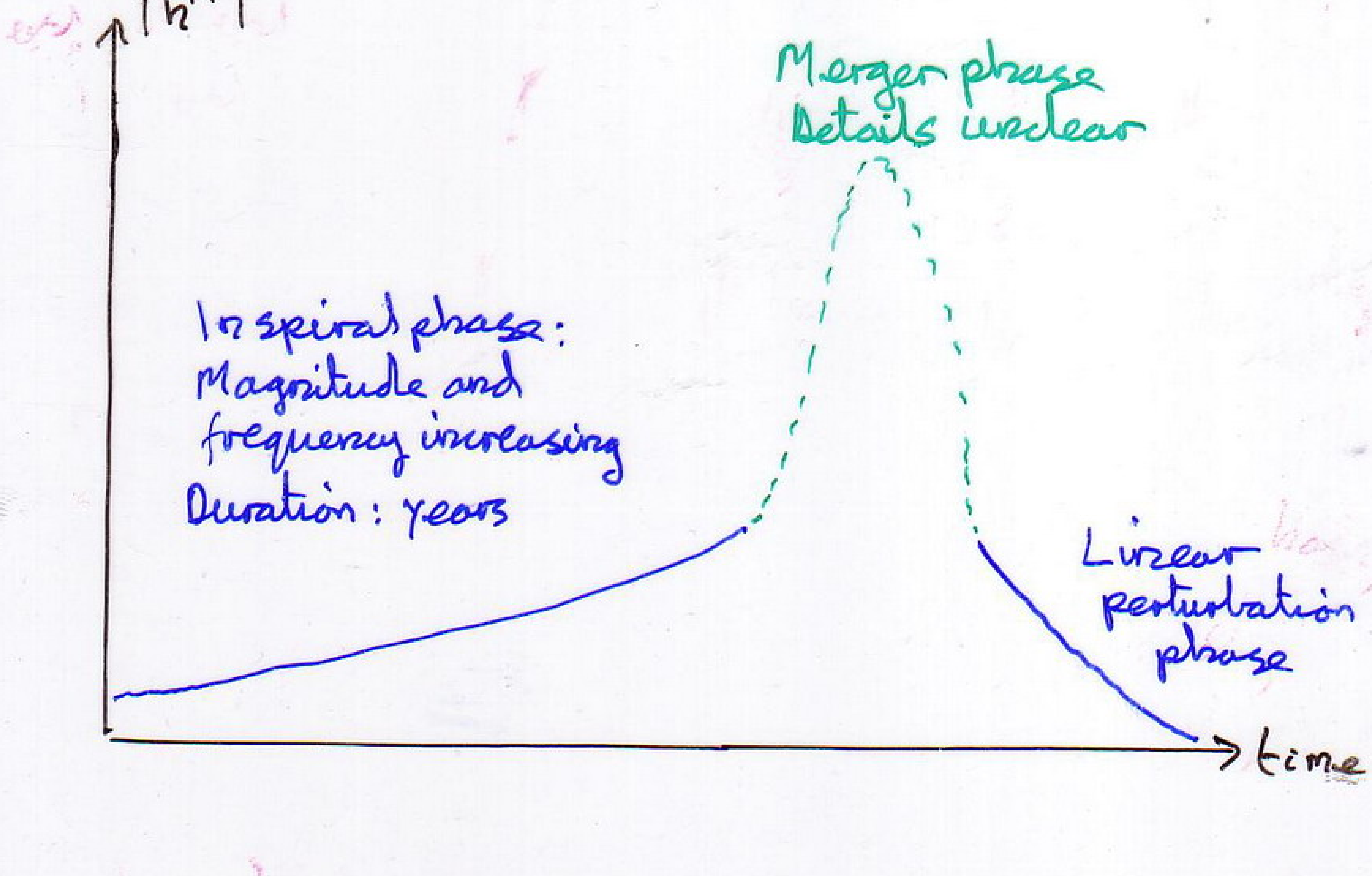
Find $h^{TT}_{+,\times} \sim \varepsilon \frac{M}{r}\left(\exp\left(-\frac{b_1 t}{M}\right) \cdot \exp\left(i\frac{b_2 t}{M}\right)\right)$

where $\varepsilon = O(m/M)$

So waveform $\sim O\left(\frac{m}{r}\right)$ with characteristic

decay time $\frac{M}{b_1} \simeq 1.1 \times 10^7 M_\odot \approx 55s$

period $\frac{2\pi M}{b_2} \simeq 16 \times 10^6 M_\odot \simeq 80s$ or $0.012\,Hz$



Exercise

A gravitational wave detector consists of two arms at right angles in the x-y plane. Each arm is 4 km long, and the detector measures the time difference for a laser beam to travel along each arm and back. Assume that each end of an arm moves on a geodesic. Suppose a gravitational wave with $h_{11} = -h_{22} = \varepsilon \exp(i\omega(t-z))$ (other components zero) meets the detector. What will the detector record?